Actions of semisimple Lie groups on circle bundles

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(joint work with Robert J. Zimmer)

$\Gamma = \mathrm{SL}(3,\mathbb{Z})$

$\quad = \{\, T \in \mathrm{Mat}_3(\mathbb{Z}) \mid \det T = 1 \,\}$

$G = \mathrm{SL}(3,\mathbb{R})$

$\quad = \{\, T \in \mathrm{Mat}_3(\mathbb{R}) \mid \det T = 1 \,\}$

Γ is a lattice in G

- Γ discrete
- G/Γ compact (or finite volume)

Ghys: every cont action of Γ on $\mathbb{T}$ has inv meas:

Thm (Ghys). $\phi\colon \Gamma \to \mathrm{Homeo}(\mathbb{T})$

$\Rightarrow \exists\ \Gamma$-*invariant prob measure on* $\mathbb{T}$.

Cor. $\phi\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}) \Rightarrow \exists$ *finite orbit.*

Proof. Assume $\mathrm{supp}(\mu) = \mathbb{T}$.
so we have a Γ-inv metric on $\mathbb{T}$:

$$d(x,y) = \mu\big([x,y]\big).$$

Then $\phi(\Gamma) \subset \mathrm{Isom}^+(\mathbb{T}) \cong \mathbb{T}$ abelian.

Because $\Gamma/[\Gamma,\Gamma]$ is finite, conclude $\phi(\Gamma)$ is finite.
So every orbit (in $\mathrm{supp}(\mu)$) is finite. $\square$

Cor (Ghys). $\phi\colon \Gamma \to \mathrm{Homeo}(\mathbb{T}) \Rightarrow \exists$ *finite orbit.*

Thm (Thurston). $\Gamma \subset \mathrm{Diff}^1_+(\mathbb{T})$

- Γ *has a fixed point*
- Γ *finitely generated*
- $\Gamma/[\Gamma,\Gamma]$ *finite*

$\Rightarrow \Gamma = e.$

Cor (Ghys). $\phi\colon \Gamma \to \mathrm{Diff}^1(\mathbb{T})$

$\Rightarrow \phi(\Gamma)$ *is finite.*

(Also proved by Burger & Monod if $H^2(\Gamma;\mathbb{R}) = 0$.)

Conj. *Same conclusion for* $\Gamma \to \mathrm{Homeo}(\mathbb{T})$.

Thm (Witte). *True if* $\Gamma \supset \mathrm{SL}(3,\mathbb{Z})$ *or* $\mathrm{Sp}(4,\mathbb{Z})$.

Thm (Ghys). $\phi\colon \Gamma \to \mathrm{Homeo}(\mathbb{T})$
$\Rightarrow \exists$ Γ*-invariant probability measure on* $\mathbb{T}$.

Thm (Ghys). *Let* E *be a circle bundle over* G/Γ. *If action of* G *on* G/Γ *lifts to cont action on* E, *then* $\exists$ G*-invariant probability measure on* E.

