What is a superrigid subgroup?

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Abstract

It is not difficult to see that every group homomorphism from $\mathbf{Z}^k$ to $\mathbf{R}^n$ extends to a homomorphism from $\mathbf{R}^k$ to $\mathbf{R}^n$. (Essentially, this is the fact that a linear transformation can be defined to have any desired action on a basis.) We will see other examples of discrete subgroups Γ of connected groups G, such that the homomorphisms defined on Γ can ("almost") be extended to homomorphisms defined on all of G.

What is a superrigid subgroup?

(1. Combinatorial superrigidity)

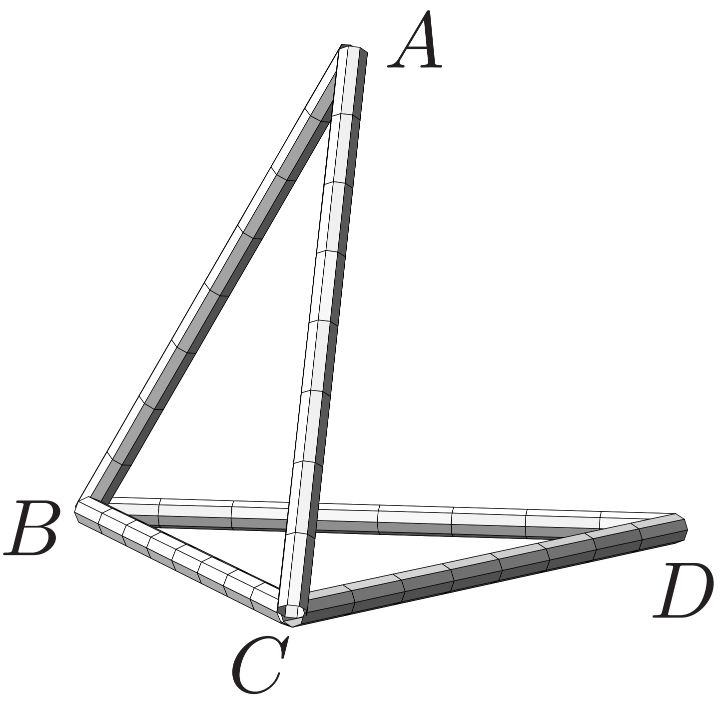
2. Group-theoretic superrigidity

(3. The analogy)

4. Superrigid subgroups

(1. Combinatorial superrigidity)

