Some discrete groups that cannot act on 1-dimensional manifolds

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Algebraic, geometric and probabilistic aspects of amenability

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Part 3

3 Major Theorems of Margulis (Thurs–Fri)
Superrigidity, Arithmeticity, Normal Subgroups

As in Part 2: interested in arithmetic groups
$\mathrm{SL}(3,\mathbb{Z})$, etc.

Theorem (Borel & Harish-Chandra)

$\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *is a lattice in* $G = \mathrm{SL}(3,\mathbb{R})$.
I.e., G/Γ *has finite volume (and* Γ *is discrete).*

Theorem (Margulis Arithmeticity Theorem)

Converse *if* $G = \mathrm{SL}(n,\mathbb{R})$, $n > 2$.
(G *simple,* $\mathbb{R}$*-rank* $G \geq 2$.)

- Γ lattice in G, (for simplicity, G/Γ **not** compact)
  $\Longrightarrow G \hookrightarrow \mathrm{SL}(\ell,\mathbb{R})$, $\Gamma = \mathrm{SL}(\ell,\mathbb{Z}) \cap G$.
- X = locally symmetric space of finite volume
  (complete, noncompact type, no flat factors, rank ≥ 2)
  $\Longrightarrow \pi_1(X)$ is arithmetic.

Margulis Arithmeticity Thm is corollary of:

Theorem (Margulis Superrigidity Thm)

$\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *is a lattice in* $G = \mathrm{SL}(3,\mathbb{R})$
$\Longrightarrow \Gamma$ *is superrigid in* G.

f.d. representations of Γ virtually extend to G:
$\rho\colon \Gamma \to \mathrm{GL}(\ell,\mathbb{R})$
$\Longrightarrow \exists\, \hat\rho\colon G \to \mathrm{GL}(\ell,\mathbb{R})$, $\hat\rho|_{\Gamma'} = \rho|_{\Gamma'}$.

Remark

Rep'ns of (connected) Lie group G are known.
(Lie algebra roots and weights)
So rep'ns of Γ are known.

Margulis Normal Subgroups Thm

Lattices are simple "modulo finite groups":

Theorem (Margulis Normal Subgroups Thm)

- $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *is a lattice in* $G = \mathrm{SL}(3,\mathbb{R})$,
- $N \triangleleft \Gamma$

$\Longrightarrow$ *N finite or Γ/N finite.*

Proof requires two facts:

Theorem (from Kazhdan's Property T)

Γ/N *amenable* $\Longrightarrow \Gamma/N$ *finite.*

Recall: $P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$.

Theorem (Margulis)

- Γ *acts on measure space* Y,
- $\psi\colon G/P \to Y$ Γ*-equivariant (a.e.)*

$$\Longrightarrow \quad \exists Q \supset P, \quad \begin{array}{ccc} G/P & \xrightarrow{\psi} & Y \\ & \searrow & \wr\| \\ & & G/Q \end{array} \quad \text{(a.e.)}$$

Proof of Margulis Normal Subgroups Thm.

Assume Γ/N is infinite, so Γ/N is *not* amenable.
Γ/N acts on cpct space X, with no inv't prob meas.

$\exists\ \psi\colon G/P \to \mathrm{Prob}(X)$ Γ-equivariant.
So $\mathrm{Prob}(X) \cong G/Q$ as a Γ-space.
Γ has no fixed point in $\mathrm{Prob}(X)$, so $Q \neq G$.

N trivial on X
$\implies N$ trivial on $\mathrm{Prob}(X)$
$\implies N$ trivial on G/Q
$\implies N \subset$ proper normal subgroup of G
$\implies N$ finite. □

Margulis Superrigidity Thm

Lemma (Topologist's Version)

For $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{R})$, let $E = (G \times \mathbb{R}^n)/\Gamma$ (diag'l action).
$E \to G/\Gamma$ is flat G-equi vector bundle over G/Γ.
(*G acts on E by left-mult on G; commutes with Γ mult on right.*)

ρ extends to G
$\iff \exists$ *G-inv't subspace $V \subset \mathrm{Sect}(E)$, $V \xrightarrow{\cong} V|_{[e]}$.*

Proof.

($\Rightarrow$) For $v \in \mathbb{R}^n$, define $\xi_v(g\Gamma) = (g, \tilde\rho(g)v)\Gamma$.

($\Leftarrow$) Have rep of G on V.
Iso to $V_{[e]}$ yields rep of G on $V_{[e]}$ (extends ρ). □

Exercise

$\exists$ *G-invariant* (nonzero) *finite-dim'l* $V \subset \mathrm{Sect}(E)$, *s.t.*

- *Γ nontrivial* (hence, faithful) *on* $V_{[e]} \subset \mathbb{R}^\ell$,
- *$V_x = V_{[e]}$ for all $x \in G/\Gamma$.*

$\implies$ *ρ extends to G.*

Notation

$$P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

Key Fact (amenability)

$\exists$ P-invariant section $\sigma\colon G/\Gamma \to E$.
(Also, letting $W = \langle \Gamma \cdot \sigma(x) \rangle \subset \mathbb{R}^\ell$, we have:
W_x is indep of x, and Γ is nontriv on W_x.)

