Touring a torus

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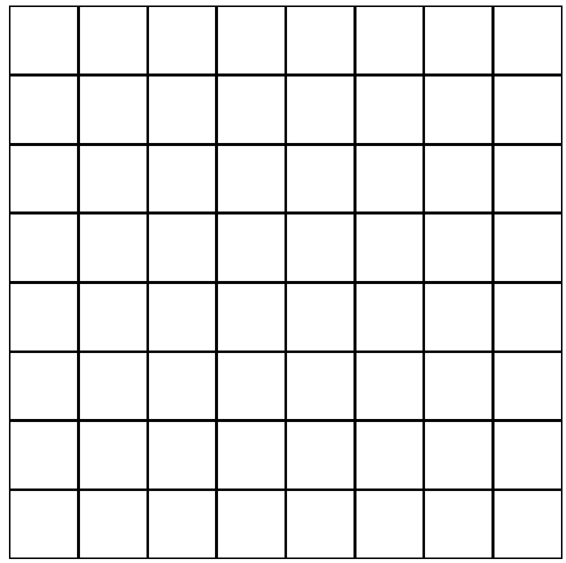
Abstract

Place a checker on one square of a checkerboard. Asking whether the checker can make a tour of the board leads to some difficult questions, and to interesting fields of mathematics, such as number theory, topology, and group theory.

The title is from a talk by J. A. Gallian on similar material.

A checker is in the Southwest corner of a standard 8×8 checkerboard.

Can the checker tour the board?



What about rectangular ($a \times b$) checkerboards?

- The checker moves North, South, East, West (not diagonally!)
- A tour must visit each square exactly once *and return to the starting point.*

Prop. *A checker can tour any board with an even number of squares.*

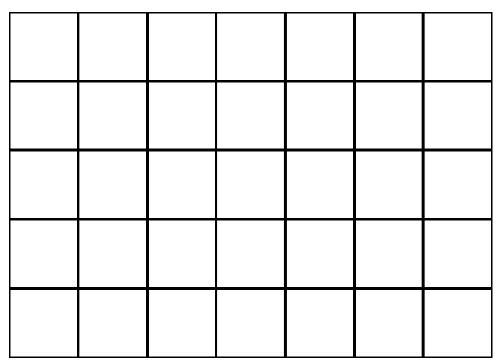
Prop. *A checker cannot tour a board with an odd number of squares.*

Proof. NORTH = SOUTH and EAST = WEST.

TOTAL
= NORTH + SOUTH + EAST + WEST
= (2 × NORTH) + (2 × EAST)
is an even number.

TOTAL = # squares on the checkerboard. □

Allow the checker to step off the edge of the board.

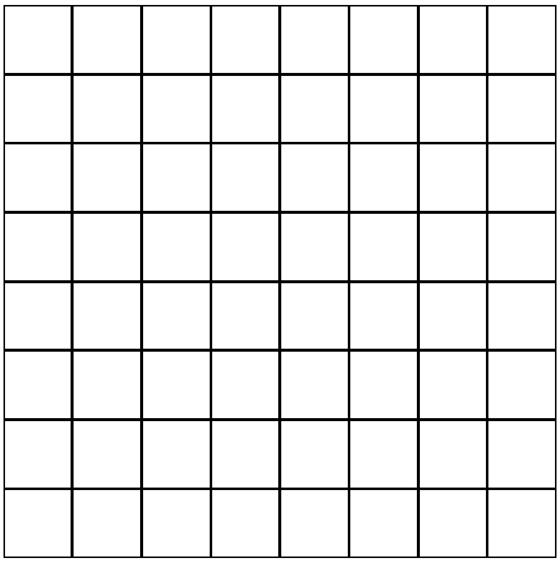


(The board is now toroidal, rather than flat.)

Prop. *A checker can tour any board if allowed to step off the edge.*

Find a route that always travels North or East.

This is easy on the 8×8 checkerboard.



Defn. A board is *hamiltonian* if it has such a tour.