Eg. Two joined triangles

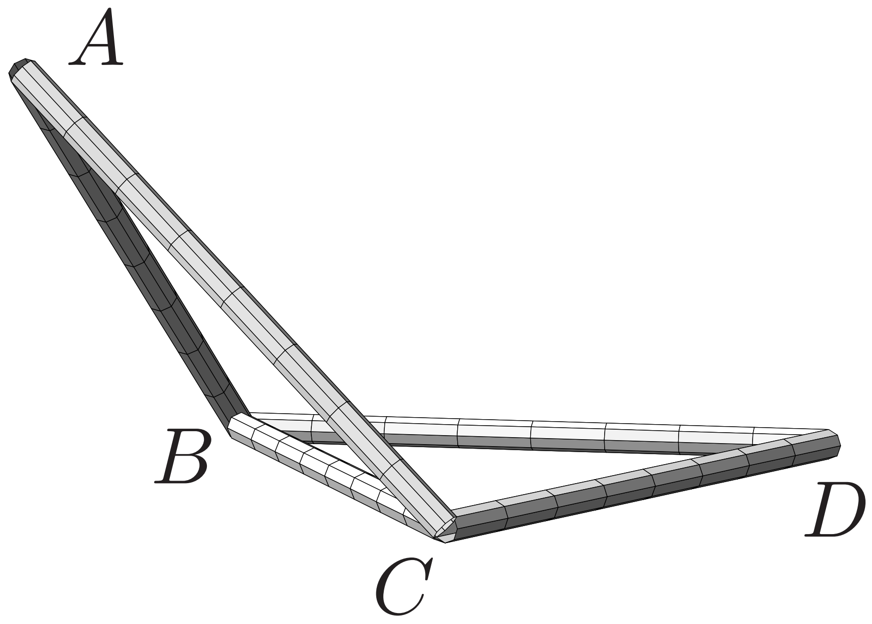


A — B A — C

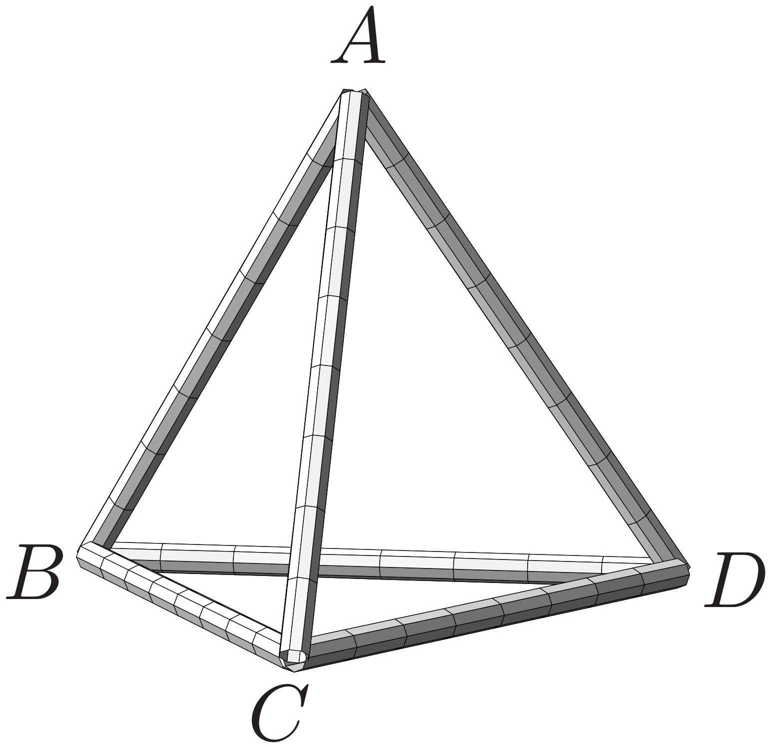
B — C B — D C — D

This is not rigid.

I.e., it can be deformed (a "hinge").


A
B
C
D

Eg. Tetrahedron

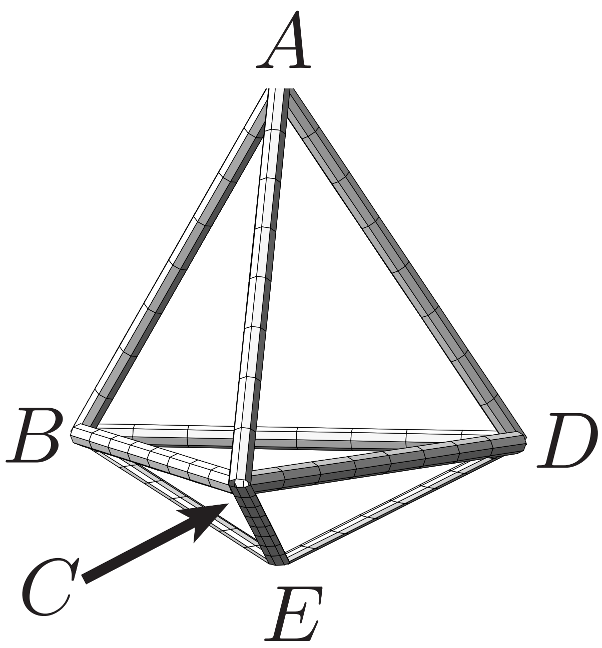


A —— B A —— C A —— D

B —— C B —— D C —— D

This is **rigid** (cannot be deformed).

