Amenable groups that act on the line

Dave Witte Morris

University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris@uleth.ca
Dave.Morris@uleth.ca

Geometry and Algebra of Solvable Groups
Universität Karlsruhe
October 16 - 20, 2006

Thanks to Étienne Ghys, Uri Bader, and Alex Furman

Introduction

Transformation groups

- G = group (countable, discrete, nontrivial)
- X = topological space

Does G have an (nontrivial, C^0) action on X?

Assume

- $X = \mathbb{R}$
- *actions are orientation preserving.*

Remark

actions on $\mathbb{R}$ $\longleftrightarrow$ actions on S^1

Observation

- *$\mathbb{Z}$ acts on $\mathbb{R}$ by translation* $(x \mapsto x + n)$
- $G \twoheadrightarrow \mathbb{Z} \implies$ *G acts on $\mathbb{R}$*

Converse?

30 years ago:

- ¿converse true for finitely generated groups?
- G acts on $\mathbb{R}$ (and f.g.) $\overset{?}{\implies}$ $G \twoheadrightarrow \mathbb{Z}$.

Counterexample

Thompson's group on S^1 (lift to universal cover)

Converse true for some classes of groups

G acts on $\mathbb{R}$ (and f.g.) $\implies$ $G \twoheadrightarrow \mathbb{Z}$:

- polycyclic (Rhemtulla, 1981)
- solvable-by-finite (Chiswell–Kropholler, 1993)
- supramenable (Kropholler, 1993)
- elementary amenable (Linnell, 1999)

Peter Linnell (2001): **amenable?**

Theorem (2006)

- *G f.g., amenable*
- *G acts on $\mathbb{R}$*

$\implies G \twoheadrightarrow \mathbb{Z}$

Corollary

- *G f.g., amenable*
- *G acts faithfully on $\mathbb{R}$*

$\implies$ *every f.g. subgroup of $G \twoheadrightarrow \mathbb{Z}$ maps onto $\mathbb{Z}$*

Converse (Conrad, 1959)

Every f.g. subgroup of G maps onto $\mathbb{Z}$
$\implies$ *G acts faithfully on $\mathbb{R}$*

Remark

Conrad's Thm. does *not* assume G is amenable

Proof of the Theorem

Assume

- *G is amenable*
- *G acts faithfully on $\mathbb{R}$*
- *G is finitely generated*

We will show $G \twoheadrightarrow \mathbb{Z}$.

1. G has a left-invariant order.
2. Ghys: G acts on the space of left-inv't orders.
3. Amenability: $\exists$ G-invariant measure.
4. Poincaré Recurrence: there is a recurrent order.
5. *Known:* G has a recurrent order $\implies G \twoheadrightarrow \mathbb{Z}$.

Step 1: G has a left-invariant order

Assume: G amenable, f.g., acts faithfully on $\mathbb{R}$

Proposition

G acts faithfully on $\mathbb{R}$
$\implies$ G has a left-invariant order:

- *$\prec$ is a total order on G*
- *$a \prec b \implies ca \prec cb, \quad \forall a, b, c \in G$*

Proof.

Define $a \prec b$ if $a(0) < b(0)$. *(can break ties)*
$c(a(0)) < c(b(0))$ bcs c preserves orientation. □

Step 2: idea of Étienne Ghys

We know: G has a left-invariant order

Definition

$\mathcal{O} = \{$ left-invariant orders on $G \} \neq \emptyset$.

Proposition

- *$\mathcal{O} \subset 2^{G \times G}$ is compact (& Hausdorff)*
- *G acts on $\mathcal{O}$ by right translation:*

$$a \prec_g b \iff ag^{-1} \prec bg^{-1}$$

This is an action by homeomorphisms.

Step 3: amenability

We know: G acts continuously on $\mathcal{O}$

Definition

- μ a *probability measure* on X: $\mu(X) = 1$
- G *amenable*:
  G acts on compact Hausdorff X
  $\implies$ $\exists$ G-invariant probability meas on X.

Corollary

$\exists$ G-invariant probability measure on $\mathcal{O}$.