Prop. *Any square checkerboard is hamiltonian.*

Eg. The 3×5 checkerboard is **not** hamiltonian.

Proof by contradiction.

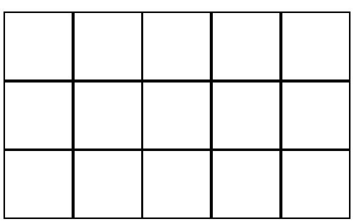
The tour must have 15 steps: $E + N = 15$.

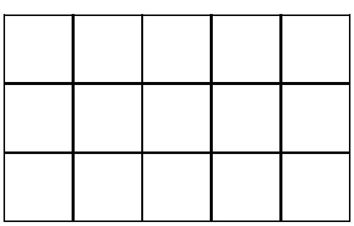
E is divisible by 5.
N is divisible by 3.

E cannot be 0 or 15,
so E is either 5 or 10,
so N is either 10 or 5.

Neither of these is divisible by 3. $\rightarrow\leftarrow$

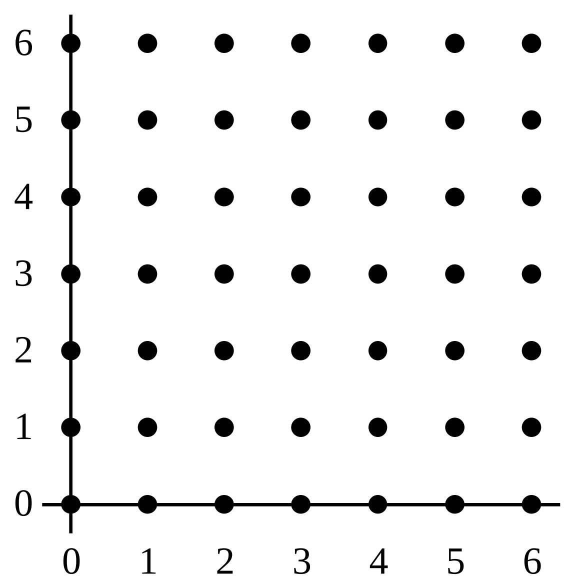
Exer. More generally, the $a \times b$ checkerboard is not hamiltonian if a and b are relatively prime (that is, if $\gcd(a, b) = 1$) and $a, b \geq 2$.

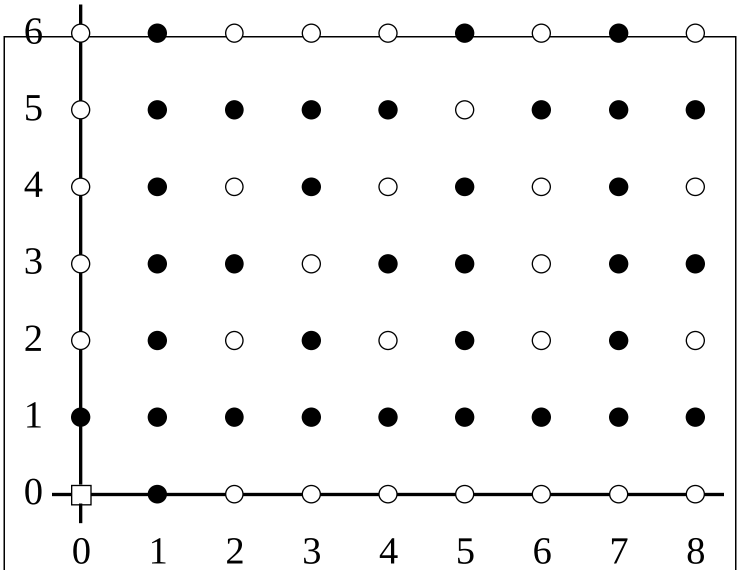




In general, deciding whether a checkerboard is hamiltonian involves the geometry of lattice points in the plane.

Defn. A *lattice point* is a point with integer coordinates.