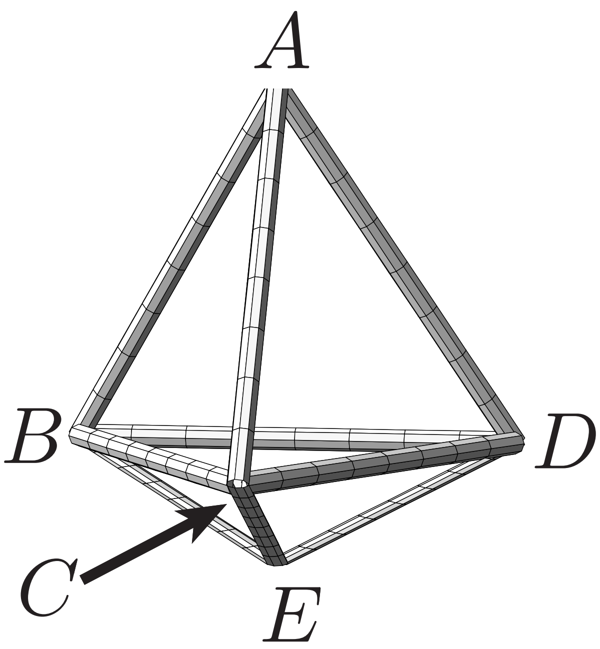
Eg. Add a small tetrahedron

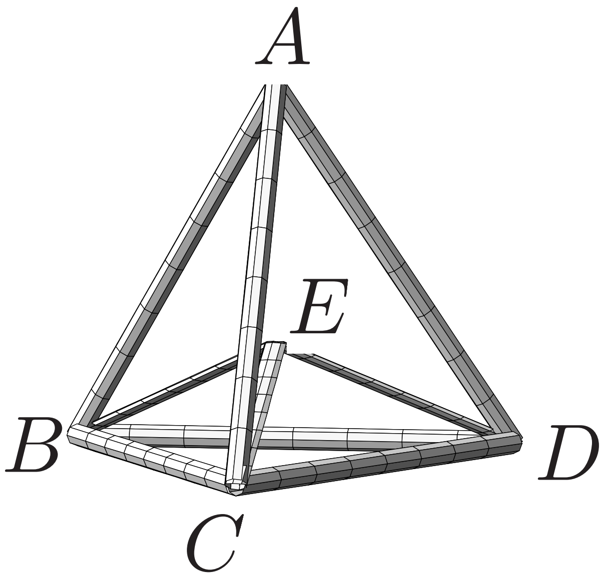


A — B	A — C	A — D
B — C	B — D	C — D
B — E	C — E	D — E

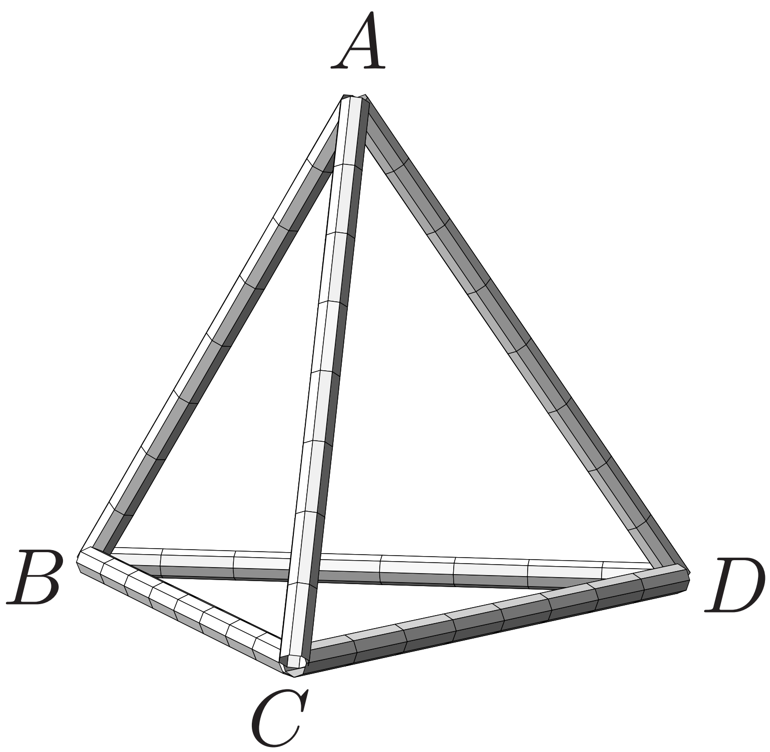
This is rigid.

However, it is not **superrigid**:
if it is taken apart,
it can be reassembled incorrectly.





A — B	A — C	A — D
B — C	B — D	C — D
B — E	C — E	D — E



A —— B $\quad$ A —— C $\quad$ A —— D

B —— C $\quad$ B —— D $\quad$ C —— D

A tetrahedron *is* superrigid: the combinatorial description determines the geometric structure.

Combinatorial superrigidity:

Make a copy of the object,
according to the combinatorial rules.
The copy is the exact same shape as the original.

This talk: analogue in group theory

2. Group-theoretic superrigidity

Group homomorphism $\phi \colon \mathbb{Z} \to \mathbb{R}^d$

(i.e., $\phi(m+n) = \phi(m) + \phi(n)$)

$\Rightarrow$ ϕ extends to a homomorphism $\hat{\phi} \colon \mathbb{R} \to \mathbb{R}^d$.

Namely, define $\hat{\phi}(x) = x \cdot \phi(1)$.

