Superrigid subgroups of solvable Lie groups

Dave Witte

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078
USA

dwitte@math.okstate.edu

http://www.math.okstate.edu/~dwitte

Solvable groups.

Defn. (by induction).

- Abelian groups are solvable.
- $N \triangleleft G$ with N, G/N solvable $\Rightarrow$ G solvable.

(G solv $\Leftrightarrow \exists N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_k$ with N_i/N_{i-1} abel)

Rem.

- Subgroups of solvable groups are solvable.
- Quotients of solvable groups are solvable.
- Semidirect products of solv grps are solv.
- "Solvable grps have lots of normal subgrps."

Prop. *A connected Lie group G is solvable iff*

$$G \hookrightarrow \begin{pmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C}^\times & \mathbb{C} \\ 0 & 0 & \mathbb{C}^\times \end{pmatrix} \subset \mathrm{GL}_d(\mathbb{C})$$

Cor. $\Rightarrow [G, G] \subset \begin{pmatrix} 1 & \mathbb{C} & \mathbb{C} \\ 0 & 1 & \mathbb{C} \\ 0 & 0 & 1 \end{pmatrix}$ (*unipotent*)

Prop. *Group homomorphism* $\phi\colon \mathbb{Z}^k \to \mathbb{R}^d$
$\Rightarrow \phi$ *extends to homomorphism* $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$.

Proof. $\hat{\Gamma} = \operatorname{graph}(\phi) \subset \mathbb{R}^k \times \mathbb{R}^d$ $\quad X = \operatorname{span} \hat{\Gamma}$.

Claim. X is the graph of a function $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$.

- $\operatorname{graph}(\hat{\phi})$ is a subgroup $\Rightarrow \hat{\phi}$ is a homo.
- $\operatorname{graph}(\phi) \subset \operatorname{graph}(\hat{\phi}) \Rightarrow \hat{\phi}$ extends ϕ.

Suffices to show:

1) X projects onto $\mathbb{R}^k$.
2) $X \cap (0 \times \mathbb{R}^d) = 0$.

1) $\pi_1(X) =$ conn subgrp of $\mathbb{R}^k$ that contains $\mathbb{Z}^k$
$\Rightarrow \pi_1(X) = \mathbb{R}^k$.

2) *Fact.* $(\operatorname{span} \hat{\Gamma})/\hat{\Gamma}$ is compact (for any closed $\hat{\Gamma}$).

$\pi_1|_{\operatorname{graph}(\phi)}$ is proper (because ϕ is continuous)
$+$ Fact $+$ π_1 is a homo $\Rightarrow \pi_1|_X$ is proper
$\Rightarrow X \cap \pi_1^{-1}(0)$ is compact.

Generalize. $\Gamma \subset G$, $\phi\colon \Gamma \to H$

$$\stackrel{?}{\Longrightarrow} \phi \text{ extends to } \hat{\phi}\colon G \to H.$$

Defn. *Syndetic hull* of Γ:
connected subgrp $X \supset \Gamma$, such that X/Γ is cpct.

Same proof if:

- Every closed $\hat{\Gamma} \subset G \times H$ has a synd hull.
- No conn, proper subgroup of G contains Γ.
- H has no nontrivial compact subgroups.

Furthermore, syndetic hulls unique $\Rightarrow \hat{\phi}$ unique.

Fact. H simply connected, solvable Lie group
$\Rightarrow H$ has no nontrivial compact subgroups.

Fact. Γ cocompact in simply conn, solvable G
$\Rightarrow$ no conn, proper subgroup of G contains Γ.

Eg. $G = \widetilde{\mathrm{SO}}(2) \ltimes \mathbb{R}^2 \qquad \Gamma = \mathbb{Z} \times (\mathbb{Z}, 0)$

$\mathbb{R} \times (\mathbb{R}, 0)$ is not a subgroup of G,
so Γ does not have a syndetic hull in G.