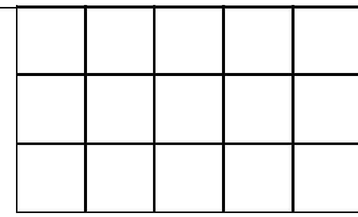




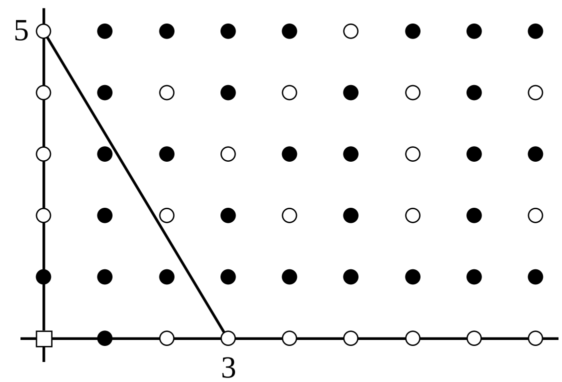
Stand at origin:

$(4,6)$ is *not* visible	$\gcd(4,6) = 2$.
$(6,3)$ is *not* visible	$\gcd(6,3) = 3$.
$(3,5)$ is visible	$\gcd(3,5) = 1$.

Defn. A lattice point is *visible* (or *primitive*) if its coordinates are relatively prime.



Recall: the 3×5 checkerboard is not hamiltonian.
(3 and 5 are relatively prime.)



There are no visible lattice points on the line segment joining $(3,0)$ and $(0,5)$.

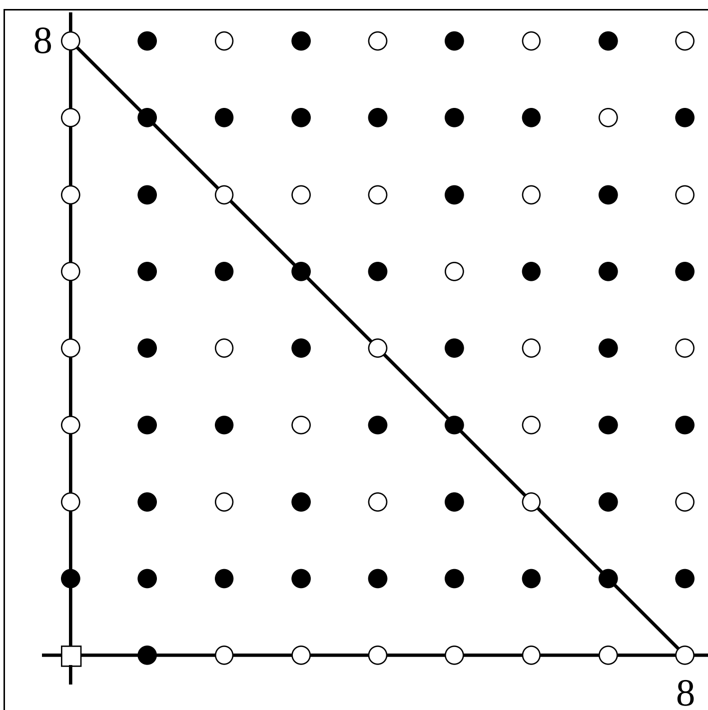
Prop. *If a and b are relatively prime, then*

- *the $a \times b$ checkerboard is not hamiltonian, and*
- *there are no visible lattice points on the line segment joining $(a,0)$ and $(0,b)$.*

Eg. • The 8×8 checkerboard is hamiltonian.
(E.g., 7E, N, 7E, N, ...)

- There are visible lattice points on the line segment joining $(8,0)$ and $(0,8)$.
(E.g., $(7,1)$.)

