Open Problems on Hamiltonian Cycles in Cayley Graphs

Dave Witte Morris

Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge, AB T1K 3M4
Dave.Morris@uleth.ca

Defn of Cayley graph.

G = finite group (e.g., dihedral grp of order 8)

$D_8 = \langle f, t \mid e = f^2 = t^4,\ ftf = t^{-1} \rangle$

S = generating set of G (e.g., $\{f, t\}$)

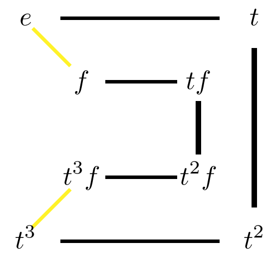
Cayley graph $\text{Cay}(G; S)$:

vertices = elements of G

edge $v - vs^{\pm 1}$

for $v \in G$ and $s \in S$

$\text{Cay}(D_{2n}; f, t)$ has a ham cycle.



Conj. $\text{Cay}(G; S)$ *has a ham cycle.* (Easy if G abelian.)

- $\text{Cay}(G; S)$ *has a hamiltonian path.*
- $\text{Cay}(G; S)$ *has a path of length* $\epsilon \#G$.
- $\text{Cay}(G; S)$ *has a ham cycle for some irredundant* S.
- [Babai] *Opposite conjecture: not always a ham path.*

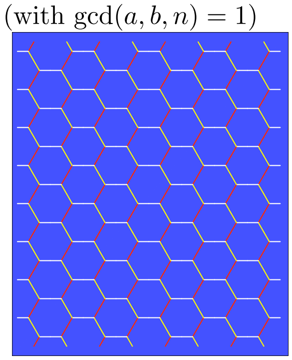
Prop.

- [Babai] $\exists$ *path (& cycle) of length* $\approx \sqrt{\#G}$.
- [Pak] $\forall G$, $\exists S$, $\text{Cay}(G; S)$ *has a ham cyc, and* $\#S \le \log_2 \#G$.
- [Witte] $\forall S$, $\exists S'$, $\text{Cay}(G; S')$ *has a ham cyc, and* $\#S' \le (\#S)^2$.

Problem. *Prove the conjecture when* G *is dihedral.*

Eg. $\text{Cay}(D_{2n}; f, ft^a, ft^b)$ (with $\gcd(a, b, n) = 1$)

- valence 3,
- embeds on torus,
- [Alspach-Zhang] has a ham cycle.



Conj. $\text{Cay}(D_{2n}; \{\textit{reflections}\})$ *has a ham cycle.*

(Then $\text{Cay}(D_{2n}; \{\text{anything}\})$ has a ham cycle.)

Thm [Witte]. $\text{Cay}(G; S)$ *has a hamiltonian cycle if* $\#G$ *is a prime power* p^n.

Problem. *Find hamiltonian cycle if* $\#G = 2p^n$.

Problem. *Find hamiltonian cycle if* $G = P \times Q$ *where* $\#P$ *and* $\#Q$ *are prime powers.*

(G is "nilpotent.")

Conj. $\text{Cay}(G; S)$ *has a hamiltonian cycle.*

True when G is "*almost*" abelian.

Defn. commutator subgroup of $G = [G, G]$

$= \langle g^{-1}h^{-1}gh \mid g, h \in G \rangle$.

Rem. G is abelian $\iff$ $[G, G] = \{e\}$.

Thm [Durnberger, Marušič, Keating-Witte].
$\text{Cay}(G; S)$ *has a ham cycle if* $[G, G]$ *has prime order or, more generally, is cyclic of prime-power order.*

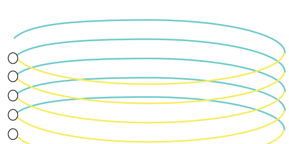
Problem. *Find ham cycle if* $[G, G]$ *is cyclic.*
Problem. *Find ham cycle if* $[G, G] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Thm [Durnberger, Marušič, Keating-Witte].
$\text{Cay}(G; S)$ *has a ham cycle if* $[G, G]$ *has prime order.*

Idea of proof. $\overline{G} = G/[G, G]$ is abelian
$\Rightarrow \text{Cay}(\overline{G}; \overline{S})$ has a ham cyc $\overline{C}$.

Lift $\overline{C}$ to a *path* P in $\text{Cay}(G; S)$.

Assume P is *not* a cycle.
["*Marušič's method*"]
Then we construct ham cyc in $\text{Cay}(G; S)$ by concatenating translates of P.



Thm [Alspach]. $\text{Cay}(G; s, t)$ *has a ham cyc if* $\langle s \rangle$ *is a normal subgroup of* G.

