Actions of arithmetic groups on the circle

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1. Some arithmetic groups that cannot act on the circle

2. Bounded generation of $\mathrm{SL}(n,\mathbb{Z})$

3. Bounded generation and actions on the circle

* Will be sloppy about passing to finite-index subgrps.

1. Some arithmetic groups that cannot act on the circle

Abstract. It is known that finite-index subgroups of the arithmetic group $\mathrm{SL}(3,\mathbb{Z})$ have no interesting actions on the circle. This naturally led to the conjecture that most other arithmetic groups (of higher real rank) also cannot act on the circle (except by linear-fractional transformations). Theorems of É. Ghys and A. Navas establish that the conjecture is true if we assume that the maps involved are differentiable. (We would prefer to assume only that they are continuous.) Both of these proofs are elegant, and they each illustrate a fundamental technique in the theory of lattice subgroups: Ghys uses the Furstenberg boundary, and Navas relies on Kazhdan's property T.

Transformation grps. Given grp Γ, cpct mfld M.
What are the actions of Γ on M?
I.e., what are homos $\phi\colon \Gamma \to \mathrm{Homeo}_+(M)$?

Simplest case: Assume $\dim M = 1$. So $M = S^1$.

In my work, Γ is an *arithmetic group*:

$\Gamma = \mathrm{SL}(3,\mathbb{Z}) = \{\, 3\times 3$ integer matrices of det $1\,\}$
(or subgroup of finite index)

Or $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ Or $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\alpha])$
α = real, irrational algebraic integer.

Or $\Gamma = \mathrm{SL}(2,\mathbb{Z}[1/2])$ Or $1/2 \mapsto 1/r$, $r > 1$.

But $\Gamma \neq \mathrm{SL}(2,\mathbb{Z})$, some other "small" grps.
(Assume Γ is irred latt in G, $\mathbb{R}$-rank $G \geq 2$.)
(Our *theorems* assume Γ *not* cocompact)

Eg. $\mathrm{SL}(2,\mathbb{Z})$ ($\approx$ free grp) has *many* actions on S^1.

$\Gamma = \mathrm{SL}(3,\mathbb{Z})$ or $\mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ or $\mathrm{SL}(2,\mathbb{Z}[1/2])$.

Eg. linear-fractional transformations $\dfrac{ax+b}{cx+d}$
provide action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}\cup\{\infty\} \sim S^1$.

So $\mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ acts by linear-fractionals.

Conj. $\phi\colon \Gamma \to \mathrm{Homeo}(S^1)$, *not* $\approx$ *lin-frac*
$\Rightarrow$ Γ^{ϕ} *is finite.*

Rem. Let $\Gamma = \pi_1(\text{hyper})$ (and go to fin-ind subgrp).
Thurston conjecture: $\exists \sigma\colon \Gamma \to \mathbb{Z}$.
Since $\mathbb{Z} \hookrightarrow \mathrm{Homeo}(S^1)$, then Γ acts on S^1.

Rem (Margulis Normal Subgrp Thm).
Γ^{ϕ} infinite $\Rightarrow$ $\ker(\phi)$ finite.
Hence, we assume $\phi\colon \Gamma \hookrightarrow \mathrm{Homeo}(S^1)$.

Evidence for the conjecture.

$\phi\colon\Gamma \to \mathrm{Homeo}(S^1)$ (*not* $\approx$ *lin-frac*) $\Rightarrow$ Γ^ϕ is finite.

Thm (Witte). Γ^ϕ *finite* *if* $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *or* $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$ *or contains either.*

I.e., $\mathbb{Q}$-rank$(\Gamma) \geq 2$.

Thm (Ghys). Γ^ϕ *has a fixed point* (*or a finite orbit*).

(Proof uses *ergodic theory* and *amenability* — or the *Furstenberg boundary*.)

Combine with Reeb-Thurston Stability Thm:

Cor (Ghys). Γ^ϕ *finite* *if* $\phi\colon\Gamma \to \mathrm{Diff}^1(S^1)$.

Thm (Navas). Γ^ϕ *finite* *if* $\phi\colon\Gamma \to \mathrm{Diff}^2(S^1)$.

(Only requires that Γ has *Kazhdan's property T*, not that Γ is arithmetic — or linear.)

Thm (Lifschitz-Morris). Γ^ϕ *finite* *if* $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ *or* $\mathrm{SL}(2,\mathbb{Z}[1/2])$.

(Proof uses bounded generation.)

The theorem of Navas.

$\Gamma \subset \mathrm{Diff}^2(S^1)$, *Kazhdan's property T* $\Rightarrow$ Γ *finite.*

Defn. Γ has *Kazhdan's property T*:

$H^1(\Gamma, \mathcal{H}) = 0$, $\forall$ unitary Γ-module $\mathcal{H}$.

I.e.: $\mathcal{H} = L^2(S^1 \times S^1)$ (square-integrable funcs).

$\mathcal{H}$ is Hilbert space (normed ∞-dim'l vec space)

$$\|F\|^2 = \left| \int_{S^1\times S^1} F(s,t)^2 \, ds\, dt \right|$$

Spse Γ acts on $\mathcal{H}$ by unitaries ($\|F^g\| = \|F\|$).

$\alpha\colon\Gamma \to \mathcal{H}$ is a 1-cocycle

$(\alpha(gh) = \alpha(g)^h + \alpha(h))$

$\Rightarrow$ α is a coboundary

$(\exists v \in \mathcal{H},\ \alpha(g) = v^g - v)$.