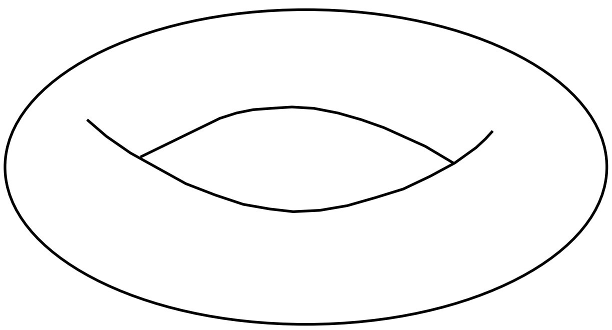
Thm (R. A. Rankin, Trotter-Erdös). *The $a \times b$ checkerboard is hamiltonian if and only if there is a visible lattice point on the line segment joining $(a,0)$ and $(0,b)$.*



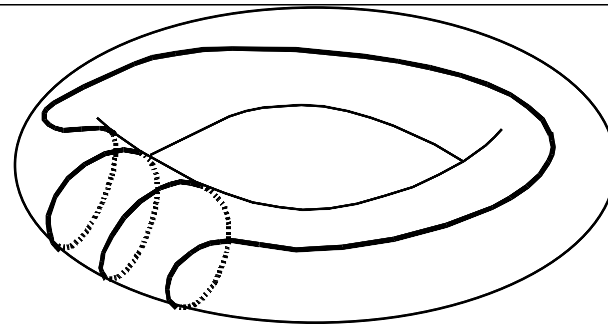
Thm (R. A. Rankin, Trotter-Erdös). *The $a \times b$ checkerboard is hamiltonian if and only if there is a visible lattice point on the line segment joining $(a, 0)$ and $(0, b)$.*

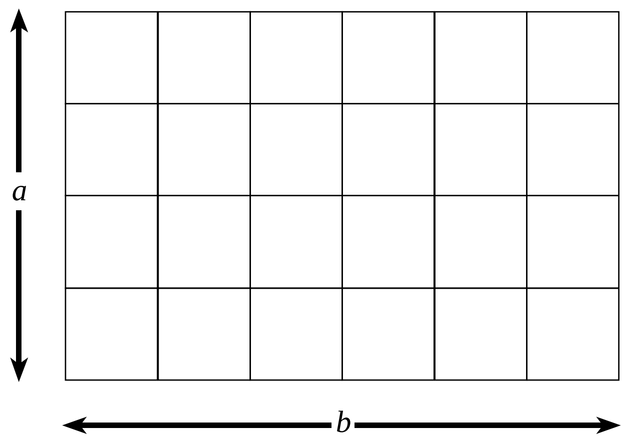
Proof. ($\Rightarrow$, Stephen Curran)

Consider the board to be toroidal.



The path traced out by the checker is a closed path on the torus — a *torus knot.*





Let $(s, t) \in \mathbb{Z} \times \mathbb{Z}$ be the knot class of this knot.
(The knot wraps s times longitudinally, the knot wraps t times meridionally.)

In other words, the checker steps off:

- the East edge of the board s times, and
- the North edge of the board t times.

The tour has

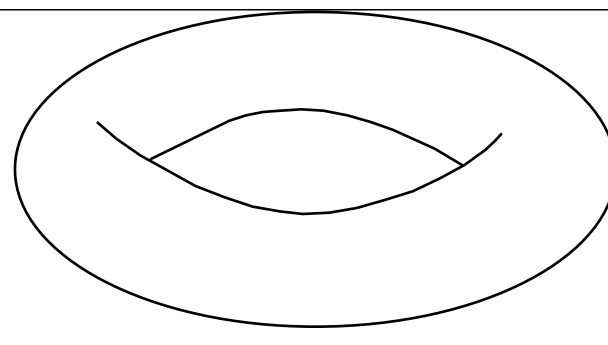
- bs steps East, and
- at steps North.

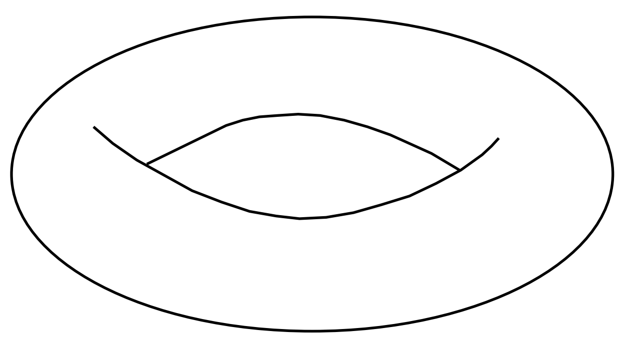
Therefore, $bs + at = ab$.