Thm (Witte-Zimmer). E = *circle bundle over* M

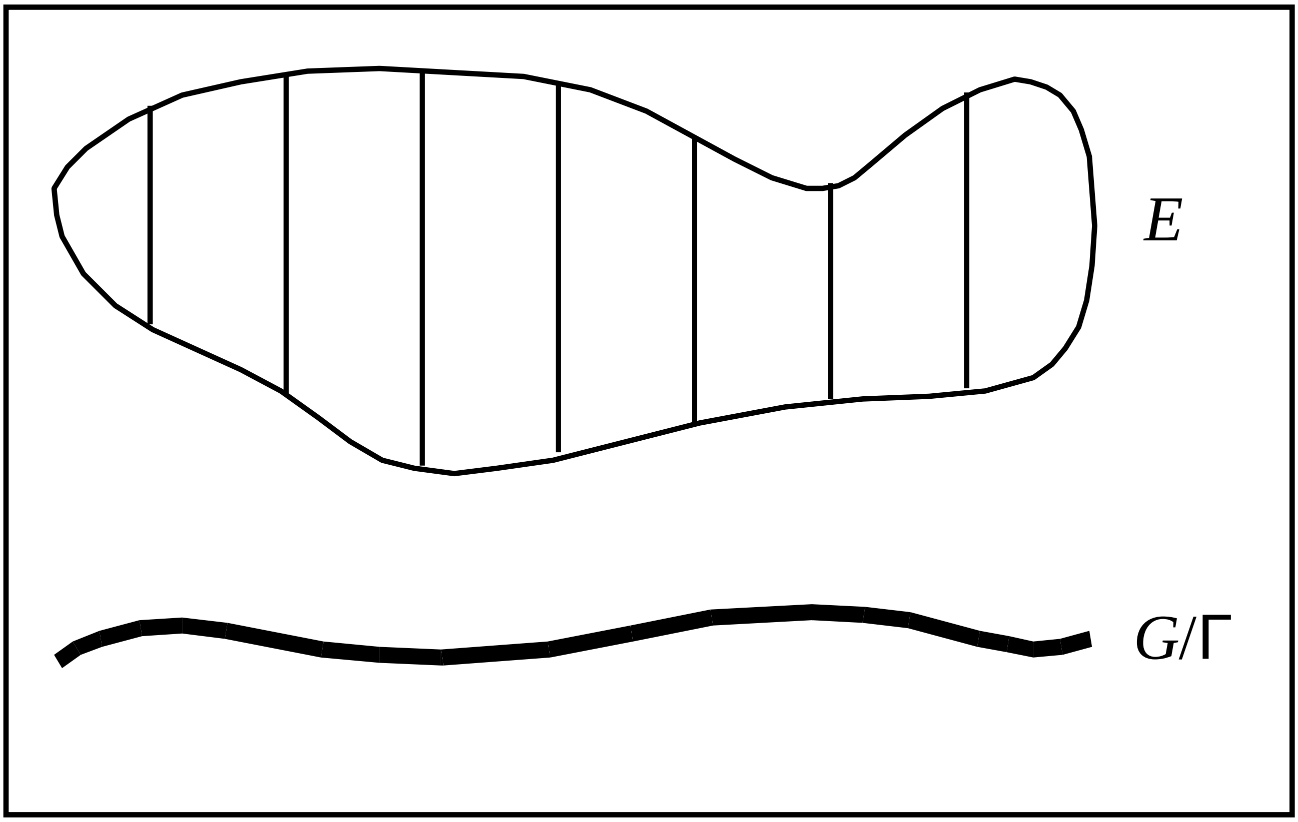
- $\phi\colon G \to \mathrm{Homeo}(E \to M)$
- $\exists$ G*-invariant prob meas* μ *on* M

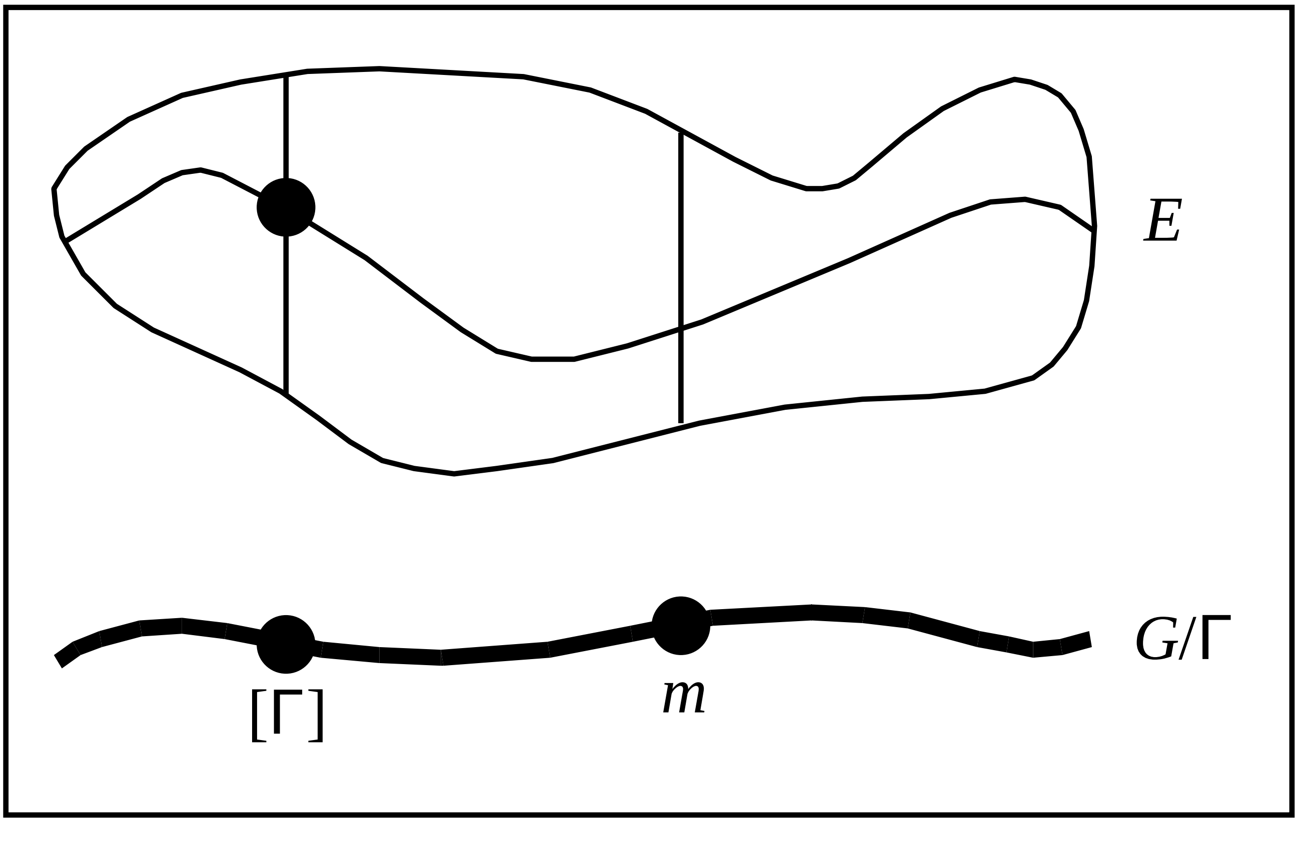
$\Rightarrow \exists$ G*-invariant probability measure* ν *on* E *such that* ν *projects to* μ *on* M.

