Some discrete groups that cannot act on 1-dimensional manifolds

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Algebraic, geometric and probabilistic aspects of amenability

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Outline

1. Actions of amenable groups (Mon–Tues)
2. Actions of arithmetic groups (Tues–Thurs)

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3. 3 Major Theorems of Margulis (Thurs–Fri)
   Superrigidity, Arithmeticity, Normal Subgroups

Part 1: Actions of amenable groups

Transformation groups

Given: group Γ, (connected) manifold M.
¿ What are the actions of Γ on M ?
I.e.: ¿ What are homos $\phi \colon \Gamma \to \mathrm{Homeo}_+(M)$?

Question

¿ $\exists$ *(nontrivial) action ?*

Simplest case

Assume $\dim M = 1$. So $M = S^1$ or $\mathbb{R}$.

Remark

Today & tomorrow: Γ is *amenable*.

Example

$\mathbb{Z}$ acts on $\mathbb{R}$. $(T_n(x) = x + n \implies T_{m+n} = T_m \circ T_n)$

Corollary

$\Gamma \twoheadrightarrow \mathbb{Z} \implies \Gamma$ *acts on* $\mathbb{R}$.

Converse true for finitely generated amenable grps.
(Γ *amenable, f.g., acts on* $\mathbb{R} \implies \Gamma \twoheadrightarrow \mathbb{Z}$.)

Corollary

Γ *amenable, acts faithfully on* $\mathbb{R}$ *(faithful: no kernel)*
$\implies$ *every f.g. subgrp of* $\Gamma \twoheadrightarrow \mathbb{Z}$. ($\Gamma$ *is locally indicable.*)

Remark (Burns & Hale, 1972)

Γ loc ind, countable $\implies \Gamma$ acts faithfully on $\mathbb{R}$.

Proof of the converse (for amen grps)

Assume

Γ *amenable, finitely generated, acts faithfully on* $\mathbb{R}$.

We show $\Gamma \twoheadrightarrow \mathbb{Z}$.

Proof has 4 easy steps, plus one fact.

1. Γ has a left-invariant order.
2. Ghys: Γ acts on the space of left-inv't orders.
3. Amenability: $\exists$ Γ-invariant probability measure.
4. Poincaré Recurrence: there is a recurrent order.
5. *Known:* Γ has a recurrent order $\implies \Gamma \twoheadrightarrow \mathbb{Z}$.

Step 1: Γ has a left-invariant order

Assume: Γ acts faithfully on $\mathbb{R}$.

Definition

$a \prec b \iff a(0) < b(0)$ or ... (break ties)

Exercise

1. $\prec$ *is a total order on* Γ.
2. $\prec$ *is left-invariant.* $(a \prec b \implies ca \prec cb)$

Hint: orientation-pres: $x < y \implies a(x) < a(y)$

Exercise (assume Γ countable)

Γ *acts faithfully on* $\mathbb{R} \iff \exists$ *left-inv't order on* Γ.

Step 2: idea of Étienne Ghys

We know: Γ has a left-invariant order.

Definition

$\mathcal{O} = \{$ left-invariant orders on $\Gamma \} \neq \emptyset$.

Proposition

- $\mathcal{O} \subset 2^{\Gamma\times\Gamma}$ *is compact (& Hausdorff)*
- Γ *acts on* $\mathcal{O}$ *by right translation:*

$$a \prec_g b \quad \iff \quad ag^{-1} \prec bg^{-1}$$

This is an action by homeomorphisms.

Proof.

Exercise. □

Step 3: Amenability

We know: Γ acts continuously on $\mathcal{O}$.

Definition

μ *probability measure* on metric space X: $\mu(X) = 1$.

Definition

Γ *amenable:*

Γ acts (continuously) on metric space X

$\implies$ ∃ Γ-invariant prob meas on X.

I.e., $\mu(g(A)) = \mu(A)$, $\forall g \in \Gamma, \forall A \subset X$.

Γ *amenable*

$\implies$ ∃ Γ-invariant prob meas on $\mathcal{O}$.

