# Ergodic actions of semisimple Lie groups on compact principal bundles

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## Abstract

Let $G = \mathrm{SL}(n, \mathbb{R})$. For any compact Lie group $K$, it is easy to find a manifold $M$, such that there is a volume-preserving, ergodic action of $G$ on some smooth, principal $K$-bundle $P$ over $M$. Can $M$ be chosen independent of $K$? We show that if $M = H/\Lambda$ is a homogeneous space, the action of $G$ on $M$ is by translations, and there is a $G$-invariant connection on $P$, then $P$ must also be a homogeneous space $H'/\Lambda'$. Consequently, there is a strong restriction on the groups $K$ that can arise over this particular $M$. This is joint work (in progress) with Robert J. Zimmer.

$G = \mathrm{SL}(3,\mathbb{R})$
$\quad = \{\, g \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid \det g = 1 \,\}$
$\quad =$ conn, noncpct, linear, simple Lie group

$\Gamma = \mathrm{SL}(3,\mathbb{Z})$
$\quad =$ lattice in $G$ (such that $G/\Gamma$ is not compact)

**Thm** (Margulis Superrigidity Thm).
$\sigma\colon \Gamma \to \mathrm{GL}(\ell,\mathbb{R})$ *homomorphism*
$\Rightarrow$ $\sigma$ *extends to a representation of* $G$.
*I.e.,* $\exists\ \hat\sigma\colon G \to \mathrm{GL}(\ell,\mathbb{R})$, $\hat\sigma|_\Gamma = \sigma$
(*modulo a finite group*).

*Defn.* $P$ = *flat principal* $K$*-bundle* over $G/\Gamma$:
- $K$ connected, linear Lie group
- $\sigma\colon \Gamma \to K$
- $P = (G \times K)/\!\sim$

$$(g\gamma, k) \sim \big(g, \sigma(\gamma)k\big)$$

$$K \to P \xrightarrow{\pi} G/\Gamma$$

**Cor.** *Every flat principal $K$-bundle over $G/\Gamma$ is trivial* $\quad (\cong (G/\Gamma) \times K)$.

*Proof.* Any princ bundle with a section is trivial.

$$\begin{array}{c} P \\ \big\downarrow \pi \\ G/\Gamma \end{array} \qquad\qquad \pi \circ \bar{\sigma} = \mathrm{Id}$$

$$P = (G \times K)/\sim \qquad (g\gamma, k) \sim \big(g, \sigma(\gamma)k\big)$$

Define $\bar{\sigma}\colon G/\Gamma \to P$ $\quad$ by $\bar{\sigma}(g) = \big(g, \hat{\sigma}(g)^{-1}\big)$.

$\bar{\sigma}$ is well defined:

$$\begin{aligned} \bar{\sigma}(g\gamma) &= \big(g\gamma, \hat{\sigma}(g\gamma)^{-1}\big) \\ &= \big(g\gamma, \sigma(\gamma)^{-1}\hat{\sigma}(g)^{-1}\big) \\ &\sim \big(g, \sigma(\gamma)\sigma(\gamma)^{-1}\hat{\sigma}(g)^{-1}\big) \\ &= \bar{\sigma}(g). \end{aligned}$$

$\bar{\sigma}$ is a section of $P$:

$$\begin{aligned} \pi\big(\bar{\sigma}(x)\big) &= \pi\big(x, \hat{\sigma}(x)^{-1}\big) \\ &= x. \end{aligned}$$