Thm (Navas). $\Gamma \subset \mathrm{Diff}^2(S^1)$, *property T* $\Rightarrow$ Γ *finite.*

Proof. For $F \in L^2(S^1 \times S^1)$ and $g \in \Gamma$,

$$F^g(r,s) = F\big(g(r), g(s)\big)\,|g'(r)|^{1/2}\,|g'(s)|^{1/2}.$$

This is unitary rep'n of Γ on $L^2(S^1 \times S^1)$.

Let $F(r,s) = f(r-s)$ on $S^1 \times S^1$,

where $f(x) = \dfrac{1}{x} + C^\infty$.

- $F \notin L^2(S^1 \times S^1)$ (bcs $1/x$ singularity)
- $\forall g \in \mathrm{Diff}^2(S^1)$, $F^g - F$ is bounded.

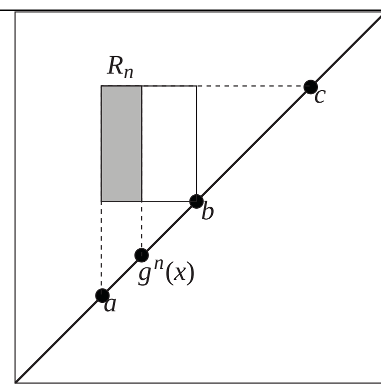
Define $\alpha(g) = F^g - F \in L^2(S^1 \times S^1))$.

α is a cocycle, so, by Kazhdan's Property T,

$\exists\ v \in L^2(S^1 \times S^1)$, $F^g - F = v^g - v$.

Then

- $F - v$ is Γ-invariant;
- $F - v \notin L^2(S^1 \times S^1)$. ($1/x$ singular on diag)



Let $\mu = (F-v)^2\, dr\, ds$, so

- μ is Γ-invariant measure on $S^1 \times S^1$;
- $\mu(\text{rect}) = \begin{cases} \infty & \text{if touches diagonal} \\ \text{finite} & \text{if away from diagonal} \end{cases}$

Choose $g \in \Gamma$, has a fixed pt.

Pass to triple cover, so g has ≥ 3 fixed points:

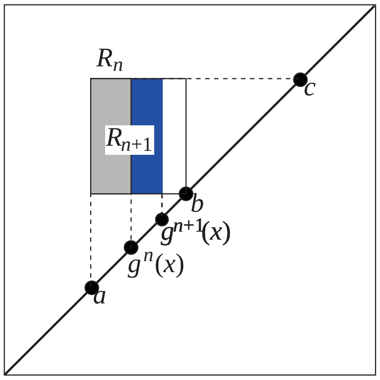
$g(a) = a$, $g(b) = b$, $g(c) = c$.

For $x \in (a,b)$, $\lim g^n(x) = \begin{cases} b & \text{as } n \to \infty \\ a & \text{as } n \to -\infty \end{cases}$

$R_n = (a, g^n(x)) \times (b,c)$

$R_n = (a, g^n(x)) \times (b, c)$.

$g(R_n) = (a, g^{n+1}(x)) \times (b, c) = R_{n+1}$.



$g(R_n) = R_{n+1}$

$\Rightarrow \mu(R_n) = \mu(R_{n+1})$

$\Rightarrow \mu(R_{n+1} - R_n) = 0$

$\Rightarrow \mu\left(\bigcup_{n=-\infty}^{\infty} (R_{n+1} - R_n)\right) = 0$

$\Rightarrow \mu((a,b) \times (b,c)) = 0$

$\rightarrow\leftarrow$ $(a,b) \times (b,c)$ touches the diagonal.

[based on ideas of Pressley-Segal and Reznikov]

The theorem of Ghys. *Let* $\Gamma \approx \mathrm{SL}(3,\mathbb{Z})$,
$\Gamma \hookrightarrow \mathrm{Homeo}(S^1) \Rightarrow \exists$ *fixed pt* (or a finite orbit).

Proof uses *ergodic theory* (meas'ble dynamics).

- $G = \mathrm{SL}(3,\mathbb{R})$,
- $\mathbf{F}$ = flag variety = $\{ (\ell, \Pi) \mid \ell \subset \Pi \subset \mathbb{R}^3 \}$.

Theory of *Furstenburg bdry:*

$\exists\ \psi \colon \mathbf{F} \to \mathrm{Prob}(S^1)$, Γ-equivariant, meas'ble.

Hard case: $\psi(x)$ is purely atomic.

Assume $\psi \colon \mathbf{F} \to S^1$, so $\vec{\psi} \colon \mathbf{F}^3 \to (S^1)^3$.

Circular order: $(S^1)^3 = X^+ \sqcup X^- \sqcup \{\text{singular}\}$.
X^+ is invariant under $\mathrm{Homeo}_+(S^1)$, so
$\vec{\psi}^{-1}(X^+)$ is Γ-invariant subset of $\mathbf{F}^3$.

Contradiction if $\nexists$ Γ-invariant subsets.
More precisely, if Γ is *ergodic* on $\mathbf{F}^3$;
i.e., A Γ-inv't $\Rightarrow \mu(A) = 0$ or $\mu(\mathbf{F}^3 \setminus A) = 0$.

Not ergodic — Ghys works with *fibered product*
$\{ (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \in \mathbf{F}^3 \mid \Pi_1 = \Pi_2 = \Pi_3 \}$.

