Some discrete groups that cannot act on 1-dimensional manifolds

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Algebraic, geometric and probabilistic aspects of amenability

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Transformation groups (simplest case)

Given a group Γ, what are the actions of Γ on S^1?
I.e., what are the homos $\phi \colon \Gamma \to \mathrm{Homeo}_+(S^1)$?

Part 2

Actions of arithmetic groups (Tues–Thurs)

Assume Γ is an *arithmetic group*:
$\Gamma = \mathrm{SL}(3,\mathbb{Z}) = \{\, 3 \times 3$ integer matrices of det $1 \,\}$
(or subgroup of finite index)
Or $\Gamma \doteq \mathrm{SL}(2,\mathbb{Z}[\alpha])$ α = real, irrat alg'ic integer.

But $\Gamma \neq \mathrm{SL}(2,\mathbb{Z})$, some other "small" grps.
(Assume Γ is irreducible lattice in G, $\mathbb{R}$-rank $G \geq 2$.
$\Gamma \not\subset \mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$, $F_{4,1}$.)

Example (linear-fractional transformations)

$\frac{ax+b}{cx+d}$ provides action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R} \cup \{\infty\} \sim S^1$.
So $\mathrm{SL}(2,\mathbb{Z}[\alpha])$ acts by linear-fractionals.

Conjecture

$\phi \colon \Gamma \to \mathrm{Homeo}(S^1)$, *not $\approx$ lin-frac* $\Rightarrow$ Γ^ϕ *is finite.*

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$\phi \colon \Gamma \to \mathrm{Homeo}(S^1)$, *not $\approx$ lin-frac* $\Rightarrow$ Γ^ϕ *is finite.*
($\Gamma = \mathrm{SL}(3,\mathbb{Z})$ or $\mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ or ...)

Example

$\mathrm{SL}(2,\mathbb{Z})$ ($\approx$ free group) has *many* actions on S^1.

Remark

Let $\Gamma = \pi_1$(hyperbolic n-mfld). (So $\Gamma \subset \mathrm{SO}(1,n)$.)
Thurston conjecture: $\exists \sigma \colon \Gamma' \twoheadrightarrow \mathbb{Z}$.
So Γ' acts on S^1. (Γ' = finite-index subgrp)

Remark (Margulis Normal Subgroup Thm)

Γ^ϕ *infinite* $\Rightarrow$ $\ker(\phi)$ *finite.* ($\Gamma' \hookrightarrow \mathrm{Homeo}_+(S^1)$.)

Theorem (Ghys, Burger-Monod)

$\phi \colon \Gamma \to \mathrm{Homeo}(S^1)$, *not $\approx$ lin-frac* $\Rightarrow$ Γ^ϕ *has finite orbit*
"fixed point"

Proposition (Reeb-Thurston Stability Thm)

- Λ = *f.g. subgroup of* $\mathrm{Diff}^1_+([0,1])$,
- *abelianization* $\Lambda/[\Lambda,\Lambda]$ *is finite.*

$\Longrightarrow$ Λ *trivial* ($\Lambda = \{e\}$)

Exercise

Prove if Λ consists of real-analytic diffeos.
Hint: $\lambda(x) = x + a_k(\lambda)x^k + a_{k+1}(\lambda)x^{k+1} + \cdots \Longrightarrow a_k$ *is homo.*

Corollary (Ghys, Burger-Monod)

$\phi \colon \Gamma \to \mathrm{Diff}^1(S^1)$ (*not $\approx$ lin-frac*) $\Longrightarrow$ Γ^ϕ *finite.*

Proof of Theorem: Γ^ϕ has a "fixed pt"

Burger-Monod: study *bounded cohomology*.
(*amenability* is a key ingredient)
Fixed point is one of many applications.

Proof of Ghys

- *ergodic theory* (transf grps + measure theory)
- *amenability* (Poisson bdry, Furstenberg bdry)

Exercise

$\exists$ Γ*-inv't prob meas on* S^1 $\Longrightarrow$ $\exists$ *finite orbit.*
Hint: See Part 1 of the lectures. Assume abelianization of Γ is finite.

Proof of Ghys

Assume

$\Gamma \hookrightarrow \text{Homeo}_+(S^1)$, $\Gamma = SL(3,\mathbb{Z})$ (for simplicity)

Key fact (amenability)

$\mathbf{F} = \begin{matrix}\text{Furstenberg}\\ \text{boundary}\end{matrix} = \begin{matrix}\text{flag}\\ \text{variety}\end{matrix} = \{(\ell,\Pi) \mid \ell \subset \Pi \subset \mathbb{R}^3\}$

$\implies \exists$ Γ-equivariant, *random* map $\mathbf{F} \to S^1$.

$\psi\colon \mathbf{F} \to \text{Prob}(S^1)$ Γ-equivariant, meas'ble.

Proof of Ghys

Show *ψ constant.* (Then $\psi(\Gamma)$ is Γ-invariant probability measure.)

Proof of key fact (amenability)

$\exists\ \psi\colon \mathbf{F} \to \text{Prob}(S^1)$ Γ-equivariant, measurable.

($\mathbf{F} = \{(\ell,\Pi) \mid \ell \subset \Pi \subset \mathbb{R}^2\}$)

$G = SL(3,\mathbb{R})$ is *transitive* on $\mathbf{F}$.

So $\mathbf{F} \cong G/P$, where $P = \text{Stab}_G(\text{flag}) = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$.

