Why arithmetic groups are lattices

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Abstract. $\mathrm{SL}(n,\mathbb{Z})$ is the basic example of a lattice in $\mathrm{SL}(n,\mathbb{R})$, and a lattice in any other semisimple Lie group G can be obtained by intersecting (a copy of) G with $\mathrm{SL}(n,\mathbb{Z})$. We will discuss the main ideas behind three different approaches to proving the important fact that the integer points do form a lattice in G.

Recall

Γ is a *lattice* in G means

- Γ is a discrete subgroup of G, and
- G/Γ has finite volume (e.g., compact)

Remark

M = hyperbolic 3-manifold of finite volume (e.g., compact)
$\iff$ $M = \mathfrak{H}^3/\Gamma$, where Γ = (torsion-free) lattice in $\mathrm{SO}(1,3)$.

More generally:

locally symmetric space of finite volume
$\longleftrightarrow$ (torsion-free) lattice in semisimple Lie group G

So it would be nice to have an easy way to make lattices.

Example

$\mathrm{SL}(n,\mathbb{Z})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$

$\mathrm{SL}(n,\mathbb{Z})$ is the basic example of an *arithmetic group*.

More generally:

For $G \subset \mathrm{SL}(n,\mathbb{R})$ (with some technical conditions):
$G_{\mathbb{Z}} = G \cap \mathrm{SL}(n,\mathbb{Z})$ is an *arithmetic subgroup* of G.
$G_{\mathbb{Z}}$ is a lattice in G.

Arithmetic groups are lattices.

Remark ("*Margulis Arithmeticity Theorem*")

Converse if $\mathbb{R}$-rank $G \geq 2$ (and Γ is irreducible).

Theorem (Siegel, Borel & Harish-Chandra, 1962)

- *G is a (connected) semisimple Lie group*
- $G \hookrightarrow \mathrm{SL}(n,\mathbb{R})$
- *G is defined over $\mathbb{Q}$*
  - i.e. *G is defined by polynomial equations with $\mathbb{Q}$ coefficients*
  - i.e. *Lie algebra of G is defined by linear eqs with $\mathbb{Q}$ coeffs*
  - i.e. *$G_{\mathbb{Q}}$ is dense in G*
  - i.e. *$G_{\mathbb{Z}}$ is Zariski dense in G [if G has no compact factors]*
    i.e. $G_{\mathbb{Z}} \not\subset$ connected, proper subgroup of G

$\implies$ *$G_{\mathbb{Z}}$ is a lattice in G.*

Remark

- It is obvious that $G_{\mathbb{Z}}$ is discrete.
- It is *not* obvious that $G/G_{\mathbb{Z}}$ has finite volume.

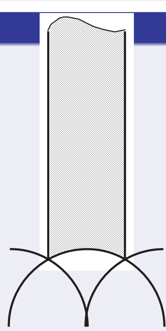
$G_{\mathbb{Z}}$ is a lattice in G

Example

$\mathrm{SL}(2,\mathbb{Z})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

Proof.

Fundamental domain:



$$\mathrm{Area}(\mathcal{F}) < \int_{\epsilon}^{\infty}\int_{-1/2}^{1/2} \frac{dx}{y}\frac{dy}{y} = \int_{\epsilon}^{\infty} \frac{1}{y^2}\,dy < \infty$$ □

A short proof for *some* lattices

Example

$\mathrm{SO}(n,1;\mathbb{Z})$ is a lattice in $\mathrm{SO}(1,n)$. *Not so easy to prove.*

Example

$B(x_1,\ldots,x_4) = x_1^2 + x_2^2 + x_3^2 - 7x_4^2$
$\implies$ $\mathrm{SO}(B(\vec{x});\mathbb{Z})$ is a lattice in $\mathrm{SO}(B(\vec{x})) \cong \mathrm{SO}(3,1)$.
In fact, $G/G_{\mathbb{Z}}$ is compact.

