Some arithmetic groups that cannot act on the line

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Transformation groups

Given: group Γ, (connected) manifold M.
¿ What are the actions of Γ on M ?
I.e.: ¿ What are homos $\phi \colon \Gamma \to \mathrm{Homeo}_+(M)$?

Question

¿ ∃ (faithful) *action ?*

Simplest case

$\dim M = 1$, so $M = S^1$ or $\mathbb{R}$.
Assume $M = \mathbb{R}$.

Example

$\mathbb{Z}$ acts on $\mathbb{R}$. $(T_n(x) = x + n \implies T_{m+n} = T_m \circ T_n)$

Question

¿ ∃ (faithful) *action of* Γ *on* $\mathbb{R}$ *?*

Today: Γ is an *arithmetic group*

Example

$\mathrm{SL}(2,\mathbb{Z})$ does *not* act on $\mathbb{R}$.

Proof.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I$. So $\mathrm{SL}(2,\mathbb{Z})$ has elt's of finite order.

But $\mathrm{Homeo}_+(\mathbb{R})$ has no elt's of finite order:
$\varphi(0) > 0 \implies \varphi^2(0) > \varphi(0) > 0 \implies \varphi^3(0) > 0$
$\implies \dots \implies \varphi^n(0) > 0$. □

Example: $\mathrm{SL}(2,\mathbb{Z})$ does *not* act on $\mathbb{R}$
because it has elements of finite order.

Example

$\Gamma \doteq \mathrm{SL}(2,\mathbb{Z})$ finite-index subgrp can be a *free group*.
Has *many* actions on $\mathbb{R}$.

Fact

There exist other examples that act on $\mathbb{R}$.
But all are "small". (I think all known are in $\mathrm{SO}(1,n)$).

Conjecture

Large arithmetic groups ($\mathbb{R}$-rank > 1) *cannot act on* $\mathbb{R}$.

Question

¿ ∃ (nontrivial) *action of* Γ *on* $\mathbb{R}$ *?*

Assume Γ is a "large" *arithmetic group*:
$\Gamma \doteq \mathrm{SL}(3,\mathbb{Z}) = \{\ 3 \times 3$ integer matrices of det 1 $\}$
(subgroup of finite index)
Or $\Gamma \doteq \mathrm{SL}(2,\mathbb{Z}[\sqrt{3}])$ or ...
Or $\Gamma \doteq \mathrm{SL}(2,\mathbb{Z}[\alpha])$ α = real, irrat alg'ic integer.

But $\Gamma \neq \mathrm{SL}(2,\mathbb{Z})$, other "small" grps. (Need $\mathrm{rank}_{\mathbb{R}}\,\Gamma > 1$.)

Conjecture

Γ *does not act on* $\mathbb{R}$.

Theorem (Witte, Lifschitz-Morris)

Γ *no action on* $\mathbb{R}$ *if* $\Gamma \doteq \mathrm{SL}(3,\mathbb{Z})$ *or* $\mathrm{SL}(2,\mathbb{Z}[\alpha])$

Proof combines bdd generation and bdd orbits.
Unipotent subgroups: $\overline{U} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$, $\underline{V} = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$.

Theorem (Carter-Keller-Paige, Lifschitz-Morris)

- $\overline{U}$ *and* $\underline{V}$ *boundedly generate* Γ (*up to finite index*).
- Γ *acts on* $\mathbb{R} \implies \overline{U}$*-orbits* (*and* $\underline{V}$*-orbits*) *are bdd.*

Bounded generation by unip subgrps

Note: Invertible matrix ⇝ Id by row operations.

Key fact: $g \in \mathrm{SL}(2,\mathbb{Z})$ ⇝ Id by integer ($\mathbb{Z}$) row ops.

Example

$$\begin{bmatrix} 13 & 31 \\ 5 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But # steps is *not bounded:*
$\overline{U}$ and $\underline{V}$ do **not** boundedly generate $\mathrm{SL}(2,\mathbb{Z})$.

Key fact: $g \in \mathrm{SL}(2,\mathbb{Z})$ ⇝ Id by integer ($\mathbb{Z}$) row ops,
but # steps is *not bounded.*

Remark: In $\mathrm{SL}(3,\mathbb{Z})$, # steps is bounded [Carter-Keller].

Theorem (Liehl, Carter-Keller-Paige)

For $\mathbb{Z}[\alpha]$ *row operations, # steps is bounded.*
$\exists n,\ \forall g \in \mathrm{SL}(2,\mathbb{Z}[\alpha]),\ g = u_1 v_1 u_2 v_2 \cdots u_n v_n$.
I.e., $\overline{U}$ *and* $\underline{V}$ *boundedly gen* $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\alpha])$.
So $\mathrm{SL}(2,\mathbb{Z}[\alpha]) = \overline{U}\underline{V}\overline{U}\underline{V} \cdots \overline{U}\underline{V}$.