Fact. In a (simply) connected, unipotent group, syndetic hulls exist and are unique.

(*Uniqueness:* $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism, so intersection of connected subgrps is connected.)

Prop (Malcev). *Suppose*

- *G and H are connected, unipotent groups*
- *Γ is a lattice in G*
- *$\phi\colon \Gamma \to H$ is a homomorphism.*

Then ϕ extends uniquely to $\hat{\phi}\colon G \to H$.

Cor (Malcev). *Γ_i lattice in conn, unipotent G_i.*

$$\Gamma_1 \cong \Gamma_2 \Rightarrow G_1 \cong G_2.$$

More generally [Saito], this is true for simply connected, **split**, solvable Lie groups, i.e.,

$$G, H \hookrightarrow \begin{pmatrix} \mathbb{R}^{\times} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R}^{\times} & \mathbb{C} \\ 0 & 0 & \mathbb{R}^{\times} \end{pmatrix}$$

Defn. Γ discrete subgroup of solvable Lie grp G
Γ is *superrigid* in G:
every finite-dimensional representation of Γ virtually extends to a representation of G
(with the same Zariski closure)

$$\forall\ \phi\colon \Gamma \to \mathrm{GL}_d(\mathbb{R}),$$
$$\exists\ \hat\phi\colon G \to \mathrm{GL}_d(\mathbb{R}), \qquad \exists\ \Gamma' \subset \Gamma,$$
$$\hat\phi|_{\Gamma'} = \phi|_{\Gamma'}, \qquad |\Gamma/\Gamma'| < \infty,$$
$$\text{and } \overline{\hat\phi(G)} = \overline{\phi(\Gamma)}^{\circ}. \qquad \text{(Zariski closure)}$$

Thm (Witte). Γ *lattice in simply conn, solv* G.
Γ *superrigid* $\Leftrightarrow$ $\overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad}G}$.

Borel Density. Γ *latt in simply conn, solv* G
$\Rightarrow \exists$ *cpct torus* $T \subset \overline{\mathrm{Ad}G}$, *s.t.* $T\,\overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad}G}$.

Cor (nilshadow). $\exists$ *simply conn* $G' \subset T \ltimes G$, *s.t.*

- $G' \supset$ *finite-index subgroup* Γ' *of* Γ, *and*
- $\overline{\mathrm{Ad}_{G'}(\Gamma')} = \overline{\mathrm{Ad}G'}$.

Example.

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overline{G} = G \qquad \overline{\Gamma} = G$$

$$G' = \begin{pmatrix} 1 & t & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i t} \end{pmatrix} \qquad \overline{G'} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{T} \end{pmatrix}$$

$$\Gamma' = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Gamma$$

Γ is a lattice in both G and G'.

$$\overline{\Gamma} = G \neq \overline{G'} \qquad \text{so } \overline{\Gamma} \neq \overline{G'}$$

Γ is *not* superrigid in G'.

E.g., the identity map $\phi\colon \Gamma \to \Gamma$
does not extend to homo $\hat{\phi}\colon G' \to \overline{\Gamma}$.

Proof. $\overline{\Gamma} = G$ is abelian, but G' is not abelian.

Thm (Witte). Γ *lattice in simply conn, solv* G.

Γ *superrigid* $\Leftrightarrow$ $\overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad} G}$.

Cor. *If* $\phi(\Gamma) \subset \overline{\phi(\Gamma)}^\circ$, *then*

- $\exists$ *finite* $F \subset Z\big(\overline{\phi(\Gamma)}^\circ\big)$,
- $\exists$ *homomorphism* $\zeta \colon \Gamma \to F$,

such that $\phi(\gamma) = \hat{\phi}(\gamma)\,\zeta(\gamma)$ *for all* $\gamma \in \Gamma$.

Cor. *If*

- $\phi(\Gamma) \subset \overline{\phi(\Gamma)}^\circ$,
- *the center of* $\overline{\phi(\Gamma)}^\circ$ *is connected, and*
- $\phi\big(\Gamma \cap [G,G]\big)$ *is unipotent,*

then $\phi = \hat{\phi}|_\Gamma$.