$G = \mathrm{SL}(3,\mathbb{R})$, $\mathbf{F} = \{ (\ell, \Pi) \mid \ell \subset \Pi \subset \mathbb{R}^2 \}$.

G is *transitive* on $\mathbf{F}$. So $\mathbf{F} \cong G/P$,

where $P = \mathrm{Stab}_G(\text{flag}) = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$.

Moore Ergodicity Thm. Γ *is ergodic on* G/H
$\Leftrightarrow$ H *is not compact.*

So Γ is ergodic on $\mathbf{F}$. (a.e. Γ-orbit is dense.)

Cor. Γ *is ergodic on* $\mathbf{F}^2 = \mathbf{F} \times \mathbf{F}$.

Proof. $\mathrm{Stab}(\mathcal{F}_1, \mathcal{F}_2) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}$. Not cpct.

Cor. Γ *is* **not** *ergodic on* $\mathbf{F}^3$.

Proof. $\mathrm{Stab}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \{\pm \mathrm{Id}\}$ finite.

Cor. *Erg on* $\{(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \in \mathbf{F}^3 \mid \Pi_1 = \Pi_2 = \Pi_3\}$.

Proof. G is transitive on $\{\text{nonsingular}\}$,
$\mathrm{Stab}_G(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \begin{bmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & 1/\lambda^2 \end{bmatrix}$ not cpct.

Furstenberg boundary theory. (amenability)
$\exists\ \psi \colon \mathbf{F} \to \mathrm{Prob}(S^1)$ Γ-*equivariant* (and meas'ble).

Becausee $\mathbf{F} = G/P$, we want $\Psi \colon G \to \mathrm{Prob}(S^1)$,
Γ-equivariant, s.t. $\Psi(gp) = \Psi(g)$, (meas'ble).

Let $\mathcal{E} = \{ \Gamma\text{-equi } \Psi \colon G \to \mathrm{Prob}(S^1) \}$.

- $\mathrm{Prob}(S^1)$ is a compact, convex set (weak*)
  $\Rightarrow \mathcal{E}$ is a compact, convex set.
- G acts on $\mathcal{E}$ by translation.

We want P to have a fixed point in $\mathcal{E}$.

Defn. group H is *amenable*:

- H acts continuously on cpct metric space X
  $\Rightarrow \exists$ H-invariant prob meas on X.
- H acts linearly on cpct convex set X
  $\Rightarrow \exists$ fixed point in X.

Prop. *Every abelian group is amenable.*

Cor. *Every solvable group is amenable.*
In particular, P is amenable.

X = compact convex subset of Banach space V.

Lem. *$T\colon V \to V$ linear (bdd), X T-invariant $\Rightarrow T$ has a fixed point in X.*

Proof. $(Tv + T^2 v + \cdots + T^n v)/n$ has a convergent subseq. Limit is fixed point.

Cor. *Abelian groups are amenable.*

Proof. Spse T_1 and T_2 commute, $F_i = \{\text{fixed pts}\}$.

F_1 cpct, convex, T_2-inv't $\Rightarrow T_2$ has fixed pt in F_1. Therefore T_1 and T_2 have a common fixed point.

Cor. *P is amenable.*

Proof. Let $P_2 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$P_3 \triangleleft P_2 \Rightarrow F_3$ is P_2-invariant. (P_3 abel $\Rightarrow F_3 \neq \emptyset$.)
P_2/P_3 abelian $\Rightarrow P_2$ has fixed point in F_3.

$F_2 \neq \emptyset$ & $P_2 \triangleleft P$ & P/P_2 abelian
$\Rightarrow P_1$ has a fixed point in F_2.

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2. Bounded generation of $\mathrm{SL}(n,\mathbb{Z})$

3. Bounded generation and actions on the circle

2. Bounded generation of $\mathrm{SL}(n,\mathbb{Z})$

Abstract. If T is an invertible $n\times n$ matrix, then a fundamental theorem of undergraduate linear algebra states that T is a product of elementary matrices. More precisely, it is not difficult to see that T is a product of n^2 (or less) elementary matrices. However, these results assume that the matrix entries belong to a field (such as $\mathbb{C}$ or $\mathbb{R}$). The situation is more interesting if we require the entries of all of our matrices to be integers, because this makes it much more difficult to show that there is a number N (analogous to n^2), such that every invertible $n \times n$ matrix is a product of N (or less) elementary matrices. For $n \geq 3$, a proof can be given that uses algebraic methods from the proof of the Congruence Subgroup Property, and then applies the Compactness Theorem of First-Order Logic. On the other hand, no such N exists for the 2×2 matrices.

Thm (Carter-Keller, Liehl, Carter-Keller-Paige).
$\Gamma = \mathrm{SL}(3,\mathbb{Z}),\ \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}]),\ \mathrm{SL}(2,\mathbb{Z}[1/2]),$
$\Rightarrow \Gamma$ *is bddly generated by elem matrices.*

Rem. Also true for

- $\mathrm{SL}(n,\mathcal{O})$ with $n \geq 3$ [C-K],
- $\mathrm{SL}(2,\mathcal{O})$ if $\mathcal{O}$ has ∞ units [C-K-P],
- $\mathbb{G}(\mathbb{Z})$ if $\mathbb{G}$ is $\mathbb{Q}$-split, $\mathbb{Q}$-rank $\mathbb{G} \geq 2$ [Tavgen],
- some orthog grps [Erovenko-Rapinchuk].

Consequences (next lecture).

