Which circulant digraphs are hamiltonian?

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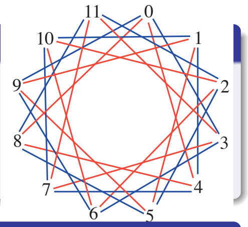
Example

Circulant graph Circ(12; 3, 4)

vertices: elements of $\mathbb{Z}_{12}$
edges: v — $v \pm 3$ & v — $v \pm 4$



Notation

$\mathrm{Circ}(n; s_1, s_2, \ldots, s_r)$.
Assume *connected*: $\gcd(s_1, s_2, \ldots, s_r, n) = 1$.

Exercise

Every circulant graph has a hamiltonian cycle.

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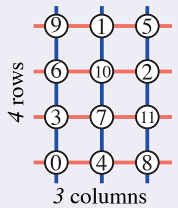
Hint

Circ(12; 3, 4) =



$\cong$ *Cartesian product* $C_3 \square C_4$

Exercise

Every circulant graph has a hamiltonian cycle.

Theorem (Chen-Quimpo)

$\mathrm{Circ}(n; S)$ *is hamiltonian connected (unless bipartite).*

Conjecture (Alspach)

$\mathrm{Circ}(n; S)$ *is hamiltonian decomposable*
(unless it has odd valence).

Theorem (Bermond-Favaron-Mahéo, Dean)

True if

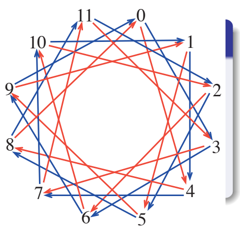
1. *valence* ≤ 5, *or*
2. *valence* $= 6$ *and* n *is odd or* $1 \in S$.

The Directed Case

Example

Circulant digraph $\overrightarrow{\mathrm{Circ}}(12; 3, 4)$

vertices: elements of $\mathbb{Z}_{12}$
arcs: $v \to v + 3$ & $v \to v + 4$



Open problem

Which circulant digraphs have *ham cycs*?

Example

$\overrightarrow{\mathrm{Circ}}(12; 3, 4)$ does *not* have a hamiltonian cycle.

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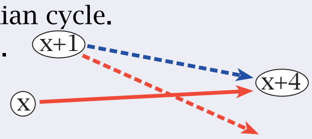
$\overrightarrow{\mathrm{Circ}}(12; 3, 4)$ does *not* have a hamiltonian cycle.

Proof.

Suppose H is a hamiltonian cycle.
Say vertex x travels by 4.



Then $x + 1$ cannot travel by 3 (collision at $x + 4$).
So $x + 1$ must travel by 4.
By induction, every vertex travels by 4. →←
So no vertex travels by 4. All travel by 3. →← □

Example

$\overrightarrow{\mathrm{Circ}}(12; 3, 4)$ does *not* have a hamiltonian cycle.

Same proof

$\nexists$ ham cyc in $\overrightarrow{\mathrm{Circ}}(n; a, b)$ if $b - a$ rel prime to n and $\gcd(a, n) \neq 1$ and $\gcd(b, n) \neq 1$.

Exercise (Rankin, 1948)

$\overrightarrow{\mathrm{Circ}}(n; a, b)$ *has a ham cyc* $\Leftrightarrow$ $\exists\, s, t \in \mathbb{Z}^{\geq 0}$, *s.t.* $s + t = \gcd(a - b, n) = \gcd(sa + tb, n)$.

Conjecture

If $\#S > 3$, *then* $\overrightarrow{\mathrm{Circ}}(n; S)$ *has a hamiltonian cycle.*

Open question

When does $\overrightarrow{\mathrm{Circ}}(n; a, b, c)$ have a hamiltonian cycle?

Theorem (Locke-Witte)

$\nexists$ *hamiltonian cycle in:*

- $\overrightarrow{\mathrm{Circ}}(12k; 6k, 6k + 2, 6k + 3)$
- $\overrightarrow{\mathrm{Circ}}(2k; a, a + 1, a + k)$ *if* $a + k$ *is even* (*and* $\gcd(a, 2k) \neq 1$ *and* $\gcd(a + 1, 2k) \neq 1$).

Main Question

Are these the only examples? (up to isomorphism)

Remark

$\overrightarrow{\mathrm{Circ}}(n; a, b, c) \cong \overrightarrow{\mathrm{Circ}}(n; ra, rb, rc)$ if $\gcd(r, n) = 1$

- $\overrightarrow{\mathrm{Circ}}(12k; 6k, 6k + 2, 6k + 3)$
- $\overrightarrow{\mathrm{Circ}}(2k; a, a + 1, a + k)$ if $a + k$ is even (and ...).