Cor. *A lattice* Γ *in a conn Lie group* G (*not necessarily solvable*) *is "superrigid" iff*

- $\overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad} G}$ mod (cpct ss normal subgrp)
- *and semisimple part of* Γ *is "superrigid."*

Thm (Witte). Γ *lattice in simply conn, solv* G.

Γ *superrigid* $\Leftrightarrow$ $\overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad} G}$.

Key Lemma. $\Gamma \subset$ *simply connected, solvable* G, *s.t.* $\overline{\mathrm{Ad}_G \Gamma} \supset$ *maximal compact torus of* $\overline{\mathrm{Ad} G}$
$\Rightarrow \Gamma$ *has a unique syndetic hull in* G.

Rem. If G is connected, but not simply connected, then syndetic hulls exist, but may not be unique.

(E.g., e and $\mathbb{T}$ are syndetic hulls of e in $\mathbb{T}$.)

Rem. G split (e.g., unipotent)

$\Rightarrow \overline{\mathrm{Ad} G}$ has no conn, compact subgroups

$\Rightarrow$ every closed subgroup has a syndetic hull

$\Rightarrow$ fact used for Malcev and Saito.

Thm (Witte). Γ *lattice in simply conn, solv* G.

$$\Gamma \textit{ superrigid } \Leftrightarrow \overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad} G}.$$

Cor. Γ *need not be a lattice for* $(\Leftarrow)$.

Proof. $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad} G}$

$\Rightarrow \overline{\mathrm{Ad}_G \Gamma} \supset$ maximal cpct subgrp of $\overline{\mathrm{Ad} G}$

$\Rightarrow \Gamma$ has syndetic hull B

So Γ is lattice in B, and is Ad-Zariski dense.

Therefore, ϕ (virt) extends to rep'n of B.

B connected, Ad-Zariski dense $\Rightarrow B \supset [G, G]$.
So not hard to extend to representation of G.

Eg. Γ superrigid in $B \Rightarrow e \times \Gamma$ superrigid in $A \times B$

Cor. *Discrete subgroup* Γ *of a simply connected solvable Lie group* G *is superrigid iff*

1) $G = A \rtimes B$;
2) *finite-index* $\Gamma' \subset B$; *and*
3) $\overline{\mathrm{Ad}_B \Gamma'} = \mathrm{Ad}_B B$.

Thm (Witte). *Γ lattice in simply conn, solv G.*

$$\Gamma \textit{ superrigid } \Leftrightarrow \overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad} G}.$$

Want syndetic hull of $\hat{\Gamma} = \mathrm{graph}(\phi)$ in $\hat{G} = G \times \overline{\Gamma^\phi}$.

Key Lemma. *$\Gamma \subset$ (simply) connected, solv G, s.t. $\overline{\mathrm{Ad}_G \Gamma} \supset$ maximal compact torus of $\overline{\mathrm{Ad} G}$ $\Rightarrow \Gamma$ has a (unique) syndetic hull in G.*

$\overline{\mathrm{Ad}_{\hat{G}} \hat{\Gamma}}^\circ$ projects onto both $\overline{\mathrm{Ad} G}^\circ$ and $\mathrm{Ad}\overline{\Gamma^\phi}^\circ$, but might not contain max'l cpct torus of the product.

Close enough: $\overline{\mathrm{Ad}_{\hat{G}} \hat{\Gamma}}^\circ$ projects onto $\overline{\mathrm{Ad} G}^\circ$

$\Rightarrow \exists$ cpct tori $S \subset \overline{\mathrm{Ad}_{\hat{G}} \hat{\Gamma}}$, $T_1 \subset \mathrm{Ad}\overline{\Gamma^\phi}$,

such that $S T_1$ is a maximal cpct torus.