Exercise

Your favorite defn of amenable $\implies$ *this defn.*

Exercises (*using this definition*)

- *finite groups are amenable* $\quad \mu = \frac{1}{\#\Gamma} \sum_{g\in\Gamma} \delta_{gx}$
- $\mathbb{Z}$ is amenable $\quad \mu = \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k} \delta_{n(x)}$
- (*amenable* × *amenable* is amenable)
- *f.g. abelian groups are amenable*
- (*subgroups*), *quotients of amen grps are amen*
- Γ *amen* $\iff$ *every f.g. subgrp of* Γ *is amen*
- $N \lhd \Gamma$ *with* $N, \Gamma/N$ *amenable* $\implies$ Γ *amenable*
- *solvable groups are amenable (!!!)*
- *free groups are not amenable*

Step 4: Poincaré Recurrence Theorem

We know: ∃ Γ-invariant probability measure on $\mathcal{O}$.

Poincaré Recurrence Theorem

- μ *prob. meas. on 2nd countable metric space* X
- T *measure-preserving homeomorphism of* X

$\implies$ *a.e.* x *is recurrent for* T: $T^{n_k}(x) \to x, \exists n_k \to \infty$.

$\therefore \forall$ *open* $U \ni x, \exists n_k \to \infty, T^{n_k}(x) \in U$.

Sketch of proof.

Given U, show a.e. $x \in U$ returns to U at least once.
Let $B = \{ b \in U \mid T^n(b) \notin U, \forall n > 0 \}$.
Spse $\mu(B) \neq 0$. $\quad \mu(T^n(B))$ const $\implies$ not all disj.
$\exists\, x \in T^n(B) \cap T^m(B)B \subset T^n(B) \cap U = \emptyset$. □

Poincaré Recurrence Theorem

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$\implies$ *a.e.* x *is recurrent for* T:

$\forall$ *open* $U \ni x, \exists n_k \to \infty, T^{n_k}(x) \in U$.

Corollary

$\forall g \in \Gamma$, *a.e.* $\prec$ *is recurrent for* g:

$a_1 \prec a_2 \prec \cdots \prec a_r$

$\implies a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \exists\, n \to \infty$

Exercise

a.e. $\prec$ *is recurrent for every element of* Γ.

Hint: Γ is ctble (f.g.). Ctble union of sets of meas 0.

Step 5: construct $\Gamma \twoheadrightarrow \mathbb{Z}$

Proposition (Hölder, 1901)

Γ *has left-inv't archimedean order* $\implies$ Γ *abelian.*

$(\forall a, b \succ e, \exists n \in \mathbb{Z}^+, a^n \succ b)$

Exercise

Γ *has left-invariant, archimedean order*

$\iff$ Γ *acts on* $\mathbb{R}$, *no element has a fixed point*

$\iff$ Γ *has bi-invariant, archimedean order*

$0 < x = a(x) < b(0) \implies a^n(0) < a^n(x) = x < b(0)$

Exercise: $\{x_n\} > 0$ subadditive $(x_{m+n} < x_m + x_n)$ $\implies \lim_{n\to\infty} \frac{x_n}{n}$ exists.

Proposition (Hölder, 1901)

Γ has left-inv't archimedean order $\implies$ Γ abelian.

- Γ acts on $\mathbb{R}$, no element has a fixed point.
- order is bi-invariant.

Proof.

Fix $a \succ e$. We may assume $a(x) = x + 1$ (no f.p.).

Define $\varphi(g) = \lim_{n\to\infty} \frac{g^n(0)}{n}$.

$g^m(0) < M$, $g^n(0) < N$

$\implies g^m \prec a^M$, $g^n \prec a^N$,

$\implies g^{m+n} = g^m g^n \prec a^M a^N = a^{M+N}$

$\implies g^{m+n}(0) < M + N$ "subadditive"

Then φ is a homomorphism $\Gamma \hookrightarrow \mathbb{R}$. □

Proposition (Hölder, 1901)

Γ has left-inv't archimedean order $\implies$ Γ abelian.

Exercise

$C \lhd \Gamma$. $\prec$ *well-defined on* Γ/C $\iff$ C *convex.*

$(\forall g \in \Gamma, c \in C,\ \ e < g < c \implies g \in C)$

Remark

$\prec$ not arch $\iff$ $\exists \langle a \rangle \prec b$ $\iff$ $\exists$ *bounded* subgroup

Corollary

$\langle$bdd subgrps of $\Gamma\rangle \subset$ *cnvx, normal subgrp of* Γ (f.g.)

$\implies \Gamma \twoheadrightarrow \mathbb{Z}$.