Check:

- $\hat{\phi}(n) = \phi(n)$
- $\hat{\phi}(x+y) = \hat{\phi}(x) + \hat{\phi}(y)$
- $\hat{\phi}$ is continuous

(only allow continuous homomorphisms)

Group homomorphism $\phi\colon \mathbb{Z}^k \to \mathbb{R}^d$

$\Rightarrow \phi$ extends to a homomorphism $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$.

Proof. Use standard basis $\{e_1, \ldots, e_k\}$ of $\mathbb{R}^k$.

"A linear transformation can
have any desired effect on a basis."

Linear transformation
$\Rightarrow$ homomorphism of additive groups

Group Representation Theory:
study homomorphisms into *Matrix Groups*.

$\mathrm{GL}_d(\mathbb{C}) = d \times d$ matrices over $\mathbb{C}$
with nonzero determinant

This is a group under multiplication.

$$\mathbb{R}^d \cong \begin{pmatrix} 1 & 0 & 0 & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

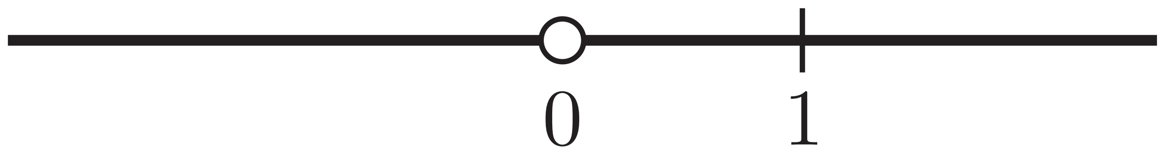
Group homomorphism $\phi\colon \mathbb{Z} \to \mathrm{GL}_d(\mathbb{R})$

(i.e., $\phi(m+n) = \phi(m) \cdot \phi(n)$)

$\not\Rightarrow$ extends to homo $\hat{\phi}\colon \mathbb{R} \to \mathrm{GL}_d(\mathbb{R})$.

(Only allow *continuous* homos.)

Proof by contradiction.



Suppose $\exists$ homo $\hat{\phi}\colon \mathbb{R} \to \mathrm{GL}_d(\mathbb{R})$
with $\hat{\phi}(n) = \phi(n)$ for all $n \in \mathbb{Z}$.

$\hat{\phi}(0) = I \qquad \Rightarrow \det\bigl(\hat{\phi}(0)\bigr) = 1 > 0$

$\mathbb{R}$ connected

$\Rightarrow \hat{\phi}(\mathbb{R})$ connected

$\Rightarrow \det\bigl(\hat{\phi}(\mathbb{R})\bigr)$ connected

$\Rightarrow \det\bigl(\hat{\phi}(\mathbb{R})\bigr) > 0$

$\Rightarrow \det\bigl(\phi(1)\bigr) > 0$

Maybe $\det\bigl(\phi(1)\bigr) < 0$. $\qquad \to\leftarrow$

(Any $A \in \mathrm{GL}_d(\mathbb{R})$, let $\phi(n) = A^n$.)

Group homo $\phi\colon \mathbb{Z} \to \mathrm{GL}_d(\mathbb{R})$

(i.e., $\phi(m+n) = \phi(m)\cdot\phi(n)$)