- Γ is *superrigid* ($< \infty$ irred reps of each dim)
- Γ has the *Congruence Subgroup Property*
- $\mathrm{SL}(3,\mathbb{Z})$ has *property T* (with explicit ϵ)
- (Ghys' Thm) action of Γ on S^1 has fixed pt (or finite orbit) if $H^2(\Gamma;\mathbb{R}) = 0$
- [Lifschitz-Morris] Γ cannot act on S^1 (except $\approx$ lin-frac)

Thm (Carter-Keller). $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *is boundedly generated by elementary matrices.*

Eg. Elementary matrices:

$$\begin{bmatrix} 1 & 25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall. Every invertible matrix can be reduced to Id by elementary column operations.

Prop. $T \in \mathrm{SL}(3,\mathbb{Z}) \Rightarrow T \rightsquigarrow \mathrm{Id}$ *by* $\mathbb{Z}$ *column ops.*

Eg.
$$\begin{bmatrix} 13 & 5 \\ 31 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 5 \\ 7 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Cor. $T \in \mathrm{SL}(3,\mathbb{Z}) \Rightarrow T =$ *product of elem mats.*

Thm (Carter-Keller). $T =$ *prod of* 48 *elem mats.*

Remark. No such bound exists for $\mathrm{SL}(2,\mathbb{Z})$:
$\mathrm{SL}(2,\mathbb{Z})$ **not** bdd gen by elem mats.

Thm (Liehl). $\mathrm{SL}(2, \mathbb{Z}[1/2])$ *bdd gen by elems.*
I.e., $T \rightsquigarrow \mathrm{Id}$ by $\mathbb{Z}[1/2]$ col ops, # steps is bdd.

Easy proof. Assume **Artin's Conjecture.**

Eg. 2 is a *primitive root* modulo 13:
$\{2^k\} = \{1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7\}$.
Complete set of residues.

Conj (Artin). $\forall r \neq \pm 1$, *perfect square,*
$\exists\, \infty$ *primes* q, *s.t.* r *is prim root modulo* q.
Assume $\exists q$ in every arith progression $\{a + nb\}$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad q = b + ka$ prime, 2 is prim root

$\begin{bmatrix} a & q \\ * & * \end{bmatrix}$ $\quad 2^\ell \equiv a \pmod{q}$; $\; 2^\ell = a + k'q$

$\begin{bmatrix} 2^\ell & q \\ * & * \end{bmatrix}$ $\quad 2^k$ unit $\Rightarrow$ can add *anything* to q

$$\begin{bmatrix} 2^\ell & 1 \\ * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

How to prove bounded generation [C-K-P].
- Compactness Theorem of 1st-order logic
- Mennicke symbols (Algebraic K-Theory)

Eg. F field $\Rightarrow T \in \mathrm{SL}(n, F)$ is prod of elem mats.
I.e., $\forall$field F, $\forall T \in \mathrm{SL}(n, F)$, $\exists r \in \mathbb{N}$,
T is a product of r elementary matrices.

Cor (by applying the Compactness Theorem).
$\exists r \in \mathbb{N}$, $\forall$*field* F, $\forall T \in \mathrm{SL}(n, F)$,
T *is a product of* $\leq r$ *elementary matrices.*

I.e., $\mathrm{SL}(n, F)$ has bdd gen by elem mats.
(& uniform for all fields—deps only on n.)

Key. Fields are defined by 1st-order statements.
- multiplication is commutative:
  $\forall x, \forall y, (xy = yx)$
- every nonzero element is a unit
  $\forall x, (x \neq 0 \rightarrow \exists y, (xy = 1))$
- the ring axioms (comm, assoc, distrib, ...)

Recall. 1st-order statements (in ring theory)
only use $\forall, \exists$, and, or, $\rightarrow, \neg, +, -, 0, 1$
(and add'l consts $c_1, c_2, \ldots$, relns R_i)
- e.g., every 2-generated ideal is principal
  $\forall x_1, \forall x_2, \exists y, (x_1 A + x_2 A = yA)$
- e.g., every finitely generated ideal is princ
  (use a list of axioms, not a single one)
- **not**: every ideal is principal
- $(x_{i,j})$ is an elementary matrix
- $(x_{i,j})$ is a product of ≤ 2 elementary mats
  $\exists$elem mats $(y_{i,j}), (z_{i,j})$,
  $x_{i,j} = y_{i,1} z_{1,j} + \cdots + y_{i,n} z_{n,j}$
- every $T \in \mathrm{SL}(n, A)$ is prod of $\leq r$ elem mats
- **not**: $\mathrm{SL}(n, A)$ gen'd by elem mats
  (or boundedly generated by elem mats)
- $(c_{i,j})$ is a product of $\leq r$ elem mats
- $(c_{i,j})$ is *not* a product of $\leq r$ elem mats
- $(c_{i,j})$ is *not* a product of elem mats

Gödel Completeness Thm. *Suppose*
- $\varphi_1, \varphi_2, \ldots$ *1st-order, and*
- $\nexists$ *ring satisfying every* φ_i.

Then one can prove a contradiction $(0 \neq 0)$ *from axioms* $\varphi_1, \varphi_2, \ldots$, *together with ring axioms.*

Proof (finite length) refers to finitely many φ_i:

Compactness Thm. *Suppose*
- $\varphi_1, \varphi_2, \ldots$ *1st-order, and*
- $\nexists$ *ring satisfying every* φ_i.