Exercise

- *$H \subset$ diag noncompact,*
- *V is H-invariant, finite-dimensional in* $\mathrm{Sect}(E)$

$\implies$ *$\langle C_G(H) \cdot V \rangle$ is finite-dimensional.*

Hint: H has a dense orbit on G/Γ (in fact, ergodic),
so what c does to a section at a single point of G/Γ
determines it everywhere.

Remark

Actually, Key Fact provides only a *measurable* inv't section, not cont. (Identify sections that agree a.e.)
For exercise, need more than a dense orbit
— use *ergodicity*.

Exercise

$\exists$ *subgroups $L_1, L_2, \ldots, L_r$, such that*

1. $G = L_r L_{r-1} \cdots L_2 L_1$,
2. *$H_i, H_i^\perp$ not compact, where*
   - $H_i = L_i \cap \mathrm{diag}$,
   - $H_i^\perp = C_{\mathrm{diag}}(L_i)$ (*i.e.,* $H_i^\perp \subset$ diag *and* $L_i \subset C_G(H_i^\perp)$)

Hint: Choose

$$L_1, \ldots, L_r \in \left\{ \begin{bmatrix} * & * & \\ * & * & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & * & * \\ & * & * \end{bmatrix}, \begin{bmatrix} * & & * \\ & 1 & \\ * & & * \end{bmatrix} \right\}.$$

Every el't of $\mathrm{SL}(3,\mathbb{R})$ is product of ≤ 10 elementary matrices.

Proof of Superrigidity.

Let $H_0 = \mathrm{diag}$, $V_0 = \langle P\text{-inv't section} \rangle$,
so V_0 is 1-dimensional and H_0-invariant.
Let $V_i = \langle L_i \cdot \mathrm{diag} \cdot L_{i-i} \cdot \mathrm{diag} \ \cdots\ L_1 \cdot \mathrm{diag} \cdot V_0 \rangle$.
Since $G = L_r L_{r-1} \cdots L_2 L_1$, we know V_r is G-inv't.

Suffices to show each V_i is finite dimensional.

- V_i is H_i-invariant.
  $\mathrm{diag} \subset C_G(H_i) \implies \mathrm{diag} \cdot V_i$ is f.d.
- $\mathrm{diag} \cdot V_i$ is $H_i^\perp$-invariant.
  $L_i \subset C_G(H_i^\perp) \implies V_{i+1} = L_i \cdot \mathrm{diag} \cdot V_i$ is f.d.

□

Proof of the Key Fact

- restate: $\exists$ Γ-equi $v: G/P \to W$, where $W = \mathbb{R}^\ell$.
- amenability: $\exists$ Γ-equi $\hat{v}: G/P \to \operatorname{Prob}(\mathbb{P}(W))$.
- proximality: Replace W with (subspace of) $\bigwedge^k W$, so $\exists\, g \in \Gamma$ with unique largest eigenval.
  ⟨steps missing here⟩
  Then $\hat{v}(x)$ is point mass, for a.e. x; $\hat{v}$ is well-defined map into $\mathbb{P}(W)$.
- algebra trick: We have $\hat{v}: G/P \to \mathbb{P}(W)$.
  Similarly, $\check{v}: G/P \to \mathbb{P}(W^*)$.
  $v = \hat{v} \otimes \check{v}: G/P \to \mathbb{P}(W \otimes W^*) = \mathbb{P}(\operatorname{End}_{\mathbb{R}}(W))$.
  Natural normalization: trace = 1.
  So $v: G/P \to \operatorname{End}_{\mathbb{R}}(W)$ (*vector space*).

Why superrigidity implies arithmeticity

Let Γ be a *superrigid* lattice in $\mathrm{SL}(n,\mathbb{R})$.
We wish to show $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$, i.e., $g_{i,j} \in \mathbb{Z}$.

Step 1. $g_{i,j}$ is algebraic

Suppose some $g_{i,j}$ is transcendental.
Then $\exists$ field auto φ of $\mathbb{C}$ with $\varphi(g_{i,j}) = ???$

Define $$\tilde{\varphi}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{bmatrix}.$$

This map $\tilde{\varphi}: \Gamma \to \mathrm{GL}(n,\mathbb{C})$ is a group homo.
Superrigid: $\tilde{\varphi}$ extends to $\hat{\varphi}: \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{C})$.
There are uncountably many different φ's,
but $\mathrm{SL}(n,\mathbb{R})$ has only finitely many n-dim'l rep'ns (up to change of basis). $\rightarrow\leftarrow$

Step 2. $g_{i,j} \in \mathbb{Q}$

Γ f.g., so $\{g_{i,j}\}$ generates finite extension of $\mathbb{Q}$.
"*algebraic number field*"
So $\Gamma \subset \mathrm{SL}(n,F)$. For simplicity, assume $\Gamma \subset \mathrm{SL}(n,\mathbb{Q})$.

Step 3. $g_{i,j}$ has no denominator

Actually, show denominators are bounded.
(Then finite-index subgroup has no denoms.)
Γ f.g., so finitely many primes appear in denoms.
Suffices to show each prime occurs to bdd power.
This is the conclusion of *p-adic superrigidity*:

Theorem (Margulis)

- Γ *a lattice in* $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$,
- $\varphi: \Gamma \to \mathrm{SL}(\ell, \mathbb{Q}_p)$ *is a group homomorphism,*

$\Longrightarrow$ $\varphi(\Gamma)$ *has compact closure.*
I.e., $\exists k$, no matrix in $\varphi(\Gamma)$ has p^k in denom.

References

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