When do subsets of $\{0,1\}^{G\times G}$ contain recurrent points?

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D. W. Morris: Amenable groups that act on the line, *Algebraic & Geometric Topology* 6 (2006) 2509–2518.
http://arxiv.org/abs/math/0606232

The Question

$C \subset \{0,1\}^{\mathbb{Z}}$ closed, invariant, nonempty.
Poincaré Recurrence: $\exists$ recurrent point for $\mathbb{Z}$-shift.
In fact, a.e. pt recurrent (w.r.t. any inv't prob meas).

$C \subset \{0,1\}^{\mathbb{Z}^n \times \mathbb{Z}^n}$ closed, invariant, nonempty.
$\exists$ pt recurrent for every $g \in \mathbb{Z}^n$.

Can replace $\mathbb{Z}^n$ with any amenable group
(or torsion grp — every el't has finite order)

Question

Replace $\mathbb{Z}^n$ with other interesting classes of groups?

Problem

G = countable group that is ☐.
$C \subset \{0,1\}^{G\times G}$ closed, invariant, nonempty.
$\exists$ pt that is recurrent for each $g \in$ diag'l $\mathbb{Z}^n$-shift.

Rem. Can replace with action on $\{0,1\}^G$ by conjugation.

Fact. amenable $\Rightarrow$ no (nonabelian) free subgroups.

Specific question of interest

Does it suffice to assume G has no free subgroups?

This would prove a conjecture of Peter Linnell.

Motivation

Question (for any group G)

¿ $\exists$ (nontrivial) action of G on the real line $\mathbb{R}$? (orientation preserving)

Example

- $\mathbb{Z}$ acts on $\mathbb{R}$
- $G \twoheadrightarrow \mathbb{Z}$ (homomorphism) $\implies$ G acts on $\mathbb{R}$.

Theorem (Morris, 2006)

G acts on $\mathbb{R}$, has rec pts in $C \subset \{0,1\}^{G\times G}$ $\Rightarrow$ $G \twoheadrightarrow \mathbb{Z}$.

Corollary for amenable G generalized several previous papers & answered question of P. Linnell.

Theorem (Morris, 2006)

G acts on $\mathbb{R}$, has rec pts in $C \subset \{0,1\}^{G\times G}$ $\Rightarrow$ $G \twoheadrightarrow \mathbb{Z}$.

Proof.

Trivial (modulo known results).

0. G acts on $\mathbb{R}$
1. $L = \{(g,h) \in G \times G \mid g(0) < h(0)\} \in \{0,1\}^{G\times G}$
   - can extend to total order on G
   - inv't under mult on left (but not on right?)
2. $C = \{\text{left-inv't orders}\} \subset \{0,1\}^{G\times G}$ [Ghys, Sikora]
3. Recurrence: $\exists$ recurrent order.
4. *Known:* G has a recurrent order ($\Rightarrow$ "Conradian")
   $\implies G \twoheadrightarrow \mathbb{Z}$. ☐