*Defn.* $K \to P \xrightarrow{\pi} M$ principal $K$-bundle

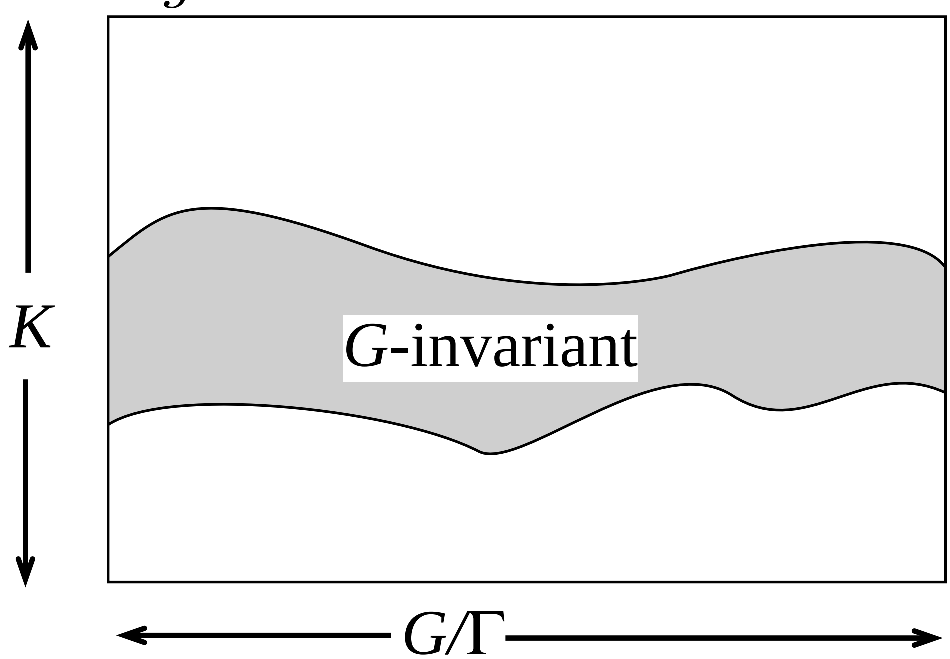
$G$ *acts* on $P$ (by bundle autos): $\quad g(kp) = k(gp)$.

$$\begin{array}{ccc} P & \xrightarrow{g} & P \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{g} & M \end{array} \qquad \pi(gp) = g\,\pi(p)$$

We usually assume the $G$-action on $M$ is given.

**Cor.** *$G$ acts on principal $K$-bundle $P$ over $G/\Gamma$ (and $K$ nontrivial)*

*$\Rightarrow G$ is not ergodic on $P$.*




*I.e.,* $P = P_1 \cup P_2$

$P_i$ *is $G$-invariant, meas'ble, nonzero measure*

**Thm** (Zimmer Cocycle Superrigidity Thm).

- *$G$ acts on principal $K$-bundle $P$ over $M$*
- *$M$ compact manifold (or finite volume)*
- *$G$ volume-preserving on $M$*
- *$K$ not compact*

$\Rightarrow$ *$G$ is not ergodic on $P$.*

**Problem.** *What if $K$ is compact?*

**Thm** (Margulis). *As in Zimmer, but:*

- *$G$ transitive on $M$* $(= G/\Gamma')$
- *$K$ compact*

$\Rightarrow$ *$K$ belongs to a finite list of groups*
*(depending only on the $G$-action on $M$).*

**Question.** *¿Is this true without transitivity?*

*Rem.* **No** if $M$ is a measure space, not a manifold:

- $K_1, K_2, \dots$
- $M = M_1 \times M_2 \times \cdots$
- $P_i' = P_i \times M_1 \times M_2 \times \cdots \hat{M}_i \cdots$

$P_i'$ is a principal $K_i$-bundle over $M$.

In fact: $\exists M, \quad \forall$ compact Lie $K$,

$G$ is ergodic on some princ $K$-bundle $P$ over $M$.

*Eg.* • $G \hookrightarrow H' = \begin{pmatrix} 1 & \mathbb{R}^n & \mathbb{T} \\ 0 & \mathrm{SL}(n,\mathbb{R}) & \mathbb{R}^n \\ 0 & 0 & 1 \end{pmatrix}$

• $\Lambda' = \begin{pmatrix} 1 & \mathbb{Z}^n & 0 \\ 0 & \mathrm{SL}(n,\mathbb{Z}) & \mathbb{Z}^n \\ 0 & 0 & 1 \end{pmatrix}$ lattice in $H'$

• $K = \begin{pmatrix} 1 & 0 & \mathbb{T} \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ cpct norm subgrp of $H'$

• $H = H'/K \cong \mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^{2n}$

• $\Lambda = \mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$ lattice in $H$

$P = H'/\Lambda'$ is principal $K$-bundle over $M = H/\Lambda$.

Moore Ergodicity Thm $\Rightarrow$ $G$ is ergodic on $P$.

**Thm.** *$M = H/\Lambda$, $G$ ergodic on $P$*

- *preserving a connection*

$\Rightarrow P = H'/\Lambda'$

*with $H \cong H'/K$,*

*so $M \cong K\backslash H'/\Lambda'$.*

*Eg.* $M = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}) \Rightarrow$

- $H = \mathrm{SL}(n,\mathbb{R})$
- $\Lambda = \mathrm{SL}(n,\mathbb{Z})$
- $H' = H \times K$
- $\Lambda' = \{\, \big(\lambda, \sigma(\lambda)\big) \mid \lambda \in \Lambda \,\}$
- $\sigma(\Lambda)$ dense in $K$

($\Leftrightarrow G$ ergodic on $P$)

$\Rightarrow$ only finitely many choices for $K$ [Margulis].