Proposition

Γ has a recurrent order (and f.g.) $\implies \Gamma \twoheadrightarrow \mathbb{Z}$

Proof (pretend bi-invariant)

1. $a_1, \ldots, a_r \preceq g \implies a_1 a_2 \cdots a_r \preceq g^{r'}$
2. $a_1^{\pm 1}, \ldots, a_r^{\pm 1} \preceq g$, $\langle g \rangle \prec h \implies \langle a_1, \ldots, a_r \rangle \prec h$.
3. $B = \langle$ bdd subgrps of Γ $\rangle$ is bdd, convex, normal.

- $\langle a_1, \ldots, a_r \rangle = \Gamma$, $g = \max\{a_i^{\pm 1}\}$: $\langle g \rangle$ unbdd
  $\exists \langle x \rangle$ unbdd
- $\langle g \rangle$ bdd, $a_1 = x$, $(h \succ \langle g \rangle)$: $\langle g \rangle \prec x$.
- $\langle a_i \rangle$ bdd, $g = \max\{a_i^{\pm 1}\}$, $h = x$: $\langle a_1, \ldots, a_r \rangle \prec x$.
  $B \prec x$
- $H \prec y \implies a^{-1} H a \prec \max\{a^{-1}, y, a\}^3$. ???

Recap: Proof of the Converse

Assume

Γ amenable, finitely generated, acts faithfully on $\mathbb{R}$.

We show $\Gamma \twoheadrightarrow \mathbb{Z}$.

Proof has 4 easy steps, plus one fact.

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Actions on the circle

Recall: Γ amenable, acts on $\mathbb{R}$, f.g. $\implies \Gamma \twoheadrightarrow \mathbb{Z}$.

Corollary

Γ amen, acts on S^1, f.g. $\implies \Gamma \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$, $\exists n > 1$.

(Conversely, $\mathbb{Z}/n\mathbb{Z}$ acts on S^1.)

Proof

We wish to show $\Gamma \twoheadrightarrow$ abelian.

Γ amenable $\implies \exists$ Γ-invariant prob meas μ on S^1.

Two cases:

1. μ has no atom: $\forall x \in S^1$, $\mu(\{x\}) = 0$.
2. μ *has* an atom.

Assume Γ amenable, f.g., $\Gamma \subset \mathrm{Homeo}_+(S^1)$.

Case 1. Assume μ has no atoms.

Assume, for simplicity, $\mathrm{supp}(\mu) = S^1$. ($\mu(\text{interval}) \neq 0$)

Exercise

$\exists \phi \in \mathrm{Homeo}_+(S^1)$, $\phi_*(\mu) = $ Lebesgue.

Wolog $\mu =$ Lebesgue. I.e., Γ preserves Lebesgue.
So Γ consists of rotations. $\therefore$ Γ is abelian. □

Exercise

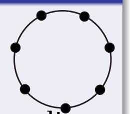
Do not assume $\mathrm{supp}(\mu) = S^1$.

Hint: Reduce to above: define $x \sim y$ if $\mu((x, y)) = 0$.
Then $S^1/\!\sim\ = \mathrm{supp}(\mu)/\!\sim\ \cong S^1$.

Assume Γ amenable, f.g., $\Gamma \subset \mathrm{Homeo}_+(S^1)$.

Case 2. Assume μ has an atom: $\mu(p) \neq 0$

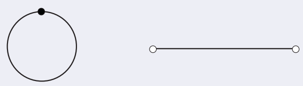
Γ preserves μ: $\mu(g(p)) = \mu(p)$.
μ prob meas: orbit Γp is finite.
Γ permutes the points cyclically (or-pres)
Homomorphism $\phi \colon \Gamma \to \mathrm{Perm}(\Gamma p)$. $\phi(\Gamma)$ cyclic.



Gap

$\phi(\Gamma)$ can be trivial: perhaps $\Gamma p = \{p\}$. (*fixed point*).

Γ acts on $S^1 \setminus \{p\} \cong \mathbb{R}$.
Therefore $\Gamma \twoheadrightarrow \mathbb{Z}$.



Recall: Γ amenable, f.g., acts on $\mathbb{R} \implies \Gamma \twoheadrightarrow \mathbb{Z}$.

Conjecture (Linnell)

Can replace amenable with no free subgroups.

Definition

Say Γ is a *Poincaré-Recurrence group* if:
$\forall$ (continuous) action of Γ on compact, metric X,
$\exists x \in X$, x is recurrent for all cyclic subgroups of Γ.

Questions

1. *¿ Γ no free subgroups $\implies$ Γ is P-R group ?*
2. *¿ $\exists$ P-R group that is not amenable ?*
3. *¿ What are the P-R groups ?*

References

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