$\overline{\Gamma^\phi}$ is Zariski closed

$\Rightarrow \exists$ cpct torus $T \subset \overline{\Gamma^\phi}$ with $\mathrm{Ad}_{\overline{\Gamma^\phi}} T = T_1$.

$S(\mathrm{Ad}_{\hat{G}} T)$ is max'l cpct subgrp of $\overline{\mathrm{Ad} \hat{G}}$

with $S \subset \overline{\mathrm{Ad}_{\hat{G}} \hat{\Gamma}}$ and cpct $T \subset \hat{G}$.

Lem. Γ *discrete* $\subset$ *conn solvable linear Lie grp* G

$\exists$ *cpct* $S \subset \overline{\mathrm{Ad}_G \Gamma}$ *and cpct* $T \subset G$ *s.t.*

$S(\mathrm{Ad}_G T)$ *is max'l cpct subgrp of* $\overline{\mathrm{Ad} G}$

$\Rightarrow \Gamma$ (*virtually*) *has simply connected syndetic hull.*

Key Fact. Γ *closed* $\subset$ *simply conn solv group* G
$\overline{\mathrm{Ad}_G \Gamma} \supset$ maximal cpct subgrp of $\overline{\mathrm{Ad} G}$
$\Rightarrow \Gamma$ *has a unique syndetic hull in* G.

Proof. By induction, $[\Gamma, \Gamma]$ has a unique syndetic hull U in $[G, G]$.

Uniqueness $\Rightarrow \Gamma \subset N_G(U)$. WOLG $G = N_G(U)$.

Mod out U, so Γ abel. WOLG $G = C_G(\Gamma)$.

I.e., $\Gamma \subset Z(G)$. WOLG $G = Z(G)$ abel. ■

Need: $N_G(U), C_G(\Gamma)$, and $Z(G)$ are connected.

Lem. *G conn solvable Lie group*

- *Q Zariski-closed subgroup of* $\mathrm{GL}_n(\mathbb{R})$
- $\rho: G \to \mathrm{GL}_n(\mathbb{R})$ *homo, such that*
- $Q \supset$ *maximal compact subgroup of* $\overline{G^\phi}$

$\Rightarrow \rho^{-1}(Q)$ *is connected.*

Let $\rho = \mathrm{Ad}_G$, $Q = N_{\overline{\mathrm{Ad} G}}(U)$, $C_{\overline{\mathrm{Ad} G}}(\Gamma)$, e.

Lem. *G conn solvable Lie group*

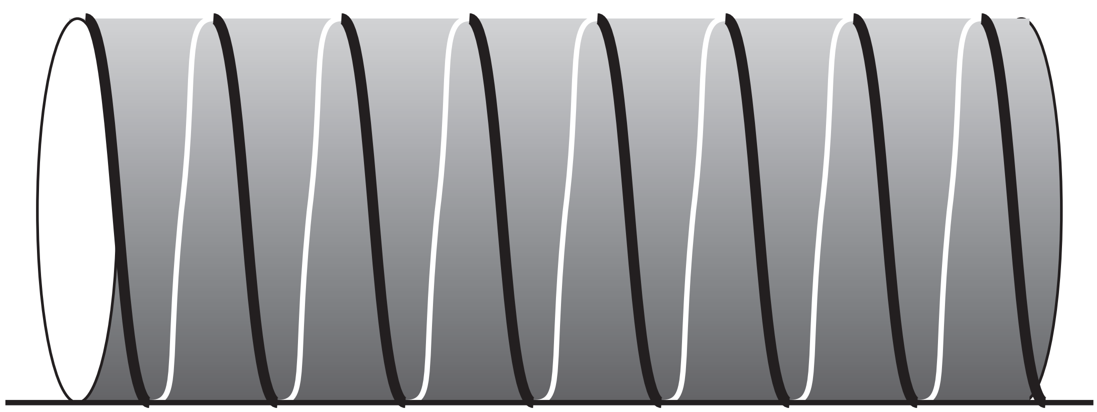
- *Q Zariski-closed subgroup of* $\mathrm{GL}_d(\mathbb{R})$
- *$\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$ homo, such that*
- *$Q \supset$ maximal compact torus of $\overline{G^\phi}$*

$\Rightarrow \phi^{-1}(Q)$ is connected.