Proposition

- $G = \mathrm{SO}(B(\vec{x}))$
- *$B(\vec{x})$ a (nondegenerate) quadratic form with $\mathbb{Z}$ coeffs*
- *$B(\vec{x})$ anisotropic: $B(v) \neq 0$ for $v \in (\mathbb{Z}^n)^{\times}$*

$\implies$ *$G/G_{\mathbb{Z}}$ is compact.*

Proof

$G/G_{\mathbb{Z}} \hookrightarrow \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$

Proof has 2 parts:

1. $G/G_{\mathbb{Z}}$ is closed in $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$.
   (True for any G defined over $\mathbb{Q}$.)
2. $G/G_{\mathbb{Z}}$ is precompact in $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$.

Part 1 of the proof.

Suppose $g_i\gamma_i \to h$ with $g_i \in G$ and $\gamma_i \in \mathrm{SL}(n,\mathbb{Z})$.
For $v \in \mathbb{Z}^n$: $B(\gamma_i v) = B(g_i\gamma_i v) \to B(hv)$,
but $B(\gamma_i v) \in B(\mathbb{Z}^n) \subset \mathbb{Z}$.
So $B(\gamma_i v) = B(hv)$ is (eventually) constant: $B(\gamma_i v) = B(\gamma_0 v)$.
For simplicity, assume $\gamma_0 = e$.
Then $B(hv) = B(\gamma_i v) = B(\gamma_0 v) = B(v)$.
So $h \in \mathrm{SO}(B(\vec{x})) = G$. □

For Part 2 of the proof, we need a lemma:

Mahler Compactness Criterion

Let $C \subset \mathrm{SL}(n,\mathbb{R})$.
The image of C in $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$ is precompact
$\iff$ *0 is not an accumulation point of $C\mathbb{Z}^n$.*

Proof ($\Rightarrow$).

Suppose $c_n z_n \to 0$ and $c_n\gamma_n \to h$. We may assume $\gamma_n = e$.
Then $hz_n \approx c_n z_n \approx 0$. But $h\mathbb{Z}^n$ is discrete. So $z_n = 0$. □

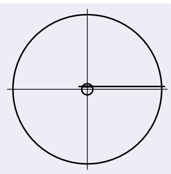
Converse is an exercise. (Try for $n = 2$ first.)

Part 2 of the proof.

$g_n v_n \in G(\mathbb{Z}^n)^{\times} \quad \Longrightarrow \; B(g_n v_n) = B(v_n) \in B((\mathbb{Z}^n)^{\times}) \in \mathbb{Z}^{\times}$
(because $B(\vec{x})$ is anisotropic with $\mathbb{Z}$ coefficients)
$\Longrightarrow \; B(g_n v_n) \not\to 0$
$\Longrightarrow \; g_n v_n \not\to 0$. □

A dynamical proof that $G_{\mathbb{Z}}$ is a lattice [Margulis]

Any line segment that reaches the boundary of the large disk has only a small fraction of its length inside the tiny disk.



Theorem (Dani-Margulis)

- $x \in \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$,
- $u \in \mathrm{SL}(n,\mathbb{R})$ *unipotent*
  I.e., u is conjugate to an element of N.

$\Longrightarrow$ *$\exists$ compact $C \subset \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$,*

$$\liminf_{R\to\infty} \frac{\#\{\,k \in [0,R] \mid u^k x \in C\,\}}{R} > 0$$

Corollary

$\exists$ u-invariant $\varphi \in L^1(\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})) \smallsetminus \{0\}$.

Proof.

$\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$ can be covered by countably many cpct sets, so $\exists$ cpct C that works for all x in a set of positive measure.

Let $\quad \varphi(x) = \liminf_{R\to\infty} \frac{\#\{k\in[0,R] \mid u^k x \in C\}}{R}$.

$\int_{G/\Gamma} \varphi = \int \liminf \frac{1}{R}(\chi_C + \chi_C \circ u^{-1} + \chi_C \circ u^{-2} + \cdots + \chi_C \circ u^{-R})$
$\le \liminf \int_{G/\Gamma} \frac{1}{R}(\chi_C + \chi_C \circ u^{-1} + \chi_C \circ u^{-2} + \cdots + \chi_C \circ u^{-R})$
(“Fatou’s Lemma”)
$= \liminf \frac{1}{R}\left(\int \chi_C + \int \chi_C \circ u^{-1} + \int \chi_C \circ u^{-R}\right)$
$= \liminf \frac{1}{R}\left(\int \chi_C + \int \chi_C + \int \chi_C + \cdots + \int \chi_C\right)$
$= \int \chi_C$
$< \infty$.