Then $\exists r \in \mathbb{N}$, $\nexists$ *ring satisfying* $\varphi_1, \ldots, \varphi_r$.

Cor. $\exists r$, $\forall$*field* F, $\forall T \in \mathrm{SL}(n, A)$,
T *is prod of* $\leq r$ *elem mats.*

Proof. φ_0: field axioms.
φ_i: $(c_{k,\ell}) \in \mathrm{SL}(n, F)$, not prod $\leq i$ elem mats.

$\nexists$ ring sat every φ_i. So $\exists r$, no field sats φ_r. $\square$

Compactness Thm.
$\nexists$ *ring satisfying* $\varphi_1, \varphi_2, \ldots$ (*1st order*).
$\Rightarrow \exists r \in \mathbb{N}$, $\nexists$ *ring satisfying* $\varphi_1, \ldots, \varphi_r$.

Defn. $\mathrm{E}(n, A) = \langle$elem mats in $\mathrm{SL}(n, A)\rangle$.

Cor. *Spse* Φ *is a set of 1st-order ring axioms, s.t.* $\forall A$ *sat* Φ, $\mathrm{E}(n, A) = \mathrm{SL}(n, A)$.
Then $\forall A$ *sat* Φ, $\mathrm{SL}(n, A)$ *bdd gen by elems.*

Proof. φ_i: $(c_{k,\ell}) \in \mathrm{SL}(n, A)$, not prod $\le i$ elems.
Not bdd gen $\Rightarrow \Phi \cup \{\varphi_1, \ldots, \varphi_r\}$ consistent
$\Rightarrow \Phi \cup \{\varphi_1, \varphi_2, \ldots\}$ consistent $\rightarrow\leftarrow$

Cor. *Spse* Φ *is a set of 1-st order ring axioms, s.t.* $\forall A$ *sat* Φ, $\mathrm{E}(n, A) \approx \mathrm{SL}(n, A)$. (finite ind)
Then $\forall A$ *sat* Φ, $\mathrm{SL}(n, A)$ *bdd gen by elems* ($\approx$).

Thm (Carter-Keller). $\mathrm{SL}(3, \mathbb{Z})$ *bdd gen by elems.*

Method of proof. $\langle$elem mats$\rangle$ fin ind in $\mathrm{SL}(3, \mathbb{Z})$.
Only use 1st-order properties of $\mathbb{Z}$ in the proof.

Thm (Carter-Keller). $\mathrm{SL}(3, \mathbb{Z})$ *bdd gen by elems.*

Prove: $\mathrm{E}(3, \mathbb{Z})$ finite index in $\mathrm{SL}(3, \mathbb{Z})$.
Let $C = C_{\mathbb{Z}} = \mathrm{SL}(3, \mathbb{Z}) / \mathrm{E}(3, \mathbb{Z})$. (finite??)

Thm. *A commutative* $\Rightarrow \mathrm{E}(3, A) \lhd \mathrm{SL}(3, A)$.
So C is a group. In fact, C is abelian.

Step 1. C has exponent dividing 12 (i.e. $x^{12} = e$).

Step 2. C is cyclic.

Let $W = W_{\mathbb{Z}} = \{\, (a, b) \in \mathbb{Z}^2 \mid \gcd(a, b) = 1 \,\}$
$= \{$1st rows of elements of $\mathrm{SL}(2, \mathbb{Z})\}$.

Define $[\ ] : W \to C$ by $\begin{bmatrix} b \\ a \end{bmatrix} \equiv \begin{bmatrix} a & b & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- $[\ ]$ is well defined (easy) and onto (SR_2).
- (MS1) $\begin{bmatrix} b + ta \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b \\ a + tb \end{bmatrix}$.
- (MS2a) $\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix}$ (need $n \ge 3$).

Step 2. C is cyclic.

Given $\begin{bmatrix} b_1 \\ a_1 \end{bmatrix}, \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$ (nontrivial).

Dirichlet: $\exists$ large prime $p \equiv b_1 \pmod{a_1}$.
$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} p \\ a_1 \end{bmatrix}$; wma $b_1 = p$ prime.
In fact, wma all a_i, b_i are large primes ($b_1 \ne b_2$).

CRT: $\exists q$, s.t. $q \equiv a_i \pmod{b_i}$; wma $a_1 = q = a_2$.

$(\mathbb{Z}/q\mathbb{Z})^\times$ cyclic $\Rightarrow \exists b, e_i$, s.t. $b_i \equiv b^{e_i} \pmod{q}$.

$$\begin{bmatrix} b_i \\ a_i \end{bmatrix} = \begin{bmatrix} b_i \\ q \end{bmatrix} = \begin{bmatrix} b^{e_i} \\ q \end{bmatrix} = \begin{bmatrix} b \\ q \end{bmatrix}^{e_i} \in \left\langle \begin{bmatrix} b \\ q \end{bmatrix} \right\rangle .$$

Any two elts of C are in same cyclic subgrp.
So C is cyclic. $\square$

Note: Since $C^{12} = e$, only need $(\mathbb{Z}/q\mathbb{Z})^\times$ cyclic modulo 12th powers.
This is a 1st-order statement:
$\exists x, \forall y, \exists z, (y = z^4 \text{ or } x z^4 \text{ or } \cdots \text{ or } x^{11} z^4)$.