Equivalent:

Lem. *G, Q conn solvable subgroups of* $\mathrm{GL}_d(\mathbb{R})$*,*
such that $\overline{Q}^\circ \supset$ maximal compact torus of $\overline{G}$
$\Rightarrow G \cap \overline{Q}^\circ$ is connected.

Eg. In $\mathbb{R} \times \mathbb{T}$, let $G =$ spiral.



$G \cap Q$ is connected if $Q = 0 \times \mathbb{T}$ or $\mathbb{R} \times \mathbb{T}$
not connected if $Q = \mathbb{R} \times 0$

Fact. Let Q be closed subgrp of conn, solv grp G. Then G/Q is simply conn $\Leftrightarrow$

- Q is connected and
- $Q \supset$ maximal compact subgroup of G.

Lem. *G, Q conn solvable subgroups of $\mathrm{GL}_d(\mathbb{R})$, such that $\overline{Q}^\circ \supset$ maximal compact torus of $\overline{G}$ $\Rightarrow G \cap \overline{Q}^\circ$ is connected.*

Proof. $\overline{G} \cap \overline{Q}$, being algebraic, is (virtually) conn. Thus, we may assume $Q = (\overline{G} \cap \overline{Q})^\circ \subset \overline{G}$.

Then Q normalizes G, so GQ is a group.

By assumption, $Q \supset$ max'l cpct torus of $\overline{G}$, so $Q \supset$ max'l cpct torus of the subgroup GQ, which implies that GQ/Q is simply connected.

Therefore $G/(G \cap Q) \sim GQ/Q$ is simply conn, so $G \cap Q$ is connected.

Lem. *G, Q conn solv subgroups of* $\mathrm{GL}_n(\mathbb{R})$.

- $\overline{Q}^{\circ} \supset$ *compact torus* S
- $G \supset$ *compact torus* T

such that ST *is a maximal compact torus of* $\overline{G}$.
$\Rightarrow G \cap \overline{Q}^{\circ}$ *is virtually connected.*

Proof. We may assume $Q = (\overline{G} \cap \overline{Q})^{\circ} \subset \overline{G}$.

Write $\overline{G}^{\circ} = \big((ST) \times A\big) \ltimes U$, where

- ST is a maximal compact torus,
- A is a maximal split torus, and
- U is the unipotent radical.

Alter S, T: WOLG $Q \subset SAU$, and $S \cap T$ is finite.

Because $T \subset G$, we have $G = \big(G \cap (SAU)\big)T$.
Also, G is conn and $(SAU) \cap T = S \cap T$ is finite.
Therefore, $G \cap (SAU)$ is virtually connected.

Since Q contains the maximal cpct torus S,
previous lemma implies $Q \cap \big(G \cap (SAU)\big)^{\circ}$ is conn.

Thm (Witte). Γ *lattice in simply conn, solv* G.

Γ *superrigid* $\Leftrightarrow$ $\overline{\mathrm{Ad}_G(\Gamma)} = \overline{\mathrm{Ad} G}$.

Lem. Γ *discrete* $\subset$ *conn solvable linear Lie grp* G

$\exists$ *cpct* $S \subset \overline{\mathrm{Ad}_G \Gamma}$ *and cpct* $T \subset G$ *s.t.*

$S(\mathrm{Ad}_G T)$ *is max'l cpct subgrp of* $\overline{\mathrm{Ad} G}$

$\Rightarrow \Gamma$ (*virtually*) *has simply connected syndetic hull.*

Proof of Theorem. Let $\phi\colon \Gamma \to \mathrm{GL}_d(\mathbb{R})$.