Problem

1. n *odd* $\overset{?}{\Rightarrow}$ $\overrightarrow{\mathrm{Circ}}(n; a, b, c)$ *has ham cycle*
2. (easy?) $\overrightarrow{\mathrm{Circ}}(n; a, b, -b)$ *has ham cycle* $(b \neq n/2)$
3. $\overrightarrow{\mathrm{Circ}}(2m; m, b, c)$ *no ham cycle* $\overset{?}{\Rightarrow}$ *known*
4. $\overrightarrow{\mathrm{Circ}}(n; a, a + 1, c)$ *no ham cycle* $\overset{?}{\Rightarrow}$ *known*

Theorem (Locke-Witte, Morris^2-Webb)

1. $\overrightarrow{\mathrm{Circ}}(2m; a, b, a + k)$ *no ham cycle* $\Rightarrow$ *known*
2. $\overrightarrow{\mathrm{Circ}}(n; 2, 3, c)$ *no ham cycle* $\Rightarrow$ *known*

Nonhamiltonian circulants with < 48 verts

$\overrightarrow{\mathrm{Circ}}(12; 2, 3, 8)$	$\overrightarrow{\mathrm{Circ}}(30; 2, 6, 21)$	$\overrightarrow{\mathrm{Circ}}(40; 4, 5, 24)$
$\overrightarrow{\mathrm{Circ}}(12; 3, 4, 6)$	$\overrightarrow{\mathrm{Circ}}(30; 2, 9, 24)$	$\overrightarrow{\mathrm{Circ}}(42; 2, 3, 24)$
$\overrightarrow{\mathrm{Circ}}(18; 2, 3, 12)$	$\overrightarrow{\mathrm{Circ}}(30; 2, 10, 25)$	$\overrightarrow{\mathrm{Circ}}(42; 2, 6, 27)$
$\overrightarrow{\mathrm{Circ}}(18; 2, 6, 15)$	$\overrightarrow{\mathrm{Circ}}(30; 3, 10, 18)$	$\overrightarrow{\mathrm{Circ}}(42; 2, 7, 28)$
$\overrightarrow{\mathrm{Circ}}(20; 2, 5, 12)$	$\overrightarrow{\mathrm{Circ}}(30; 5, 6, 20)$	$\overrightarrow{\mathrm{Circ}}(42; 2, 12, 33)$
$\overrightarrow{\mathrm{Circ}}(24; 2, 3, 14)$	$\overrightarrow{\mathrm{Circ}}(36; 2, 3, 20)$	$\overrightarrow{\mathrm{Circ}}(42; 2, 15, 36)$
$\overrightarrow{\mathrm{Circ}}(24; 2, 9, 12)$	$\overrightarrow{\mathrm{Circ}}(36; 2, 9, 20)$	$\overrightarrow{\mathrm{Circ}}(42; 2, 18, 39)$
$\overrightarrow{\mathrm{Circ}}(24; 3, 4, 16)$	$\overrightarrow{\mathrm{Circ}}(36; 2, 15, 20)$	$\overrightarrow{\mathrm{Circ}}(42; 3, 14, 24)$
$\overrightarrow{\mathrm{Circ}}(28; 2, 7, 16)$	$\overrightarrow{\mathrm{Circ}}(36; 3, 8, 18)$	$\overrightarrow{\mathrm{Circ}}(42; 6, 7, 28)$
$\overrightarrow{\mathrm{Circ}}(30; 2, 3, 18)$	$\overrightarrow{\mathrm{Circ}}(40; 2, 5, 22)$	$\overrightarrow{\mathrm{Circ}}(44; 2, 11, 24)$

Conjecture

If $\#S > 3$, *then* $\overrightarrow{\mathrm{Circ}}(n; S)$ *has a hamiltonian cycle.*

Exercise

Conjecture true

$\Rightarrow$ $\mathrm{Cay}\,(D_{2n}; S)$ *has a hamiltonian cycle*

$\Rightarrow$ $\overrightarrow{\mathrm{Cay}}\,(D_{2n}; S)$ *has a hamiltonian cycle.*

Theorem (Alspach-Zhang)

$\mathrm{Cay}(D_{2n}; S)$ *has a hamiltonian cycle if* $\#S = 3$.

Some references

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