*Instead of* $H/\Lambda$*, consider* $M = C\backslash H/\Lambda$*.*

*Eg.* • $G = \begin{pmatrix} G & 0 \\ 0 & \mathrm{Id}_{n\times n} \end{pmatrix} \hookrightarrow \mathrm{SL}(n+3,\mathbb{R}) = H$

• $\Lambda =$ lattice in $H$

• $P = H/\Lambda$

• $K = \begin{pmatrix} \mathrm{Id}_{3\times 3} & 0 \\ 0 & \mathrm{SO}(n) \end{pmatrix} \hookrightarrow H$

(compact subgroup that centralizes $G$)

• $M = K\backslash H/\Lambda$

$P$ is a principal $K$-bundle over $M$.

$G$-action on $P$ factors thru to $M$:

$$g\,(Kh\Lambda) = K(gh)\Lambda$$

(because $G$ centralizes $K$).

*Rem.* Similar eg. for *any* compact Lie group $K$.

**Cor.** *$M = C\backslash H/\Lambda$, $G$ ergodic on $P$*

- *preserving a connection*

$\Rightarrow P = C'\backslash H'/\Lambda'$

*with $M \cong K'\backslash H'/\Lambda'$*

*and $K \cong K'/C'$.*

Possibilities for $K$ determined by group theory.

**Thm.** *$M = H/\Lambda$, $G$ ergodic on $P$*

- *preserving a connection*

$\Rightarrow P = H'/\Lambda'$

*with $H \cong H'/K$*

*so $M \cong K\backslash H'/\Lambda'$.*

*Proof.* $\mathfrak{h}$ = Lie algebra of vector fields on $M$ associated to the $H$-action on $M$.

$\overline{\mathfrak{h}}$ = lift of $\mathfrak{h}$ to horizontal vector fields on $P$.

**Assume the connection is flat**,

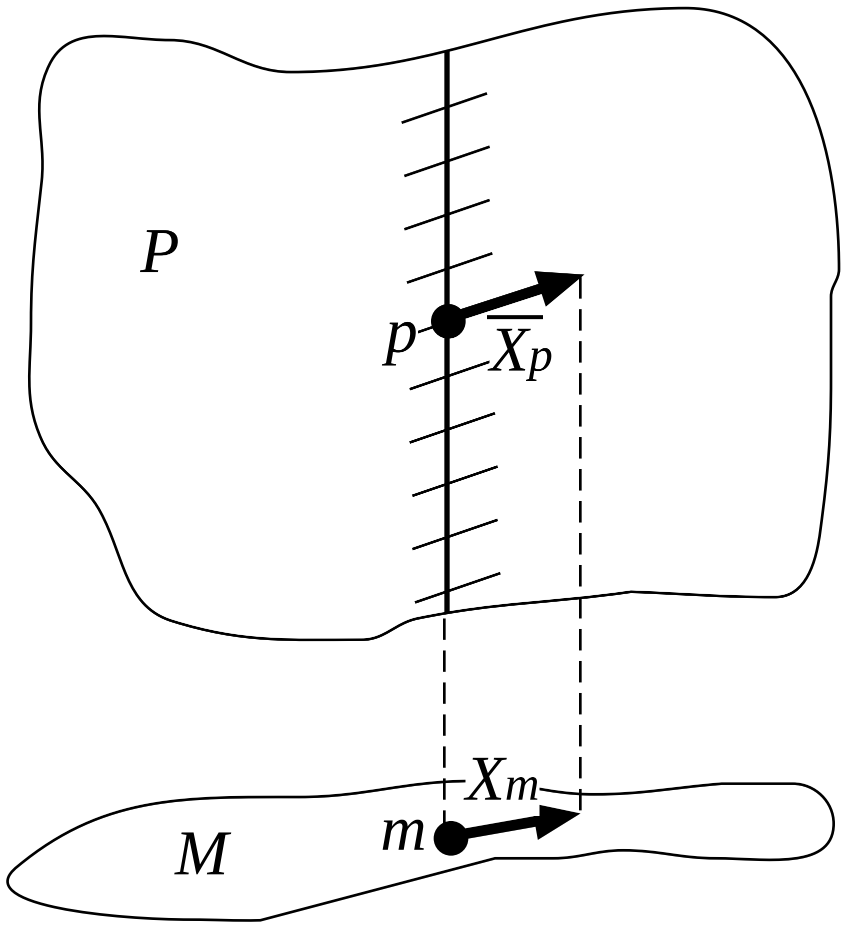
i.e., $X, Y$ horizontal $\Rightarrow [X, Y]$ horizontal.