$\hat{G} = G \times \overline{\phi(\Gamma)}^{\circ}$, $\hat{\Gamma} = \mathrm{graph}(\phi) \subset \hat{G}$.

Lemma: $\hat{\Gamma}$ virt'ly has a simply conn synd hull X.

Pass to finite index: $\hat{\Gamma} \subset X$.

Claim. X is graph of a function $\hat{\phi}\colon G \to \mathrm{GL}_d(\mathbb{R})$.

- $\mathrm{graph}(\hat{\phi})$ is a subgroup $\Rightarrow \hat{\phi}$ is a homo.
- $\mathrm{graph}(\phi) \subset \mathrm{graph}(\hat{\phi}) \Rightarrow \hat{\phi}$ extends ϕ.

Suffices to show:

1) X projects onto G.

2) $X \cap \left(e \times \overline{\phi(\Gamma)}^{\circ}\right)$ is trivial.

Suffices to show:

1) X projects onto G.

2) $X \cap \left(e \times \overline{\phi(\Gamma)}^{\circ}\right)$ is trivial.

$\hat{\Gamma} = \text{graph}(\phi) \subset G \times \overline{\phi(\Gamma)}^{\circ}$

$X =$ simply connected syndetic hull of $\hat{\Gamma}$

1) $\pi_1(X) =$ conn subgrp of G that contains Γ

$\Rightarrow \pi_1(X) = G$.

2) $\pi_1|_{\hat{\Gamma}}$ is proper (because ϕ is continuous)

$+$ $X/\hat{\Gamma}$ is compact

$+$ π_1 is a homomorphism

$\Rightarrow \pi_1|_X$ is proper

$\Rightarrow X \cap \pi_1^{-1}(e)$ is compact.

Every compact subgroup of X is trivial, because X is simply connected.

Mostow Rigidity.

- Γ_1, Γ_2 lattices in simply conn solv G_1, G_2
- $\overline{\mathrm{Ad}_{G_1}\Gamma_1} = \overline{\mathrm{Ad}G_1}$
- $\pi\colon \Gamma_1 \to \Gamma_2$ isomorphism

$\Rightarrow \pi$ extends to $\sigma\colon G_1 \to \overline{G_2} = T \ltimes G_2$

where T is a compact subgroup of $\overline{\mathrm{Ad}G_2}$

Project to factors: $\tau\colon G_1 \to T, \quad \phi\colon G_1 \to G_2$

τ is homo, but ϕ is a **crossed homomorphism**:

$(xy)^\phi = (x^\phi)^{y^\tau} y^\phi$

We have $\Gamma_1^\tau = e$, so $\quad (x\gamma)^\phi = (x^\phi)^{\gamma^\tau}\gamma^\phi = x^\phi\gamma^\phi$.

Therefore, ϕ induces a diffeo $G_1/\Gamma_1 \to G_2/\Gamma_2$.

Thm (Mostow 1954). $\Gamma_1 \cong \Gamma_2$

$\Rightarrow$ *G_1/Γ_1 is diffeomorphic to G_2/Γ_2.*

In gen'l, ϕ **doubly** crossed: $(xy)^\phi = \left((x^{y^\mu})^\phi\right)^{y^\tau} y^\phi$

Nilshadow [Auslander (et al)]. "G^Δ"

Spse $G = T \ltimes U$ (T torus, U nilp)

$G^\Delta = T \times U$ (kill the action of T) or:

T acts on G by conj: form $T \ltimes G = T \ltimes (T \ltimes U)$

Embed G^Δ in $T \ltimes G : (t, u) \mapsto (t^{-1}, t, u)$

Anti-diag embedding of T sends it into $Z(G)$.

Define $g^\Delta = \big((g^\pi)^{-1}, g\big)$, where $\pi\colon G \to T$ is proj.