Step 4: Poincaré Recurrence Theorem

We know: $\exists$ G-inv't prob meas on $\mathcal{O}$

Theorem

- *μ prob. meas. on 2nd countable metric space X*
- *T measure-preserving homeo of X*

$\implies$ a.e. point of X is recurrent for T:
$\forall$ open $U \supset x$, $\exists n_k \to \infty$, $T^{n_k}(x) \in U$

Corollary

$\forall g \in G$, a.e. $\prec$ is recurrent for g:

$$a_1 \prec a_2 \prec \cdots \prec a_r$$
$$\implies a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \exists\, n \to \infty$$

Corollary

$\forall g \in G$, a.e. $\prec$ is recurrent for g:

$$a_1 \prec a_2 \prec \cdots \prec a_r$$
$$\implies a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \exists\, n \to \infty$$

Corollary

a.e. $\prec$ is recurrent for every element of G.

Proof.

G is countable (f.g.).

Union of countably many sets of measure 0
still has measure 0. □

Step 5: recurrent order $\implies$ $G \twoheadrightarrow \mathbb{Z}$

Theorem (Conrad, 1959)

G has a recurrent order (and f.g.) $\implies$ $G \twoheadrightarrow \mathbb{Z}$

Definition

$\prec$ is *recurrent*:

$$a_1 \prec a_2 \prec \cdots \prec a_r$$
$$\implies a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \exists n \to \infty.$$

Definition

$\prec$ is *bi-invariant*:

$$a_1 \prec a_2 \prec \cdots \prec a_r$$
$$\implies a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \forall n g.$$

Proposition (Conrad, 1959)

G has archimedean order

$(\forall a, b \succ e, \exists n \in \mathbb{Z}^+, a^n \succ b)$

$\Rightarrow$ *G abelian*

Remark

Archimedean and f.g. $\Rightarrow \; G \twoheadrightarrow \mathbb{Z}$

Proof.

Fix $a \succ e$. Define $\varphi(g) = \lim_{n\to\infty} \frac{\underline{\log}_a g^n}{n}$,

where $\underline{\log}_a g = m$ means $a^m \preceq g \prec a^{m+1}$.

φ is a homomorphism $G \hookrightarrow \mathbb{R}$. □

Proposition

G has a bi-invariant order (and f.g.) $\Rightarrow \; G \twoheadrightarrow \mathbb{Z}$

Lemma

$a_1, \ldots, a_r \preceq g \;\Rightarrow\; a_1 a_2 \cdots a_r \preceq g^r$

Corollary

$B = \langle$ bounded subgroups of $G \rangle$ *is bounded, normal.*

Proof.

$\langle a_i \rangle \prec g \quad \Rightarrow \langle a_1, a_2, \ldots, a_r \rangle \preceq \langle \max a_i \rangle \prec g.$ □

Proof of Proposition.

Corollary: $\prec$ induces an archimedean order on G/B.

So $G/B \mapsto \mathbb{Z}$. □

Recap: Proof of the Theorem

Assume

- *G is amenable*
- *G acts faithfully on $\mathbb{R}$*
- *G is finitely generated*

We show $G \twoheadrightarrow \mathbb{Z}$.

1. G has a left-invariant order.
2. Ghys: G acts on the space of left-inv't orders.
3. Amenability: $\exists$ G-invariant measure.
4. Poincaré Recurrence: there is a recurrent order.
5. *Known:* G has a recurrent order $\Rightarrow \; G \twoheadrightarrow \mathbb{Z}$.

Open questions

Theorem (Farb-Franks)

G nilpotent, acts on $\mathbb{R}$ $\Rightarrow$ G has C^1 action on $\mathbb{R}$.

Question

Can every action of a nilp grp on $\mathbb{R}$ be made C^1?

Question (Calegari)

Can every action of an amen grp on $\mathbb{R}$ be made C^1?

Question (Plante, Farb-Franks, Navas, . . .)

Which solvable (or nilpotent or amenable) groups have C^∞ actions on $\mathbb{R}$?

Theorem (Reznikov)

Thompson's group on the circle
does not have Kazhdan's property T

Question (well known)

G acts on $\mathbb{R}$ $\overset{?}{\Rightarrow}$ G not Kazhdan (T).

Remark

Navas has a result for differentiable actions on S^1

Theorem

Assume G amenable.

G has a faithful action on $\mathbb{R}$

$\Leftrightarrow$ *every f.g. subgroup of G maps onto $\mathbb{Z}$*

Question

Which amenable (or solvable) groups have $C^{\square}$ actions on an n-manifold? $(n > 1)$