Ghys' proof generalizes in a natural way.
We give a more unified proof based on Ghys' ideas (plus a bit more Lie theory).

Rem. Only assume actions on M and E are measurable, but continuous in the fiber direction.

Rem. The analogue of a fixed point of a Γ-action is a choice of a point in each fiber of E.
I.e., G-invariant (measurable) section $M \to E$.
May not exist!


E
G/Γ


E
G/Γ
[Γ]
m

Eg. • $G \times \mathbb{T} \subset \mathrm{SL}(n, \mathbb{R}) = H$
• $\Lambda = \mathrm{SL}(n, \mathbb{Z})$ (lattice in H)
• $E = H/\Lambda$
• $M = \mathbb{T} \backslash H/\Lambda = \mathbb{T} \backslash E$

So E is a principal $\mathbb{T}$-bundle over M.

$G \subset \mathrm{Diff}^\infty(E \to M)$
(because G commutes with $\mathbb{T}$)

Suppose $\exists$ G-equivariant $\sigma \colon M \to E$;
Then $t\sigma$ is G-equivariant, for $t \in \mathbb{T}$.
So $E \cong \mathbb{T} \times M$ (trivial G-action on $\mathbb{T}$)

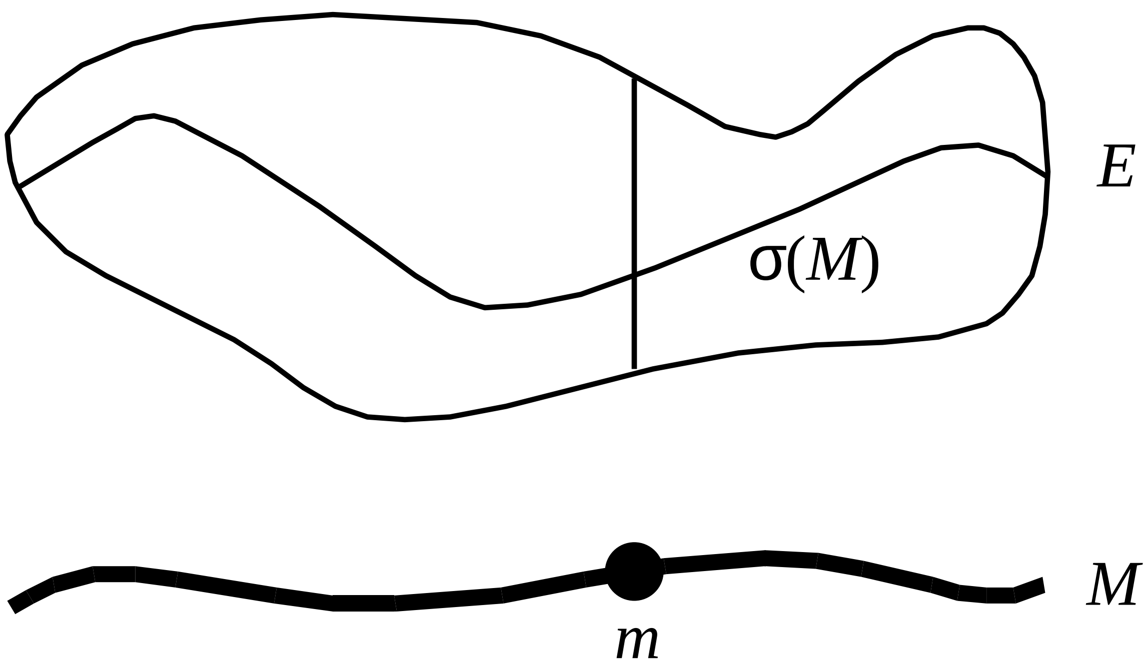
So G is not ergodic on E.