Lem. $[\ ]$ *is onto.*

Proof. $\mathbb{Z}$ satisfies:
$\forall x, y, z, (xA + yA + zA = A$
$\to \exists y' \equiv y \pmod{z}, (xA + y'A = A))$.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ x & y & z \end{bmatrix} \rightsquigarrow \begin{bmatrix} * & * & * \\ * & * & * \\ x & y' & z \end{bmatrix} \rightsquigarrow \begin{bmatrix} * & * & * \\ * & * & * \\ x & y' & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}.$$

Lem (MS2a). $\begin{bmatrix} b \\ a \end{bmatrix} \begin{bmatrix} b' \\ a \end{bmatrix} = \begin{bmatrix} bb' \\ a \end{bmatrix}$.

Proof. Since $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}^{\pm 1} \in \mathrm{E}(3, A)$ (verify!!),

$$\begin{bmatrix} b' \\ a \end{bmatrix} = \begin{bmatrix} a & b' & 0 \\ c' & d' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\equiv \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b' & 0 \\ c' & d' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} d' & 0 & -c' \\ 0 & 1 & 0 \\ -b' & 0 & a \end{bmatrix}.$$

$$\begin{bmatrix} b \\ a \end{bmatrix} \begin{bmatrix} b' \\ a \end{bmatrix} \equiv \begin{bmatrix} a & b & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d' & 0 & -c' \\ 0 & 1 & 0 \\ -b' & 0 & a \end{bmatrix}$$
$$= \begin{bmatrix} ad' & b & -ac' \\ * & * & * \\ -b' & 0 & a \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & b & 0 \\ * & * & * \\ -b' & 0 & a \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & b & 0 \\ 0 & * & * \\ 0 & bb' & a \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & bb' & a \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} a & bb' & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} bb' \\ a \end{bmatrix}.$$

Step 1. C has exponent dividing 12 (i.e. $x^{12} = e$).

Lem. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, A) \Rightarrow \begin{bmatrix} b \\ a \end{bmatrix}^{-1} = \begin{bmatrix} c \\ a \end{bmatrix}$.

Given a, b, we choose $f_i, g_i, b_i \in \mathbb{Z}$
to make following calculation valid:

$$\begin{bmatrix} b \\ a \end{bmatrix}^{-12} = \begin{bmatrix} c \\ a \end{bmatrix}^{12} = \prod_{i=1}^{2} \begin{bmatrix} c \\ f_i + g_i a \end{bmatrix}^2$$
$$= \prod_{i=1}^{2} \begin{bmatrix} c g_i \\ f_i + g_i a \end{bmatrix}^2 = \prod_{i=1}^{2} \begin{bmatrix} b_i g_i \\ f_i + g_i a \end{bmatrix}^{-2}$$
$$= \prod_{i=1}^{2} \begin{bmatrix} b_i \\ f_i + g_i a \end{bmatrix}^{-2} = \prod_{i=1}^{2} \begin{bmatrix} b_i \\ 1 \end{bmatrix}^{-2} = 1.$$

It suffices to know that:

- $(f_1 + g_1 a)(f_2 + g_2 a) \equiv a^6 \pmod{c}$.
- $f_i^2 \equiv 1 \pmod{g_i}$.
- $f_i \mathrm{Id} + g_i \begin{bmatrix} a & b_i \\ c & * \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.
- $f_i + g_i a \equiv 1 \pmod{b_i}$.

To obtain these properties, let

- $b_1 = b$.
- $b_2 \equiv b \pmod{a}$, $\gcd(\phi(b_1), \phi(b_2)) \le 6$.*
- α_i = exponent of a modulo b_i
  (so $\gcd(\alpha_1, \alpha_2) \le 6$).
- $t_1, t_2 \in \mathbb{Z}$ with $\alpha_1 t_1 + \alpha_2 t_2 = 6$.
- $f_i, g_i \in \mathbb{Z}$ with $T_i^{\alpha_i t_i} = f_i \mathrm{Id} + g_i T_i$,
  where $T_i = \begin{bmatrix} a & b_i \\ c & d_i \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

Then:

- $a^6 = a^{\alpha_1 t_1} a^{\alpha_2 t_2}$
  $\equiv (f_1 + g_1 a)(f_2 + g_2 a) \pmod{c}$.
- $1 = \det(T_i^{\alpha_i t_i}) \equiv \det(f_i \mathrm{Id}) = f_i^2 \pmod{g_i}$.
- $f_i \mathrm{Id} + g_i \begin{bmatrix} a & b_i \\ c & * \end{bmatrix} = T_i^{\alpha_i t_i} \in \mathrm{SL}(2, \mathbb{Z})$.
- $f_i + g_i a \equiv a^{\alpha_i t_i} \equiv 1^{t_i} \equiv 1 \pmod{b_i}$.

*$\forall$ prime $p > 3$, write $b \equiv x_p y_p \pmod{p}$, where $x_p, y_p \not\equiv 1 \pmod{p}$. Let $b_2 = xy$, where x, y prime and $x \equiv x_p \pmod{p}$ for prime divisors p of $\phi(b)$.

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Actions of arithmetic groups on the circle

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1. Some arithmetic groups that cannot act on the circle

2. Bounded generation of $\mathrm{SL}(n, \mathbb{Z})$

3. Bounded generation and actions on the circle

* Will be sloppy about passing to finite-index subgrps.