And $\varphi > 0$ on a set of positive measure, so $\varphi \ne 0$ in L^1. □

Combine with decay of matrix coefficients:

Theorem (Howe-Moore)

$\varphi, \psi \in L^2(G/\Gamma) \ominus$ constants $\Longrightarrow \; \lim\limits_{g\to\infty} \langle \varphi, \psi \circ g\rangle = 0$.

In fact, valid for vectors in any unitary representation of G.

Corollary (Mautner Phenomenon)

- *H noncompact, closed subgroup of G,*
- *$\varphi \in L^p(G/\Gamma)$, H-invariant*

$\Longrightarrow$ *φ is constant.*

Proof.

Wolog assume $p = 2$ and $\varphi \perp$ constants.
$0 = \lim\limits_{g\to\infty} \langle \varphi, \varphi \circ g\rangle = \lim\limits_{h\to\infty} \langle \varphi, \varphi \circ h\rangle = \lim\limits_{h\to\infty} \langle \varphi, \varphi\rangle = \|\varphi\|_2^2$
$\Longrightarrow \; \varphi = 0$ is constant. □

Corollary

$\mathrm{SL}(n,\mathbb{Z})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$.

Proof.

We know $\exists$ u-invariant $\varphi \in L^1(\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})) \smallsetminus \{0\}$.
Howe-Moore: φ is constant.
So $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$ has finite measure. □

Proof generalizes easily to general $G_{\mathbb{Z}}$.
Just need to know $G/G_{\mathbb{Z}}$ is closed in $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{Z})$.

Chevalley’s Theorem: G defined over $\mathbb{Q}$
$\Longrightarrow \; \exists\, \varphi \colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(N,\mathbb{R})$ with

- $G = \mathrm{Stab}(\text{vector in } \mathbb{Q}^N)$, and
- $\varphi(\mathrm{SL}(n,\mathbb{Z})) \mathrel{\dot\subset} \mathrm{SL}(N,\mathbb{Z})$.

Proof of Howe-Moore Theorem.

Let $\{g_j\} \subset G$, with $\|g_j\| \to \infty$.

For simplicity, assume $G = \mathrm{SL}(2,\mathbb{R})$ and $\{g_j\} \subset \begin{bmatrix} * & \\ & * \end{bmatrix} = A$.

Passing to subseq, wma g_j converges weakly, to some E:

$\langle g_j\,\varphi \mid \psi\rangle \to \langle E\varphi \mid \psi\rangle \quad \forall \varphi,\psi \in L^2$.

Also assume $g_j u g_j^{-1} \to e$ for $u \in N = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$. So

$$\langle Eu\varphi \mid \psi\rangle = \lim\langle g_j u\varphi \mid \psi\rangle = \lim\langle (g_j u g_j^{-1})\,g_j\varphi \mid \psi\rangle$$
$$= \lim\langle g_j\varphi \mid \psi\rangle = \langle\, E\varphi \mid \psi\rangle,$$

so $Eu = E$; i.e., $E(u - \mathrm{Id}) = 0$.

$\nexists$ u-inv't vectors in $L^2 \implies \mathrm{img}(u - \mathrm{Id})$ dense, so $E = 0$. □

Actual proof:

- E annihilates $\mathrm{img}(u - \mathrm{Id})$.
- E^* annihilates $\mathrm{img}(v - \mathrm{Id})$ for $v \in N^- = \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$.
- $E^*E = E\,E^*$, so $\ker E = \ker E^*$.
- $\mathrm{img}(u - \mathrm{Id}) + \mathrm{img}(v - \mathrm{Id})$ dense in $L^2 \ominus$ constants.

The traditional proof

Example

$\mathrm{SL}(n,\mathbb{Z})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$.

Proof uses *Iwasawa decomposition*: $G = KAN$, where

- $K = \mathrm{SO}(n)$,
- A = diagonal matrices (with positive entries, so A is connected),
- N = upper triangular, with 1s on the diagonal.

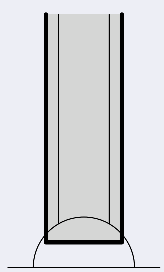
(To show $g = k\,a\,n$, use Gram-Schmidt orthogonalization.)