So (s, t) is on the line segment joining $(a, 0)$ and $(0, b)$.

Since (s, t) is the knot class,
$\gcd(s, t) = 1$.

$\therefore$ (s, t) is a visible latt pt on the line segment. $\square$





Thm (R. A. Rankin, Trotter-Erdös). *The $a \times b$ checkerboard is hamiltonian if and only if there is a visible lattice point on the line segment joining $(a, 0)$ and $(0, b)$.*

Proof ($\Leftarrow$). We have $bs + at = ab$
with $\gcd(s, t) = 1$.

Hence, there are e and n with
$e + n = \gcd(a, b)$,
$\gcd(e, b) = 1$, and
$\gcd(n, a) = 1$.

Tour the board in a periodic pattern:
e steps East, n steps North;
e steps East, n steps North;
e steps East, n steps North;
...
until return to the start. □

Change the rules:

A tour must visit each square exactly once *but need not return to the starting place.*

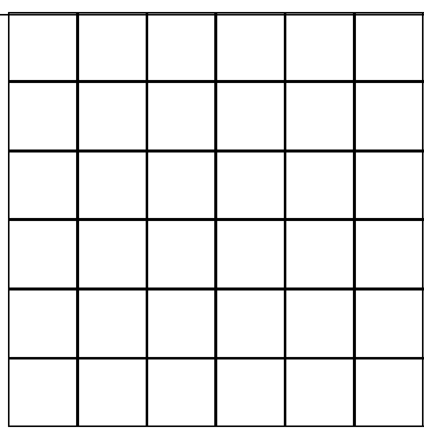
Where can tours end? (starting in SW corner)

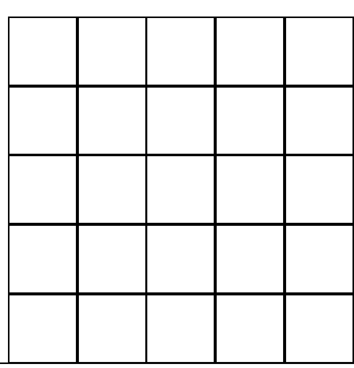
Thm. *On an $n \times n$ (square) checkerboard:*
- *Tours always end on the main diagonal.*
- *n even $\Rightarrow \exists$ tour to anywhere on main diag.*
- *n odd $\Rightarrow$ only to every other vertex.*

Harder if the checkerboard is not square, but solved in terms of the geometry of lattice points.

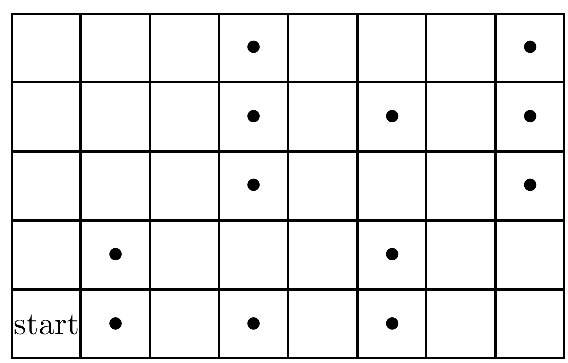
Eg. If a and b are relatively prime, then
\# endpoints
= \# visible latt pts in triangle $T(a, b)$
− 1.