$\to\leftarrow$

Generalization of the Mautner phenomenon (geodesic flow on surf of const neg curv is ergodic):

Thm (Moore Ergodicity Theorem).
A = noncompact, closed subgroup of H
$\Rightarrow$ A is ergodic on H/Λ

$$\begin{pmatrix} \boxed{G} & & \\ & \boxed{\mathrm{SO}(2)} & \\ & & \boxed{\mathrm{Id}} \end{pmatrix}$$



Thm (Witte-Zimmer). *E = circle bundle over M*

- $\phi: G \to \mathrm{Diff}^1(E \to M)$
- *$\exists$ G-invariant prob meas μ on M*

$\Rightarrow$ *$\exists$ G-invariant prob meas ν on E*
such that ν projects to μ on M
and ν is equivalent to Lebesgure measure.

Cor.

- *M = ergodic G-space with inv prob meas*
- $\alpha: G \times M \to \mathrm{Diff}^1(\mathbb{T})$ *Borel cocycle*

$$(I.e., \alpha(gh, x) = \alpha(g, hx)\alpha(h, x))$$

$\Rightarrow$ $\alpha \sim$ *cocycle with values in* $\mathrm{Isom}(\mathbb{T})$
(*as a cocycle into* $\mathrm{Homeo}(\mathbb{T})$).

Thm (Thurston). $\Gamma \subset \mathrm{Diff}^1_+(\mathbb{T})$

- Γ *has a fixed point*
- Γ *finitely generated*
- $\Gamma/[\Gamma,\Gamma]$ *finite*

$\Rightarrow \Gamma = e$.

Γ has a fixed point on $\mathbb{T} \Rightarrow \Gamma$ acts on $I = [0,1]$.

Analogue:

Thm (Witte-Zimmer). $G \subset \mathrm{Diff}^1_+(M \times I \to M)$

- $\exists$ *G-inv prob meas μ on M*

$\Rightarrow \exists$ *G-inv prob meas ν on $M \times I$,*
such that ν is equiv to $\mu \times$ Leb.

Zimmer has a nice proof of Thurston's Theorem, using the fact that Γ has Kazhdan's property (T):

- $\langle \gamma_1, \ldots, \gamma_r \rangle = \Gamma$
- $\rho\colon \Gamma \to U(\mathcal{H})$ unitary representation
- $\forall \epsilon > 0$, $\exists v \in \mathcal{H}$, $\forall i$, $\|\rho(\gamma_i)v - v\| < \epsilon \|v\|$

$\Rightarrow \exists w \in \mathcal{H} \setminus \{0\}$, $\forall \gamma \in \Gamma$, $\rho(\gamma)w = w$.

This proof generalizes to our setting.

Thm. $\Gamma \subset \mathrm{Diff}^1_+(I)$, *Kazhdan* $\Rightarrow \Gamma = e$.

Proof. Suffices: the fixed pts are dense in I.
Spse not: assume no fixed pts in interior of I.

Unitary representation ρ of Γ on $L^2(I)$:

$$\big(\rho(\gamma)f\big)(t) = [\gamma'(t)]^{1/2} f(\gamma^{-1}t).$$

$\Gamma/[\Gamma,\Gamma]$ finite

$\Rightarrow$ any homomorphism $\Gamma \to \mathbb{R}$ is trivial

$\Rightarrow \gamma'(0) = 1, \ \forall \gamma \in \Gamma$.

Choose a small interval J containing 0.
Let χ = characteristic function of J.