Then $\overline{\mathfrak{h}}$ is a Lie algebra (bcs $[\overline{X}, \overline{Y}] = \overline{[X, Y]}$).

- $\overline{H}$ = corresponding Lie group of diffeos of $P$
- $H' = \overline{H}\, K$.

Since $\overline{H}$ centralizes $K$, we know $H'$ is a group and $K$ is a normal subgroup.

$H'$ is transitive on $P$, so $P = H'/\Lambda'$.


P
p
$\overline{X_p}$
$X_m$
m
M

*All that remains:*

For $G$-equivariant diffeos, wish to show $G \subset H'$.

Suff to show $\mathfrak{g}$ horizontal. (Then $\mathfrak{g} \subset \overline{\mathfrak{h}}$.)

Fix $p \in P$.

Define $\omega_p \colon \mathfrak{g} \to \mathfrak{k}$ by

$\omega_p(X) =$ vertical component of $X_p$.

($\omega$ is the *connection form*.)

Flat $\Rightarrow$ $\omega_p$ is a Lie algebra homomorphism:

$$\omega_p\big([X,Y]\big) = \big[\omega_p(X), \omega_p(Y)\big] - 2\Omega_p(X_p, Y_p).$$

$G$ noncompact, $K$ compact

$\Rightarrow$ $\nexists$ nontrivial homomorphism $G \to K$

$\Rightarrow$ $\omega_p(\mathfrak{g}) = 0$

$\Rightarrow$ $\mathfrak{g}$ is horizontal.

*What if the connection is not flat?*

$\Omega$ = curvature form of the connection:

- $\hat{\Omega}\colon P \to \operatorname{Hom}_{\mathbb{R}}(\mathfrak{h} \wedge \mathfrak{h}, \mathfrak{k})$,
- $\hat{\Omega}_{gp}\big((\operatorname{Ad}_H g)X, (\operatorname{Ad}_H g)Y\big) = \hat{\Omega}_p(X, Y)$.

$\hat{\Omega}$ is constant, because:

- $\hat{\Omega}$ is $G$-equivariant,
- $G$ is ergodic on $P$,
- $G$-action on $\operatorname{Hom}_{\mathbb{R}}(\mathfrak{h} \wedge \mathfrak{h}, \mathfrak{k})$ is algebraic,
- Borel Density Theorem.

$X, Y \in \mathfrak{h}$

$\Rightarrow \omega_p\big([\overline{X}, \overline{Y}]\big) = -2\hat{\Omega}_p(X, Y)$
is constant.

So $[\overline{\mathfrak{h}}, \overline{\mathfrak{h}}] \subset \overline{\mathfrak{h}} + \mathfrak{k}$.

$\overline{\mathfrak{h}} + \mathfrak{k}$ is a Lie algebra:

$H'$ = corresponding Lie group of diffeos of $P$.

*All that remains:* show $\mathfrak{g}$ horizontal.

Suff to show $\Omega_p(X_p, Y_p) = 0$ for all $X, Y \in \mathfrak{g}$.

(Then $\omega_p$ is a homomorphism on $\mathfrak{g}$.)

$D\Omega = 0$

$$\Rightarrow \sum_{\substack{\text{cyclic}\\\text{perms}}} \hat{\Omega}\big([X,Y],Z\big) = 0 \quad \forall X, Y, Z \in \mathfrak{h}$$

$\Rightarrow \hat{\Omega}$ is Lie algebra 2-cocycle.

Whitehead's Lemma: $H^2(\mathfrak{g}; V) = 0$.

$\exists \sigma \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}; \mathfrak{k})$, s.t. $\forall X, Y \in \mathfrak{g}$,

$$\sigma\big([X,Y]\big) = \hat{\Omega}(X,Y)$$
$$= \sigma\big((\mathrm{Ad}_H g)[X,Y]\big).$$

Therefore $0 = \sigma\big([G, \mathfrak{g}]\big) = \sigma(\mathfrak{g})$.

So $\hat{\Omega}(\mathfrak{g}, \mathfrak{g}) = 0$. $\square$