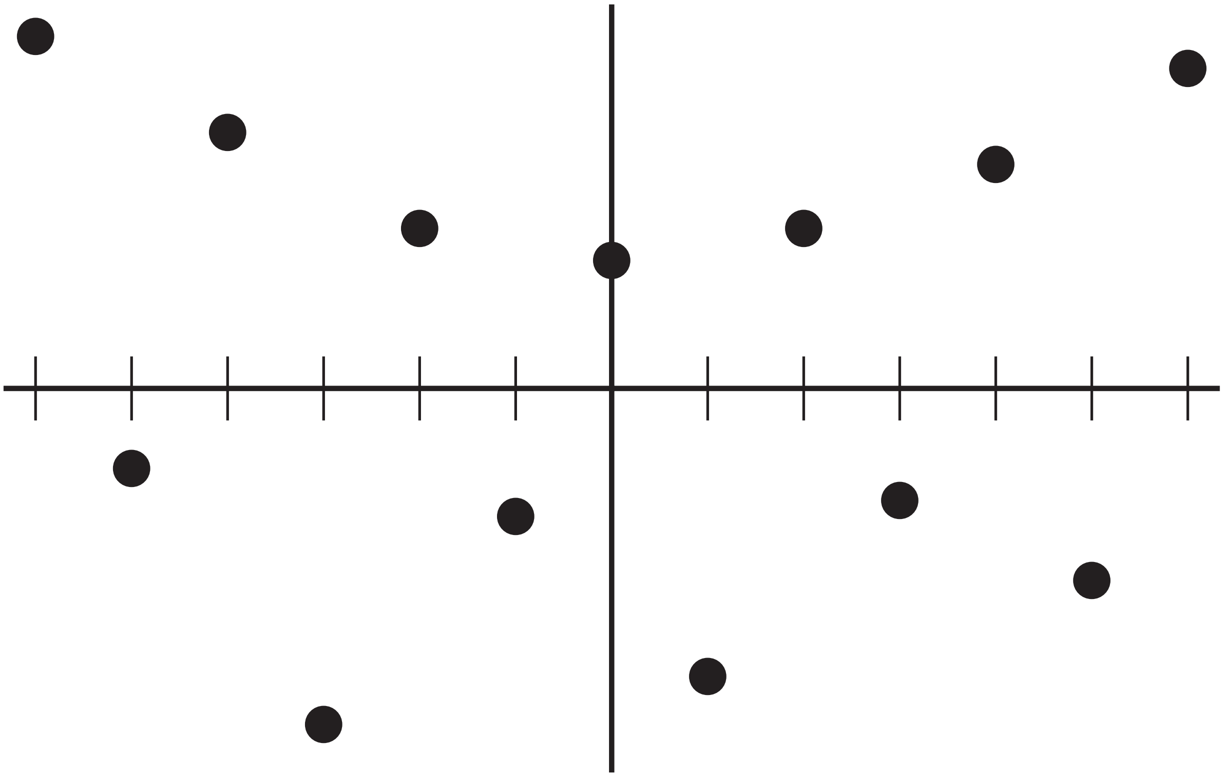
$\not\Rightarrow$ extends to homo $\hat{\phi}\colon \mathbb{R} \to \mathrm{GL}_d(\mathbb{R})$.

Because: maybe $\det\big(\phi(1)\big) < 0$.

However, $\det\big(\phi(\text{even})\big) > 0$.

$$\begin{aligned}
&\det\big(\phi(2m)\big)\\
&\qquad = \det\big(\phi(m+m)\big)\\
&\qquad = \det\big(\phi(m)\cdot\phi(m)\big)\\
&\qquad = \Big(\det\big(\phi(m)\big)\Big)^2\\
&\qquad > 0.
\end{aligned}$$

May have to ignore odd numbers:
restrict attention to even numbers.



Analogously, may need to restrict to multiples of 3
(or 4 or 5 or . . .)

Restrict attention to multiples of N

$\{$multiples of $N\}$ is a subgroup of $\mathbb{Z}$

"Restrict attention to a finite-index subgroup"

Prop. *Group homomorphism* $\phi\colon \mathbb{Z}^k \to \mathrm{GL}_d(\mathbb{R})$
$\Rightarrow \phi$ *"almost" extends to homo* $\hat{\phi}\colon \mathbb{R}^k \to \mathrm{GL}_d(\mathbb{R})$
such that $\hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}$. (*"Zariski closure"*)

This means $\mathbb{Z}^k$ is *superrigid* in $\mathbb{R}^k$.

> "Homomorphisms defined on $\mathbb{Z}^k$ almost extend to be defined on $\mathbb{R}^k$"