3. Bounded generation and actions on the circle

Abstract. Joint work with Lucy Lifschitz provides many new examples of arithmetic groups that have no interesting actions on the circle. This includes all finite-index subgroups of $\mathrm{SL}(2, \mathbf{Z}[\sqrt{r}])$ or $\mathrm{SL}(2, \mathbf{Z}[1/r])$, where $r > 1$ is square free. The proofs are based on the fact, proved by D. Carter, G. Keller, and E. Paige, that every element of these groups is a product of a bounded number of elementary matrices.

$\Gamma = \mathrm{SL}(3, \mathbb{Z})$ or $\mathrm{SL}(2, \mathbb{Z}[\sqrt{5}])$ or $\mathrm{SL}(2, \mathbb{Z}[1/5])$

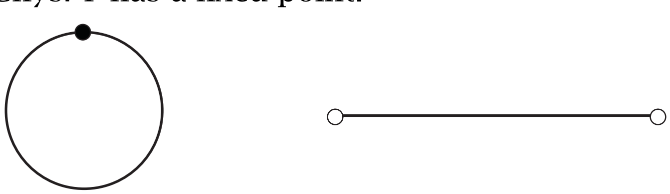
Thm (Witte, Lifschitz-Morris). *No actions on S^1.*
$\phi: \Gamma \to \mathrm{Homeo}(S^1)$, *not $\approx$ lin-frac** $\Rightarrow \Gamma^\phi$ *is finite.*

(and other applications of *bounded generation*)

Thm (Ghys). $\phi: \Gamma \to \mathrm{Homeo}(S^1)$, *not $\approx$ lin-frac**
$\Rightarrow \Gamma^\phi$ *has a fixed point* (*or finite orbit*).

Cor. $\nexists$ *action on* $\mathbb{R} \Rightarrow \nexists$ *action on* S^1 (*).

Proof. Suppose Γ acts on S^1 (not $\approx$ lin-frac).
Ghys: Γ has a fixed point.



So Γ acts on $S^1 \smallsetminus \{x\} \approx \mathbb{R}$. □

Thm (Witte, Lifschitz-Morris). *No actions on* $\mathbb{R}$.
$\phi: \Gamma \to \mathrm{Homeo}_+(\mathbb{R}) \quad \Rightarrow \Gamma^\phi$ *is trivial.*

Thm (Lifschitz-Morris). $\Gamma = \mathrm{SL}(2, \mathbb{Z}[1/5])$,
$\phi: \Gamma \to \mathrm{Homeo}_+(\mathbb{R}) \quad \Rightarrow \Gamma^\phi$ *is trivial.*

Proof: Combine **bdd generation** *and* **bdd orbits**.

$$\overline{U} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}, \qquad \underline{V} = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}.$$

Thm (Liehl, Carter-Keller-Paige). $\Gamma \approx \mathrm{SL}(2, \mathcal{O})$,
$\Rightarrow (\approx)\ \Gamma$ *has bdd generation by* $\overline{U}$ *and* $\underline{V}$.
I.e., $\Gamma \approx \overline{U}\underline{V}\overline{U}\underline{V} \cdots \overline{U}\underline{V}$.

Thm (Lifschitz-Morris). $\Gamma \approx \mathrm{SL}(2, \mathcal{O})$ *acts on* $\mathbb{R}$
$\Rightarrow$ *every* $\overline{U}$*-orbit is bdd* (*and every* $\underline{V}$*-orbit is bdd*).

Cor. *Every* Γ*-orbit is bdd, so* Γ *has a fixed pt.*

Cor [L-M]. $\nexists$ *nontrivial* (*or-pres*) *action of* Γ *on* $\mathbb{R}$.

Proof. Let $F = \{\text{fixed points}\}$ (closed).

I = component of $\mathbb{R} \smallsetminus F$ (Γ-invariant).
Γ acts on I = open interval $\approx \mathbb{R}$.
Γ has no fixed points in I. $\to\leftarrow$

Orbits of unipotent subgroups are bounded.

Thm. $\mathrm{SL}(2,\mathbb{Z}[1/p])$ *acts on* $\mathbb{R} \Rightarrow \overline{U}$*-orbits bdd.*

$$\overline{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix},\ \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix},\ \not{p} = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix}$$

Commutation relations.

- $\not{p}^{-n}\overline{1}\not{p}^{n} \to \overline{0}$,
- $\not{p}^{-n}\underline{1}\not{p}^{n} \to \underline{\infty}$.

Spse $\overline{U}$-orbit and $\underline{V}$-orbit of x not bdd above.

Assume $\not{p}$ fixes x. ($\not{p}$ does have fixed pts, so ok.)

Wolog $x\,\overline{1} > x\,\underline{1}$. So $(x\,\overline{1})\not{p}^{n} > (x\,\underline{1})\not{p}^{n}$.

LHS $= (x\,\overline{1})\not{p}^{n} = x(\not{p}^{-n}\,\overline{1}\not{p}^{n}) \to x\ (\overline{0}) = x < \infty$.

RHS $= (x\,\underline{1})\not{p}^{n} = x\ (\not{p}^{-n}\underline{1}\not{p}^{n}) \to x\ (\underline{\infty}) \to \infty$.

$\to\leftarrow$

Similar for $\mathrm{SL}(2,\mathbb{Z}[\alpha])$*.*

$\not{p} \mapsto \not{\omega}$, ω = unit of infinite order.

Other applications of bounded generation.