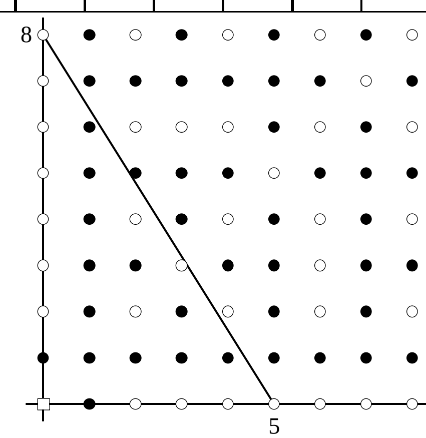
Cor. *If a and b are large (and rel prime), then*
$$\#\ \textit{endpoints} \approx \frac{3}{\pi^2} ab \approx .304\, ab.$$



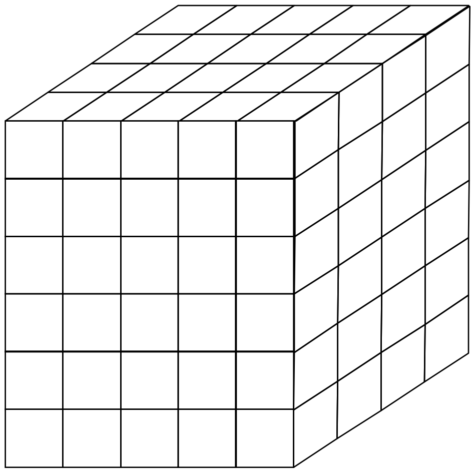


Tours can end at the marked squares





Let's look at higher dimensions.



Which 3-dimensional checkerbds are hamiltonian?

(North, East, Up)

Answer: all of them.

Same for 4D, 5D, 6D, ...

The proof depends on two-dimensional boards.

Each level of a 3D board can be thought of as a 2D board.

We can tour a 3-dimensional board level-by-level:
traverse all the cubes in a level,
then move up to the next level.

The idea is that we can choose various paths in the various levels so that we end up at any desired cube in the top level.

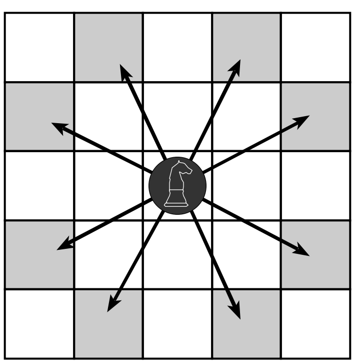
In particular, there is a tour that ends directly above the Southwest corner of the bottom level.

Hence the 3D board is hamiltonian. □

Conj. *If* $\gcd(a, b, c) = 1$ (*and* $a, b, c \geq 2$), *then tours in the* $a \times b \times c$ *checkerbd can end anywhere.*

Different contraints on the motion of the checker.
(Still on a toroidal checkerboard.)

Eg. Knight moves (on a 2D board):



Only consider constraints that allow the checker to get to every square. ("generating set")

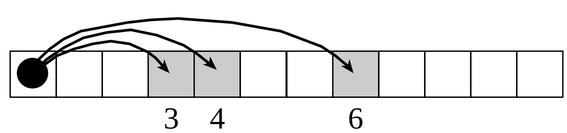
Eg. If
- a and b are even, and
- checker moves diag'ly (NE, NW, SE, SW),

then the checker cannot get to every square.

If there are only two generators, then visible lattice points again provide the answers.

When there are more than two generators, mathematicians do not yet know a good general method to tell whether the checkerboard is hamiltonian, even for one-dimensional checkerboards.

Eg. $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 4, 6)$ is not hamiltonian.

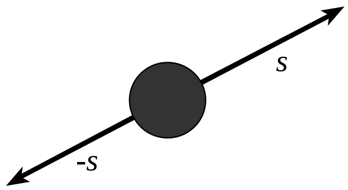


Note: 6 is a *redundant* generator in this example.
(Can get everywhere using only 3 and 4.)

Conj. *Any checkerboard is hamiltonian if*
- *there are at least three generators, and*
- *none of the generators are redundant.*

Not known even for 1D checkerboards!

Prop. *If the generating set is "symmetric," then the checkerboard is hamiltonian.*



Defn. A generating set is *symmetric* if, for each allowable move, the inverse is also allowable.

These results are for checkerboards that are in the shape of a torus.

One can also consider checkerboards that are in the shape of a projective plane. This is obtained by applying a twist when gluing the east edge to the west edge, and the north edge to the south edge.