Theorem (Liehl)

$\mathrm{SL}(2,\mathbb{Z}[1/p])$ *bddly gen'd by elem mats.*
I.e., T ⇝ *Id by* $\mathbb{Z}[1/p]$ *col ops, # steps is bdd.*

Easy proof

Assume Artin's Conjecture:
$\forall r \neq \pm 1$, perfect square,
∃ ∞ primes q, s.t. r is primitive root modulo q:
$\{r, r^2, r^3, \dots\} \bmod q = \{1, 2, 3, \dots, q-1\}$
Assume $\exists q$ in every arith progression $\{a + kb\}$.

$\exists\ q = a + kb$, $\underline{\ p\ }$ is a primitive root modulo q.

Theorem (Liehl)

$\mathrm{SL}(2, \mathbb{Z}[1/p])$ *bddly gen'd by elem mats.*
I.e., $T \rightsquigarrow \mathrm{Id}$ *by* $\mathbb{Z}[1/p]$ *col ops, # steps is bdd.*

Proof.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad q = a + kb \text{ prime, } p \text{ is prim root}$$

$$\rightsquigarrow \begin{bmatrix} q & b \\ * & * \end{bmatrix} \qquad p^\ell \equiv b \pmod q; \;\; p^\ell = b + k'q$$

$$\rightsquigarrow \begin{bmatrix} q & p^\ell \\ * & * \end{bmatrix} \qquad p^\ell \text{ unit: can add \textit{anything} to } q$$

$$\rightsquigarrow \begin{bmatrix} 1 & p^\ell \\ * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

- Bdd generation: $\Gamma = \overline{U}\underline{V}\overline{U}\underline{V} \cdots \overline{U}\underline{V}$.
- Bdd orbits: $\overline{U}$-orbits and $\underline{V}$-orbits are bounded.

Corollary

$\phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$
$\Rightarrow$ *every* Γ*-orbit on* $\mathbb{R}$ *is bounded*
$\Rightarrow$ Γ *has a fixed point.*

Corollary

Γ *cannot act on* $\mathbb{R}$.

Corollary

Γ *cannot act on* $\mathbb{R}$.

Proof.

Suppose there is a nontrivial action.

It has fixed points:

Remove them:

Take a connected component:

Γ acts on open interval ($\approx \mathbb{R}$) with no fixed point.

$\to\leftarrow$ □

Bounded orbits

Theorem (Lifschitz-Morris)

$\Gamma = \mathrm{SL}(2, \mathbb{Z}[1/p])$ *acts on* $\mathbb{R}$ $\Rightarrow$ *every* $\overline{U}$*-orbit bdd.*

$$\overline{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \; \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \; \not{p} = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix}$$

Assume $\overline{U}$-orbit and $\underline{V}$-orbit of x not bdd above.
Assume $\not{p}$ fixes x. ($\not{p}$ does have fixed pts, so not an issue.)

- Wolog $\overline{u}(x) < \underline{v}(x)$.
- Then $\not{p}^n(\overline{u}(x)) < \not{p}^n(\underline{v}(x))$.
- LHS $= \not{p}^n(\overline{u}(x)) = (\not{p}^n \overline{u} \not{p}^{-n})(x) \to \overline{\infty}(x) \to \infty$.
- RHS $= \not{p}^n(\underline{v}(x)) = (\not{p}^n \underline{v} \not{p}^{-n})(x) \to \underline{0}(x) < \infty$.

$\to\leftarrow$ □

Other arithmetic groups of higher rank

Proposition

Suppose $\Gamma_1 \subset \Gamma_2$.

- *If* Γ_2 *acts on* $\mathbb{R}$*, then* Γ_1 *acts on* $\mathbb{R}$.
- *If* Γ_1 *does not act on* $\mathbb{R}$*, then* Γ_2 *does not act on* $\mathbb{R}$.

Our methods require Γ to have a unipotent subgrp.
Such arithmetic groups are called *noncocompact*.

Theorem (Chernousov-Lifschitz-Morris)

Spse Γ *is a noncocompact arith group of higher rank.*
Then $\Gamma \dot{\supset} \mathrm{SL}(2, \mathbb{Z}[\alpha])$
or noncocpct arith grp in $\mathrm{SL}(3, \mathbb{R})$ *or* $\mathrm{SL}(3, \mathbb{C})$.

Open Problem

Show noncocpct arith grps in $\mathrm{SL}(3, \mathbb{R})$ *and* $\mathrm{SL}(3, \mathbb{C})$
cannot act on $\mathbb{R}$.

Conjecture (Rapinchuk, ~1990)

These arith grps are boundedly generated by unips.

Rapinchuk Conjecture implies no action on $\mathbb{R}$
if Γ noncocompact of higher rank.

Cocompact case will require new ideas.

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