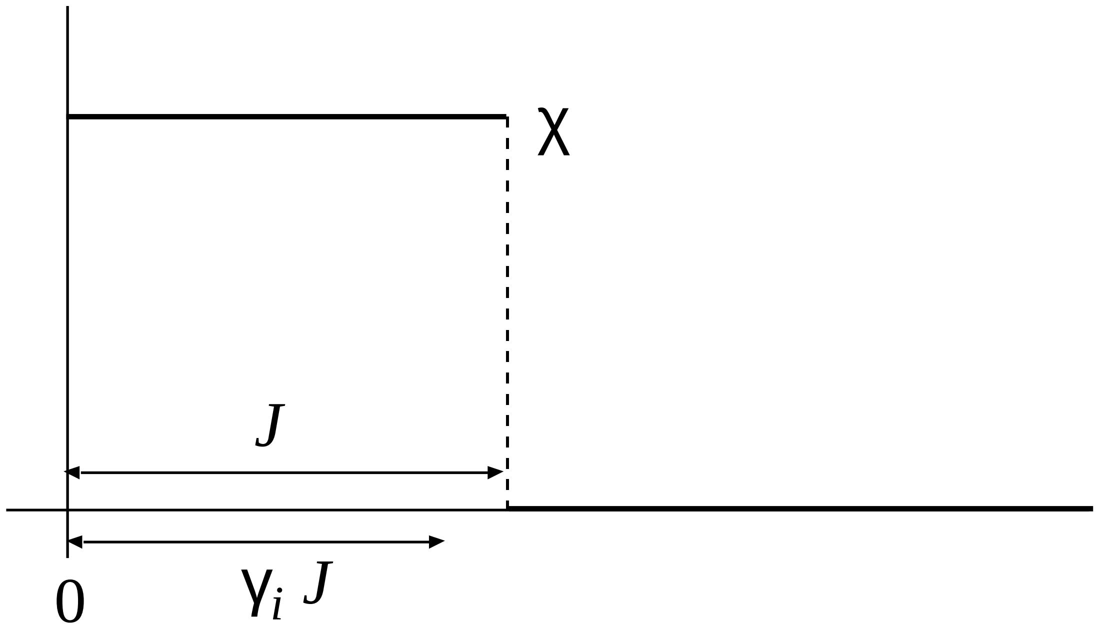
Action is $C^1 \Rightarrow \gamma_i'(t) \approx 1$ for $t \in J$ and $1 \leq i \leq r$

$\Rightarrow \rho(\gamma_i)\chi \approx \chi$

$\Rightarrow \exists$ $\rho(\Gamma)$-invariant $\phi \in L^2(I)$ [Kazhdan]

Then $|\phi|^2 dt$ is a Γ-invariant measure on I,
so every point in its support is fixed by Γ.

$\rightarrow\leftarrow$ no fixed pts in interior of I.


χ
J
0
γ_i J

Thm (Ghys). $\phi\colon \Gamma \to \mathrm{Homeo}(\mathbb{T})$
$\Rightarrow \exists\ \Gamma$-*invariant prob measure on* $\mathbb{T}$.

Proof. Let

- $\mathrm{Prob}(\mathbb{T}) = \{\text{prob meas on } \mathbb{T}\}$
- $P = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\}$

Fact. $\exists\ \Gamma$-equi (meas'ble) $\Psi\colon G/P \to \mathrm{Prob}(\mathbb{T})$.
[Furstenberg boundary theory]

I.e., $\Psi(\gamma g p) = \gamma \Psi(g)$, for $\gamma \in \Gamma$, $g \in G$, $p \in P$.

We wish to show Ψ is essentially constant.
(Then $\Psi(G/P)$ must be a fixed point:
i.e., an invariant measure.)

Case 1. Assume $\Psi(x)$ *has no atoms, for a.e.* x.

Case 2. Assume $\Psi(x) \in \mathbb{T}$, *for a.e.* x.

Case 1. Assume $\Psi(x)$ *has no atoms, for a.e. x.*

- $\mathrm{Prob}_0(\mathbb{T}) = \{\text{atomless prob meas on } \mathbb{T}\}$
- metric $D(\mu_1, \mu_2) = \sup_J |\mu_1(J) - \mu_2(J)|$
- $\Psi^2 \colon (G/P)^2 \to \mathrm{Prob}_0(\mathbb{T})^2$
- $A = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$

Moore Ergodicity Theorem

$\Rightarrow A$ is ergodic on G/Γ

$\Rightarrow$ no nonconstant A-inv funcs on G/Γ

$\Rightarrow$ no nonconstant (A, Γ)-inv funcs on G

$\Rightarrow$ no nonconstant Γ-inv funcs on G/A

$\Rightarrow$ no nonconstant Γ-inv funcs on $(G/P)^2$.