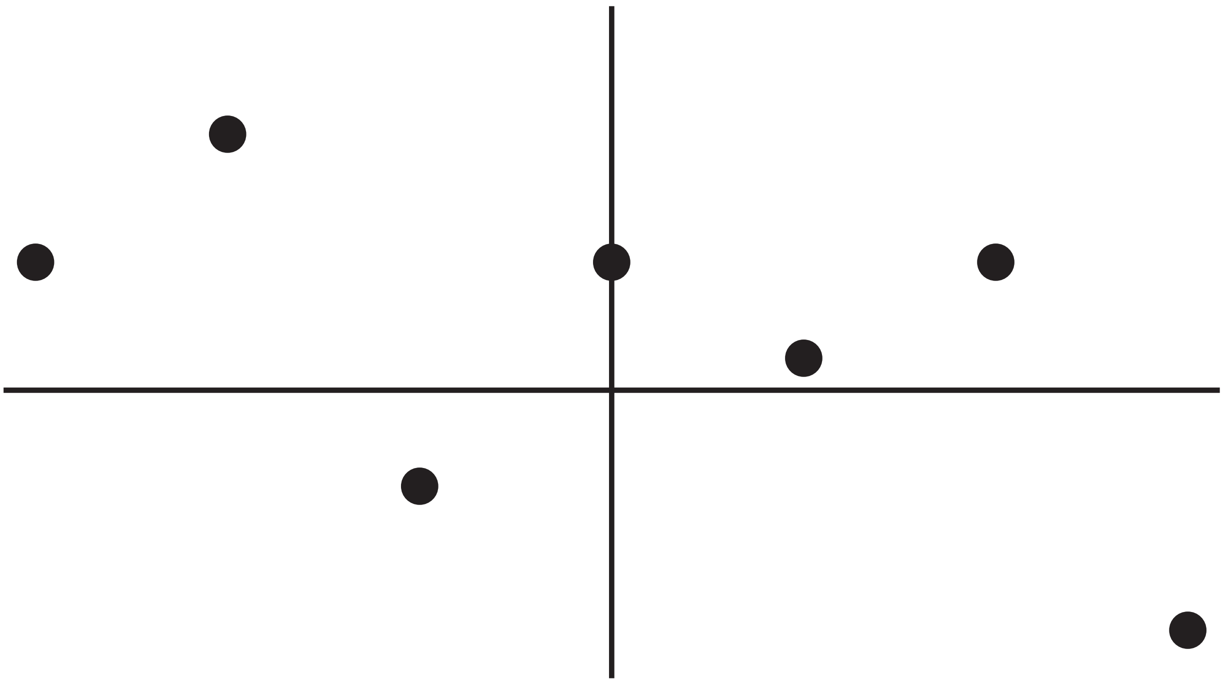
Generalize to nonabelian groups.

Lagrange interpolation:

there is a polynomial curve

$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

through any $n + 1$ points.



Idea: Zar closure is like *convex hull*.

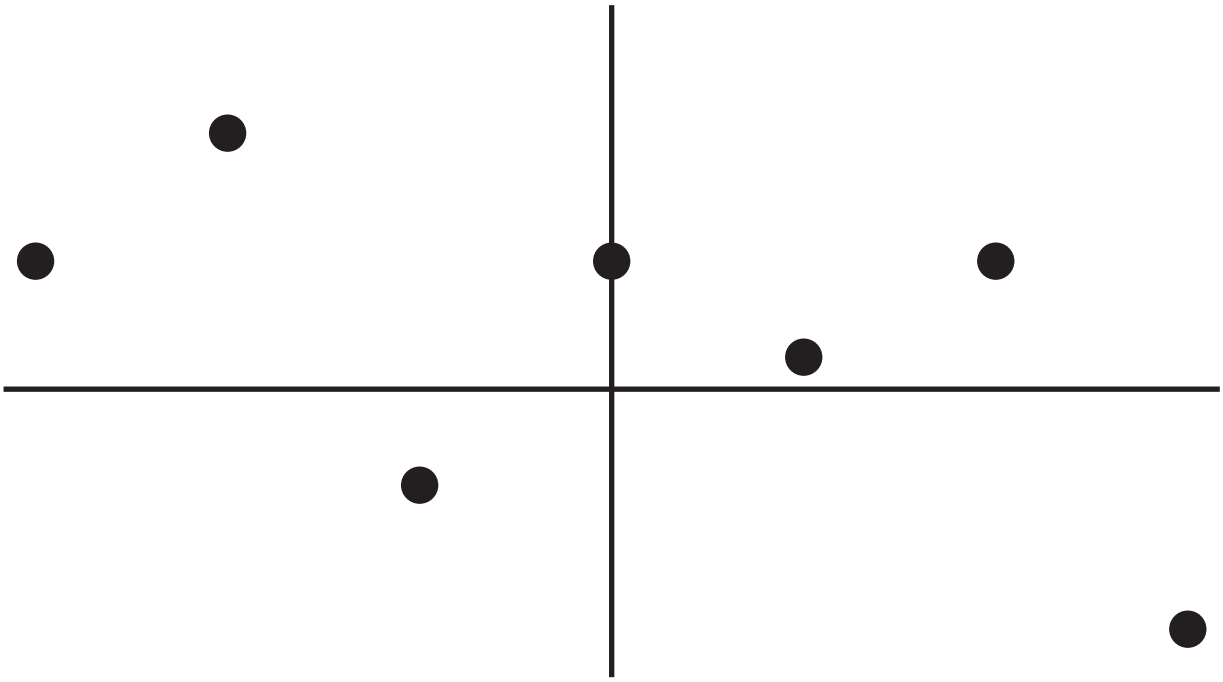


Image of ϕ controls image of $\hat{\phi}$.