G^Δ is the img of Δ ("anti-diag embedding")

In general: $\overline{\mathrm{Ad} G} = T \ltimes U$ $\Delta\colon G \to T \ltimes G$

Kill only subtorus S of T: $\pi\colon G \to S$ (proj onto S)

We kill complement to $\overline{\mathrm{Ad}_G \Gamma}$ (assume conn).

$\overline{\mathrm{Ad}_{G^\Delta} \Gamma^\Delta} = \overline{\mathrm{Ad} G^\Delta}$, $\Gamma^\pi = e$

Cor. Γ_1, Γ_2 *latt in sc solv* G_1, G_2.

$+ \ \overline{\mathrm{Ad}_{G_1} \Gamma_1} = \overline{\mathrm{Ad} G_1}$

$\Gamma_1 \cong \Gamma_2 \Rightarrow \exists \pi\colon G_1 \to T \subset \mathrm{Aut}\, G_1$, Γ_1^π *finite,*

$$G_2 \cong G_1^\Delta \subset T \ltimes G_1.$$

Lem. Δ *is a crossed homo:* $(ab)^\Delta = (a^\Delta)^{b^\pi} b^\Delta$

Cor. $\Gamma_1^\pi = e \Rightarrow \Delta$ *is* Γ_1*-equivariant:*

$$(a\gamma)^\Delta = a^\Delta \gamma^\Delta$$

In particular, $\Gamma \cong \Gamma^\Delta$.

Cor. Γ_1, Γ_2 *latt in sc solv* G_1, G_2.

$$+\ \overline{\mathrm{Ad}_{G_1}\Gamma_1} = \overline{\mathrm{Ad}G_1}$$

Any iso $\pi\colon \Gamma_1 \to \Gamma_2$ *extends to a* Γ_1*-equivariant crossed iso* $\hat{\pi}\colon G_1 \to G_2$.

Cor. Γ_1, Γ_2 *latt in sc solv* G_1, G_2.
Any iso $\pi\colon \Gamma_1 \to \Gamma_2$ *extends to a* Γ_1*-equivariant doubly-crossed iso* $\hat{\pi}\colon G_1 \to G_2$.

Def. doubly-crossed: $(ab)^\Delta = \left((a^{b^\pi})^\Delta\right)^{b^{\pi'}} b^\Delta$

References

• V. V. Gorbatsevich, On Lie groups with lattices and their properties, *Soviet Math. Dokl.* 33 (1986) 321–325.

• M. Gromov and R. Schoen, Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one, *Publ. Math. IHES* 76 (1992) 165–246.

• F. Grunewald and V. Platonov, On finite extensions of arithmetic groups. *C. R. Acad. Sci. Paris Sér. I Math.* 325 (1997) 1153–1158.

• A. I. Malcev, On a class of homogeneous spaces, *Amer. Math. Soc. Transl.* No. 39 (1951) = 9 (1962) 276–307.

• G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, New York, 1991.

• M. V. Milovanov, A description of solvable Lie groups with a given uniform subgroup, *Math. USSR Sbornik* 41 (1982) 83–99.

• M. S. Raghunathan, *Discrete Subgroups of Lie Groups,* Springer-Verlag (New York) 1972.

• M. Saito, Sur certains groupes de Lie résolubles II, *Sci. Papers Coll. Gen. Ed. Tokyo* 7 (1957) 157–168.

• A. N. Starkov, Rigidity problem for lattices in solvable Lie groups, *Proc. Indian Acad. Sci.* (*Math. Sci.*) 4 (1994).

• D. Witte, Tessellations of solvmanifolds, *Trans. Amer. Math. Soc.* (1998).

• D. Witte, Superrigidity of lattices in solvable Lie groups, *Inventiones Math.* 122 (1995) 147-193.

• D. Witte, Superrigid subgroups of solvable Lie groups, *Proc. Amer. Math. Soc.* 125 (1997) 3433–3438.

• D. Witte, Archimedean superrigidity of solvable S-arithmetic groups, *J. Algebra* 187 (1997) 268–288.

• R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.