- Γ is *superrigid* ($< \infty$ irred reps of each dim)
- Γ has the *Congruence Subgroup Property*
- $\mathrm{SL}(3,\mathbb{Z})$ has *property T* (with explicit ϵ)
- (Ghys' Thm) action of Γ on S^1 has fixed pt (or finite orbit) if $H^2(\Gamma;\mathbb{R}) = 0$

Thm (Ghys). *Action of* Γ *on* S^1 *has fixed pt* ($\approx$) *if* $H^1(\Gamma;\mathbb{R}) = 0$*.*

Proof. Univ cover $\widetilde{S^1} = \mathbb{R}$ has action of $\widetilde{\Gamma}$.

$\pi_1(S^1) = \mathbb{Z} \quad \Rightarrow e \to \mathbb{Z} \to \widetilde{\Gamma} \to \Gamma \to e$.

Extension defined by $\omega \in H^2(\Gamma;\mathbb{Z})$.

$$H^1(\Gamma;\mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H^2(\Gamma;\mathbb{Z}) \to H^2(\Gamma;\mathbb{R}) = 0$$

$\omega = \delta(\alpha)$ for some homo $\alpha\colon \Gamma \to \mathbb{R}/\mathbb{Z}$

$\mathbb{R}/\mathbb{Z}$ abel $\Rightarrow$ α trivial on Γ'

$\Rightarrow$ ω trivial on Γ' (finite index)

So $\widetilde{\Gamma} \doteq \Gamma \times \mathbb{Z}$.

Action of Γ on S^1 lifts to action of Γ on $\mathbb{R}$.

Has fixed point [Lifschitz-Morris].

Thm. $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *has Kazhdan's property* T*.*

Idea of proof. Wish to show $H^1(\Gamma;\mathcal{H}) = 0$.

Equivalent: Γ acts on $\mathcal{H}$ by affine isometries ($\gamma(v) = Tv + w$, T linear, $w \in \mathcal{H}$) $\Rightarrow$ Γ has a fixed point.

Enough to show Γ has a bounded orbit (then centroid of convex hull is fixed).

By bounded generation, enough to show $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$ has bounded orbit.

Use $\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix} \cong \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, show $\mathbb{Z}^2$ has bounded orbit.

Thm. $SL(2, \mathbb{Z}[\sqrt{5}])$ *is superrigid.*
I.e., $< \infty$ irreducible reps of each dimension.

Idea of proof. Suppose $\alpha\colon \Gamma \to GL(d, \mathbb{C})$.

Show eigenvals of $\alpha(g)$ are alg'ic. (ctbly many)
$\therefore$ trace$(\alpha(g))$ cannot deform continuously.
$\therefore$ α cannot deform continuously.
So $< \infty$ possibilities for α.

Note. Eigenvaluess of $\alpha(g)$ are algebraic
$\Leftrightarrow$ $\alpha(g)$ satisfies a poly with $\mathbb{Q}$-coeffs
$\Leftrightarrow$ $\mathbb{Q}$-span of $\langle \alpha(g) \rangle$ is finite dim'l.

Key. Eigenvalues of $\alpha(\overline{u})$ are algebraic.
Therefore $\mathbb{Q}$-span of $\alpha(\overline{U})$ is finite dim'l.

Since $\Gamma = \overline{U}\underline{V}\overline{U}\underline{V} \cdots \overline{U}\underline{V}$,
then $\mathbb{Q}$-span of $\alpha(\Gamma)$ is finite dim'l.
So $\mathbb{Q}$-span of $\langle \alpha(g) \rangle$ is finite dim'l.

Key. Eigenvalues of $\alpha(\overline{u})$ are algebraic.

Consider $SL(2, \mathbb{Z}[1/p])$.

$$\overline{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \quad \not{p} = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix}$$

$$\not{p}^{-1}\overline{u}\not{p} = \overline{u/p^2}$$
$$\Rightarrow (\not{p}^{-1}\overline{u}\not{p})^{p^2} = \overline{u}$$
$\Rightarrow$ eigenval of $\overline{u}$ sats $\lambda^{p^2} = \lambda$

Apply α to calculation:
eigenval of $\alpha(\overline{u})$ sats $\lambda^{p^2} = \lambda$
So λ is algebraic.

Recall. $\Gamma \supset SL(3, \mathbb{Z})$ or $Sp(4, \mathbb{Z})$ $\Leftrightarrow$ $\mathbb{Q}$-rank $\Gamma \geq 2$
$\Rightarrow$ Γ does not act on S^1 (or $\mathbb{R}$).

Work in progress.

- In $\{\, \Gamma \mid \mathbb{Q}\text{-rank}\, \Gamma = 1 \ \ (\&\ \mathbb{R}\text{-rank}\, G \geq 2) \,\}$,
  find shortest possible list of grps $\Gamma_1, \Gamma_2, \ldots$
  s.t. $\forall \Gamma$, $\exists\, \Gamma_i \subseteq \Gamma$ $\quad(\approx)$.
  Almost finished (is ${}^2E_6$ on the list?)
  (with V. Chernousov and L. Lifschitz)

$SL(2, \mathcal{O})$, $SU(|x|^2 - |y|^2 + a|z|^2)_{\mathcal{O}}$, ${}^{3,6}D_4$, ${}^2E_6$???

- Prove unip subgrps of Γ_i have bdd orbits.
  Finished (with L. Lifschitz).

- Prove unip subgrps bddly gen Γ_i.
  Not done.

Conclusion: Conjecture true when $\mathbb{Q}$-rank $\Gamma \geq 1$.

After that: Case where $\mathbb{Q}$-rank $\Gamma = 0$. (cocpct)

Need new methods! (no unip elements in Γ)

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