Eg. If all matrices in $\phi(\mathbb{Z})$ commute, then all matrices in $\hat{\phi}(\mathbb{R})$ commute.

Eg. If all matrices in $\phi(\mathbb{Z})$ fix a vector v, then all matrices in $\hat{\phi}(\mathbb{R})$ fix v.

(3. The analogy)

Combinatorial superrigidity:

Make a copy of the object,
according to the combinatorial rules.

The copy is the exact same shape as the original.

Maybe not exactly the same object:
may be rotated from the original position;
may be translated from original position.

These are trivial modifications:
rotations and translations are symmetries of the whole universe (Euclidean space $\mathbb{R}^3$).

Combinatorial superrigidity:

Make a copy of the object, according to the combinatorial rules.

The same result can be obtained by keeping the original object and moving the whole universe to a new position.

> "If the object can be moved somewhere,
> then the whole universe can be moved there."

Let H be a subgroup of a group G.
Superrigid means:

homomorphism $\phi\colon H \to \mathrm{GL}_d(\mathbb{R})$ extends
to homomorphism $\hat{\phi}\colon G \to \mathrm{GL}_d(\mathbb{R})$

Group-theoretic superrigidity:

Make a copy of H as a group of matrices.

The same copy of H can be obtained by moving all of G into a group of matrices.

4. Superrigid subgroups

Prop. *Group homomorphism* $\phi \colon \mathbb{Z}^k \to \mathrm{GL}_d(\mathbb{R})$
$\Rightarrow \phi$ *"almost" extends to homo* $\hat{\phi} \colon \mathbb{R}^k \to \mathrm{GL}_d(\mathbb{R})$
such that $\quad \hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}$. (*"Zariski closure"*)

This means $\mathbb{Z}^k$ is *superrigid* in $\mathbb{R}^k$.

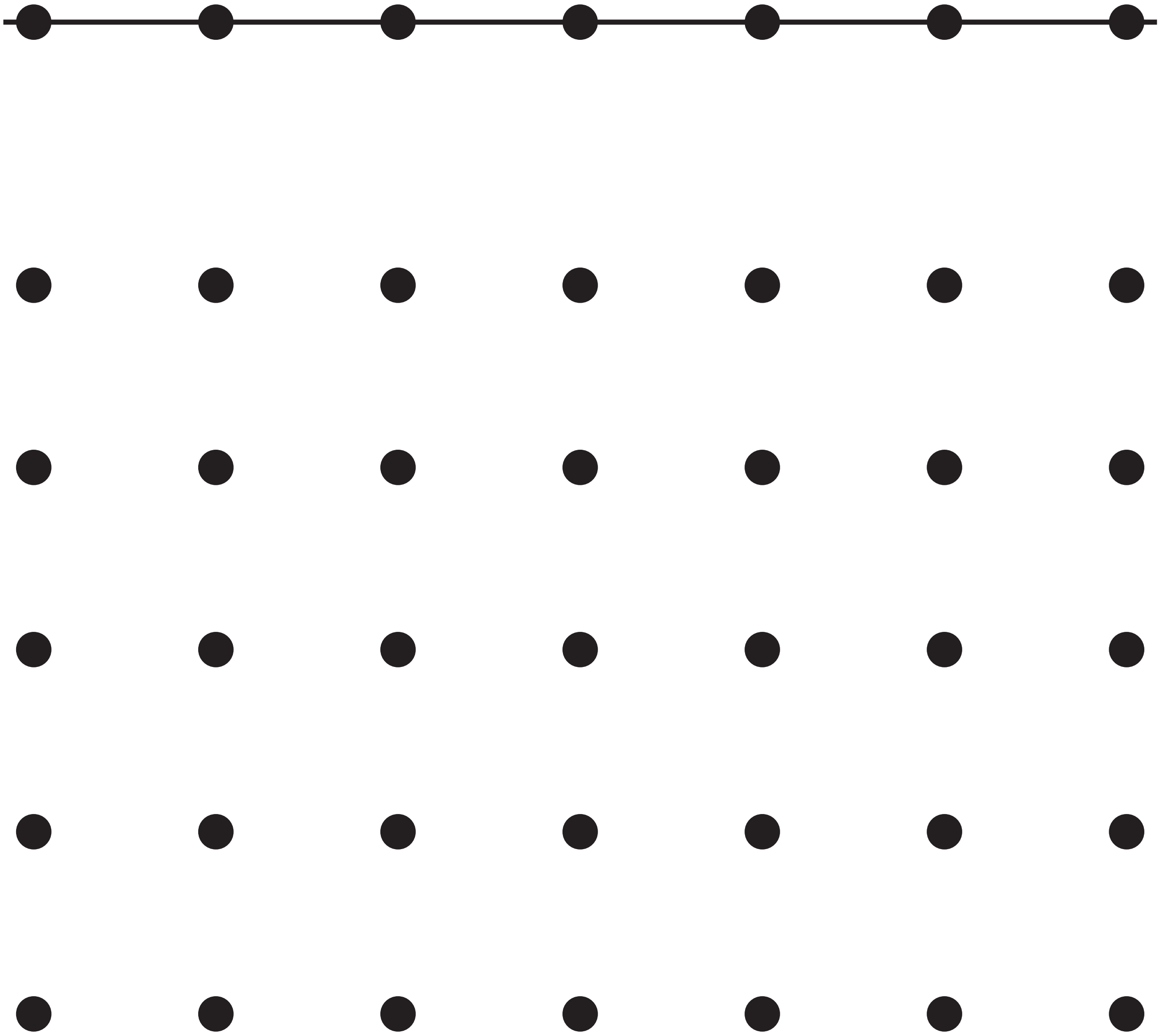
Generalize to nonabelian groups.