D is $\mathrm{Homeo}(\mathbb{T})$-invariant

$\Rightarrow D \circ \Psi^2$ is Γ-invariant

$\Rightarrow D \circ \Psi^2$ is constant.

$(D \circ \Psi^2)(x, x) = 0$

$\Rightarrow D \circ \Psi^2 \equiv 0$

$\Rightarrow \Psi$ is constant.

Case 2. Assume $\Psi(x) \in \mathbb{T}$, *for a.e.* x.

$$L = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1/(ad-bc) \end{pmatrix},$$

$$B = L \cap P = \begin{pmatrix} a & 0 & 0 \\ c & d & 0 \\ 0 & 0 & 1/(ad) \end{pmatrix}$$

Recall. $\mathrm{GL}(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \det g \neq 0 \right\}$

acts on $\mathbb{R} \cup \{\infty\} \sim \mathbb{T}$ by $gx = \dfrac{ax+b}{cx+d}$.

- This action is triply transitive.
- $\mathrm{Stab}_{\mathrm{GL}(2,\mathbb{R})}(0) = \{b = 0\} \approx B$.

So L is triply transitive on L/B.

Ghys: contrast triple transitivity of L with the fact that $\mathrm{Homeo}^+(\mathbb{T})$ is not triply transitive, to show that $\Psi(g) = \Psi(gL)$.

The L's generate $G \Rightarrow \Psi$ is constant.

Define:

- $X = \left\{ \begin{array}{c} (x_1, x_2, x_3) \\ \in (G/B)^3 \end{array} \middle| \begin{array}{c} x_1 L = x_2 L = x_3 L \\ x_i \neq x_j \end{array} \right\}$
- $\Psi^3 \colon X \to \mathbb{T}^3$
- $Z = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a^2 \end{pmatrix}$,

G is transitive on X

and $\operatorname{Stab}_G\big(([l_1],[l_2],[l_3])\big) \supset Z$ is not compact

$\Rightarrow$ no nonconstant Γ-inv funcs on X

$\Psi^3(\Gamma\text{-orbit}) \subset \operatorname{Homeo}^+(\mathbb{T})\text{-orbit}$

$\Rightarrow \overline{\Psi^3} \colon X \to \mathbb{T}^3 / \operatorname{Homeo}^+(\mathbb{T})$ is Γ-invariant

$\Rightarrow \Psi^3(X) \subset$ single $\operatorname{Homeo}^+(\mathbb{T})$ orbit O in $\mathbb{T}^3$

Since Ψ^3 is S_3-equivariant [symmetric group], conclude that O is invariant under action of S_3.

But $\operatorname{Homeo}^+(\mathbb{T})$ preserves the circular order:

conclude that $O = \{(t,t,t)\}$.

So $\Psi(x_1) = \Psi(x_2)$ whenever $x_1 \in x_2 L$.

Thus, $\Psi(x_1) = \Psi(x_1 L)$. $\square$

Lem. *G is transitive on X.*

Proof. Given $x, y \in X$.

G is transitive on G/L,
so we may assume $x_1, y_1 \in L$.
Then, from the defn of X, we conclude $x_i, y_i \in L$.

Because L is triply transitive on L/B,
$\exists l \in L$, $lx = y$. $\square$

Notation. Let $S = \mathrm{Stab}_G\big(([l_1], [l_2], [l_3])\big)$.

Lem. *$S \supset Z$.*

Proof.
$$\begin{aligned} z([l_1], [l_2], [l_3]) &= ([zl_1], [zl_2], [zl_3]) \\ &= ([l_1 z], [l_2 z], [l_3 z]) \\ &= ([l_1], [l_2], [l_3]) \qquad \text{(because } z \in B\text{)} \end{aligned}$$
$\square$

Lem. $\nexists$ *nonconstant* Γ*-inv funcs on* X

Proof. Let $S = \mathrm{Stab}_G\big(([l_1],[l_2],[l_3])\big)$.

Moore Ergodicity Theorem

$\Rightarrow$ S is ergodic on G/Γ

$\Rightarrow$ no nonconstant S-inv funcs on G/Γ

$\Rightarrow$ no nonconstant (S,Γ)-inv funcs on G

$\Rightarrow$ no nonconstant Γ-inv funcs on G/S

$\Rightarrow$ no nonconstant Γ-inv funcs on X $\square$

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