Want $\Psi\colon G \to \text{Prob}(S^1)$, Γ-equi, s.t. $\Psi(gp) = \Psi(g)$.

Let $\mathcal{E} = \{\Gamma\text{-equivariant } \Psi\colon G \to \text{Prob}(S^1)\}$.

- $\text{Prob}(S^1) \subset C(S^1)^*$ is compact, convex
  $\Rightarrow \mathcal{E}$ is a compact, convex set.
- G acts on $\mathcal{E}$ by translation.

We want P to have a fixed point in $\mathcal{E}$.

We want P to have a fixed point in cpct, cnvx set $\mathcal{E}$.

Exercise

Group H is amenable:

- *H acts continuously on cpct metric space X*
  $\Rightarrow \exists$ H-invariant prob meas on X.
- *H acts linearly on cpct convex set $\mathcal{E}$ $\subset$ Banach*
  $\Rightarrow \exists$ fixed point in $\mathcal{E}$.

Exercise

P is amenable.

Hint: Show P is solvable. (Recall: solvable groups are amenable.)

$P = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \triangleright \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \triangleright \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \triangleright \{\text{Id}\}$. Abelian quotients

Key fact: $\exists\ \psi\colon \mathbf{F} \to \text{Prob}(S^1)$ Γ-equi, measurable.

($\mathbf{F} = \{(\ell,\Pi) \mid \ell \subset \Pi \subset \mathbb{R}^2\}$)

Ghys: ψ is constant a.e.

Fact (from Moore Ergodicity Thm)

Γ is ergodic on $\mathbf{F}$:

- *Γ-invariant meas'ble **function** is const a.e.*
- *Γ-invariant meas'ble **set** is null or conull.*
- *(a.e. orbit is dense)*

Key fact: $\exists\ \psi\colon \mathbf{F} \to \text{Prob}(S^1)$ Γ-equi, measurable.

($\mathbf{F} = \{(\ell,\Pi) \mid \ell \subset \Pi \subset \mathbb{R}^2\}$)

Ghys: ψ is constant a.e. ($\exists$)

Observation

Let $\psi_{\text{atom}}(x) =$ atomic part of $\psi(x)$, so
$\psi(x) = \psi_{\text{atom}}(x) + \psi_{\text{no atom}}(x)$.

Then $\psi_{\text{atom}}, \psi_{\text{no atom}}$ are Γ-equi, measurable.

Only two cases:

1. $\psi(x)$ has no atoms, $\forall x$.
2. $\psi(x)$ purely atomic, $\forall x$.

No need to consider mixed case.

Case 1. $\psi(x)$ has no atoms.

- $\text{Prob}_0(S^1) = \{\mu \in \text{Prob}(S^1) \mid \mu \text{ has no atoms}\}$.
- $\psi_2\colon \mathbf{F}^2 \to (\text{Prob}_0(S^1))^2$ ($\psi_2(x,y) = (\psi(x),\psi(y))$)
  measurable, Γ-equivariant.
- $d\colon (\text{Prob}_0(S^1))^2 \to \mathbb{R}$
  $d(\mu_1,\mu_2) = \sup_{J \text{ interval}} |\mu_1(J) - \mu_2(J)|$.
  continuous, Γ-invariant.

Composition $d \circ \psi_2$ is Γ-invariant; hence const a.e.:
$d(\psi_2(x,y)) = c$, for a.e. $x,y \in \mathbf{F}$.

Exercise

$c = 0$. *Hint:* $d(\psi_2(x,x)) = 0$. *d continuous, Lusin's Thm.*

Case 2. $\psi(x)$ is purely atomic.

Assume $\psi\colon \mathbf{F} \to S^1$,   so $\psi_3\colon \mathbf{F}^3 \to (S^1)^3$.

Circular order: $(S^1)^3 = X^+ \sqcup X^- \sqcup \{\text{singular}\}$.
X^+ is invariant under $\text{Homeo}_+(S^1)$, so
$\psi_3^{-1}(X^+)$ is Γ-invariant subset of $\mathbf{F}^3$.
Contradiction if $\nexists$ Γ-invariant subsets.
More precisely, if Γ is *ergodic* on $\mathbf{F}^3$.

Not ergodic — Ghys works with *fibered product*
$\{ (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \in \mathbf{F}^3 \mid \Pi_1 = \Pi_2 = \Pi_3 \}$
instead of the cartesian product.

Theorem (Moore Ergodicity Thm)

Γ is ergodic on G/H $\Leftrightarrow$ H is not compact.

Corollary

Γ is ergodic on $\mathbf{F}$.

$$\text{Stab}(\mathcal{F}) = P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}. \quad \text{Not cpct.}$$

Corollary

Γ is ergodic on $\mathbf{F}^2 = \mathbf{F} \times \mathbf{F}$.

$$\text{Stab}(\mathcal{F}_1, \mathcal{F}_2) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}. \quad \text{Not cpct.}$$

Theorem (Moore Ergodicity Thm)

Γ is ergodic on G/H $\Leftrightarrow$ H is not compact.

Corollary

Γ is **not** *ergodic on* $\mathbf{F}^3$.

$\text{Stab}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \{\pm \text{Id}\}$ finite.

Corollary

Γ is ergodic on $\{(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \in \mathbf{F}^3 \mid \Pi_1 = \Pi_2 = \Pi_3\}$.

$$\text{Stab}_G(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \begin{bmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & 1/\lambda^2 \end{bmatrix} \text{ not cpct.}$$

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