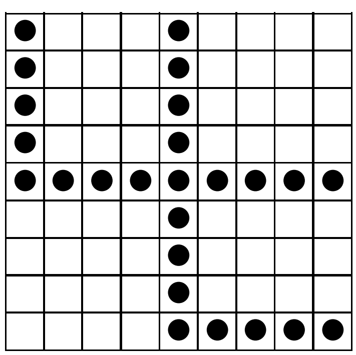
On these boards, it is (usually) not possible to find a tour that starts in the southwest corner. So a natural question to to ask where tours can start ("initial squares"), besides where they can end ("terminal squares").

These problems have been solved for square ($n \times n$) checkerboards. The basic shape of the answer depends on whether n is even or odd.

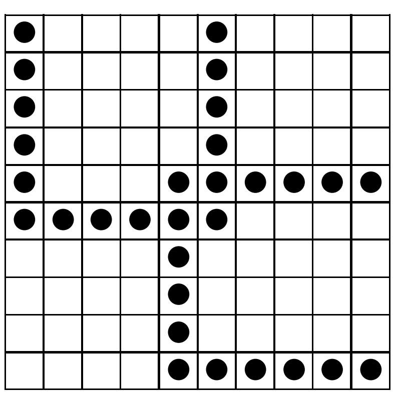
The answers are not yet known for $a \times b$ checkerboards, but it should be feasible to find them.

Initial squares in $n \times n$ projective checkerboards

odd

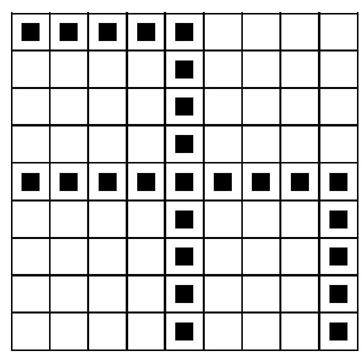


even

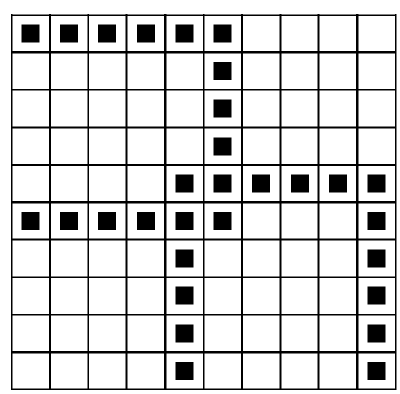


Terminal squares in projective checkerboards

odd



even



For those who have studied group theory:

Let S be a generating set for a finite group G.

Can we tour G by using the generators from S?

List elements of G:

$g_0, g_1, \ldots, g_n$ (with $g_n = g_0$),

s.t. $g_{i+1} = g_i s_i$ for some $s_i \in S$.

Conj. *If S is symmetric, then G is hamiltonian.*

We are nowhere near a proof of this conjecture.

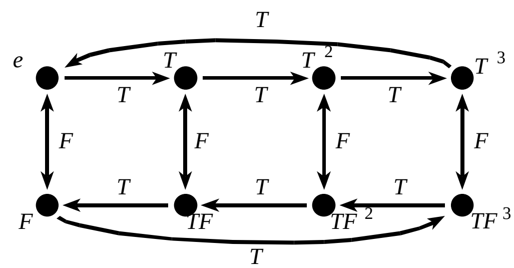
It is true if

- G is abelian, or
- G has prime-power order, or
- the commutator subgroup of G is cyclic of prime-power order.

Not known, even for some dihedral groups.

Eg. The dihedral group of order 8 is generated by a rotation and a reflection.

$$D_8 = \langle T, F \mid T^4 = F^2 = (TF)^2 = e \rangle$$



We define the *Cayley digraph* $\mathrm{Cay}(G;S)$ as follows.

The vertices of the digraph are the elements of G.

There is a directed edge from g to gs for $g \in G$ and $s \in S$.

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