Example

For $n = 2$, let

- $A_c = \left\{ \begin{bmatrix} t & \\ & 1/t \end{bmatrix} \in A \;\middle|\; 0 < t \le c \right\}$
- $N_c = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in N \;\middle|\; |t| \le c \right\}$
- ("*Siegel set*") $\mathfrak{S}_c = KA_cN_c$

The picture of $\mathfrak{S}_c$ in $\mathfrak{h}^2$ is:



Proof that $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a lattice:

1. $\mathfrak{S}_c\Gamma = \mathrm{SL}(2,\mathbb{R}) \quad (\exists c)$
2. $\mathfrak{S}_c$ has finite measure $\quad (\forall c)$

Generalize to $\mathrm{SL}(n,\mathbb{R})$.

Write $g = k\,a\,n$, with $\quad a = a(g) = \begin{bmatrix} a_{11}(g) & & & \\ & a_{22}(g) & & \\ & & \ddots & \\ & & & a_{nn}(g) \end{bmatrix}$.

Note that $a_{11}(g) = \|ge_1\|$.

Key Lemma

$a_{11}(g) \le a_{11}(g\gamma),\ \forall \gamma \in \mathrm{SL}(n,\mathbb{Z}) \implies a_{11}(g) < 2\,a_{22}(g)$.

Corollary

For $n = 2$: $\quad \mathfrak{S}_c\Gamma = \mathrm{SL}(2,\mathbb{R})$ if $c \ge \sqrt{2}$.

Proof.

Given $g \in \mathrm{SL}(2,\mathbb{R})$. Assume wolog that g minimizes a_{11} on $g\Gamma$.

So $a_{11} < 2a_{22}$.

Then $2 = 2a_{11}\,a_{22} > a_{11}^2$, so $a_{11} < \sqrt{2}$.

Multiplying g by an element of $N_{\mathbb{Z}}$, assume $|n(g)| \le 1/2$.

So $g \in KA_{\sqrt{2}}N_{1/2} \subset \mathfrak{S}_c$. □

For general n, let

- $A_c = \{\, a \in A \mid a_{ii} < c\,a_{i+1\,i+1}, \quad \forall i \,\}$
- $N_c = \{\, n \in N \mid \|n - \mathrm{Id}\| < c \,\}$
- ("*Siegel set*") $\mathfrak{S}_c = KA_cN_c$

Corollary (of Key Lemma)

$\mathrm{SL}(n,\mathbb{R}) = \mathfrak{S}_c\ \mathrm{SL}(n,\mathbb{Z})$ *for* $c \ge 2$.

Proof is by induction.

Exercise

$\mathfrak{S}_c$ has finite volume (for all c)

Key Lemma

$a_{11}(g) \le a_{11}(g\gamma),\ \forall \gamma \in \mathrm{SL}(n,\mathbb{Z}) \implies a_{11}(g) < 2\,a_{22}(g)$.

Proof.

$$a_{11} = \|ge_1\| \le \|ge_2\| = \|\,(kan)e_2\| = \|\,ane_2\| = \|\,a(e_2 + te_1)\|$$
$$= \|a_{22}e_2 + ta_{11}e_1\| = \sqrt{a_{22}^2 + t^2a_{11}^2} \le \sqrt{a_{22}^2 + \tfrac{1}{4}a_{11}^2}$$ □

This lemma is easy, but stating and proving an analogous result for general G requires substantial algebra (the theory of algebraic groups).

The big advantage of this proof is that it provides a coarse (or approximate) fundamental domain that can be used to study the topology and geometry of G/Γ.

References

Short proof: pp. 57–60 of
Alan Reid, *Hyperbolic Manifolds* (unpublished lecture notes).
http://www.its.caltech.edu/~dmcreyn/HyperbolicManifolds.ps

This must also be in
Colin Maclachlan and Alan Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, Springer, New York, 2003. ISBN: 0-387-98386-4 MR1937957 (2004i:57021)

Dynamical proof: forthcoming chapter of
my book-in-progress on *Introduction to Arithmetic Groups*
http://people.uleth.ca/~dave.morris/books/IntroArithGroups.html

Traditional proof: pp. 13–17 of
Armand Borel, *Introduction aux Groupes Arithmétiques,* Hermann, Paris 1969. MR0244260 (39 #5577)