$\mathbb{Z}^k$ is a *lattice* in $\mathbb{R}^k$. I.e.,

- $\mathbb{R}^k$ is a (simply) connected grp
  ("Lie group")
- $\mathbb{Z}^k$ is a discrete subgroup
- all of $\mathbb{R}^k$ is within a bounded distance of $\mathbb{Z}^k$

  $\exists C,\ \forall x \in \mathbb{R}^k,\ \exists m \in \mathbb{Z}^k, \quad d(x,m) < C.$

H is a *lattice* in G



All of $\mathbb{R}^k$ is within $\sqrt{k}/2$ of $\mathbb{Z}^k$

Let us consider *solvable* groups.

A connected subgroup G of $\mathrm{GL}_d(\mathbb{C})$ is **solvable** if it is upper triangular

$$G \subset \begin{pmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C}^\times & \mathbb{C} \\ 0 & 0 & \mathbb{C}^\times \end{pmatrix}$$

(or is after a change of basis).

Eg. All abelian groups are solvable.

Proof. Every matrix can be triangularized over $\mathbb{C}$.

Pairwise commuting matrices can be simultaneously triangularized.

Prop. *H superrigid in G*

$$\Rightarrow \overline{H} = \overline{G} \pmod{\overline{Z(G)}}.$$

Proof. The inclusion $H \hookrightarrow \mathrm{GL}_d(\mathbb{R})$
must extend to $G \hookrightarrow \mathrm{GL}_d(\mathbb{R})$
with $G \subset \overline{H}$.

Converse:

Thm (Witte). *A lattice H in a solvable grp G is superrigid iff* $\overline{H} = \overline{G} \pmod{\overline{Z(G)}}$.

Examples of lattices.

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} \mathbb{R}^+ & 0 & 0 \\ 0 & \mathbb{R}^+ & 0 \\ 0 & 0 & \mathbb{R}^+ \end{pmatrix}$$

$$H = \begin{pmatrix} 2^{\mathbb{Z}} & 0 & 0 \\ 0 & 2^{\mathbb{Z}} & 0 \\ 0 & 0 & 2^{\mathbb{Z}} \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \overline{G} = G$$

$$H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z}+\mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \overline{H} = G$$

$$G' = \begin{pmatrix} 1 & t & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i t} \end{pmatrix}$$

$$\overline{G'} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{T} \end{pmatrix}$$

$$H' = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z}+\mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = H$$

H is a lattice in both G and G'.

$$\overline{H} = G \neq \overline{G'} \qquad \text{so } \overline{H} \neq \overline{G'}$$

H is *not* superrigid in G'.

E.g., the identity map $\phi\colon H \to H$
does not extend to homo $\hat{\phi}\colon G' \to \overline{H}$.

Proof. Note that $\overline{H} = G$ is abelian
but G' is not abelian.

A nonabelian group cannot be embedded in an abelian one.

$\overline{H} \neq \overline{G'}$: some of the rotations associated to G' do not come from rotations associated to H

$$\mathrm{rot}\begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha}{|\alpha|} & 0 \\ 0 & \dfrac{\beta}{|\beta|} \end{pmatrix}$$

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