Problem. *Show* $\text{Cay}(G; S)$ *has a ham cyc if*

- $\langle s \rangle \triangleleft G$, *for some* $s \in S$, *and*
- $\text{Cay}(G/\langle s \rangle; S)$ *has a ham cyc.*

Thm [Paulraja].
The prism over X has a hamiltonian cycle if X is cubic and 3-connected.

(Short proof: [Čada-Kaiser-Rosenfeld-Ryjáček])

Problem. *Find ham cyc in prism* $\mathrm{Cay}(G;S) \square P_2$.

Paulraja: Case where valence is three.

Many Cayley digraphs do not have hamiltonian cycles.

Eg. (with G cyclic): $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12};3,4)$ has no ham cyc.
[Rankin]: $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n;\ s,s+1)$ *has no ham cyc unless* $\gcd(n,s)=1$ *or* $\gcd(n,s+1)=1$.
In general, $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n;s,t)$ has ham cyc
$\iff \gcd(n,ks+\ell t)=1$, with $k+\ell=\gcd(n,s-t)$.

Problem. *When does* $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n;a,b,c)$ *have a ham cyc?*
Thm [Locke-Witte]. $\exists\ \infty$ *non-hamiltonian examples.*

Conj [Curran-Witte]. $\{a,b,c\}$ *irredundant* $\Rightarrow \exists$ *ham cyc.*
Rem. G abelian $\Rightarrow \overrightarrow{\mathrm{Cay}}(G;S)$ has ham **path**.

Survey articles.

B. Alspach,
The search for long paths and cycles in vertex-transitive graphs and digraphs.
Combinatorial mathematics, VIII (Geelong, 1980),
Lecture Notes in Math. #884,
Springer, Berlin-New York, 1981, pp. 14–22.
MR 83b:05080

D. Witte and J. A. Gallian,
A survey: Hamiltonian cycles in Cayley graphs,
Discrete Math. 51 (1984), no. 3, 293–304.
MR 86a:05084

S. J. Curran and J. A. Gallian,
Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey,
Discrete Math. 156 (1996), no. 1–3, 1–18.
MR 97f:05083

Some other references.

B. Alspach,
Lifting Hamilton cycles of quotient graphs,
Discrete Math. 78 (1989), no. 1-2, 25–36.
MR1020643 (91c:05091)

B. Alspach and C.–Q. Zhang,
Hamilton cycles in cubic Cayley graphs on dihedral groups,
Ars Combin. 28 (1989), 101–108.
MR1039136 (91b:05124)

L. Babai,
Long cycles in vertex-transitive graphs,
J. Graph Th. 3 (1979), no. 3, 301–304.
MR0542553 (80m:05059)

L. Babai,
Automorphism groups, isomorphism, reconstruction,
Handbook of Combinatorics, Vol. 2,
Elsevier, Amsterdam, 1995, pp. 1447–1540.
(See end of §3.3, pp. 1472–1474.)
MR1373683 (97j:05029)

R. Čada, T. Kaiser, M. Rosenfeld, and Z. Ryjáček,
Hamiltonian decompositions of prisms over cubic graphs,
Discrete Math. 286 (2004), no. 1-2, 45–56.
MR2084278 (2005d:05095)

S. J. Curran and D. Witte,
Hamilton paths in Cartesian products of directed cycles,
Cycles in graphs (Burnaby, B.C., 1982),
North-Holland, Amsterdam, 1985, pp. 35–74.
MR0821505 (87h:05139)

K. Keating and D. Witte,
On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup,
Cycles in graphs (Burnaby, B.C., 1982),
North-Holland, Amsterdam, 1985, pp. 89–102.
MR0821508 (87f:05082)

S. C. Locke and D. Witte,
On non-Hamiltonian circulant digraphs of outdegree three,
J. Graph Theory 30 (1999), no. 4, 319–331.
MR1669452 (99m:05069)

P. Paulraja,
A characterization of hamiltonian prisms,
J. Graph Theory 17 (1993), no. 2, 161–171.
MR1217391 (94e:05215)

R. A. Rankin,
A campanological problem in group theory,
Proc. Cambridge Philos. Soc. 44 (1948) 17–25.
MR0022846 (9,267f)

D. S. Witte,
On Hamiltonian circuits in Cayley diagrams,
Discrete Math. 38 (1982), no. 1, 99–108.
MR0676525 (83k:05055)

D. Witte,
Cayley digraphs of prime-power order are hamiltonian,
J. Combin. Theory Ser. B 40 (1986), no. 1, 107–112.
MR0830597 (87d:05092)