Tessellations of homogeneous spaces of $\mathrm{SU}(2,n)$

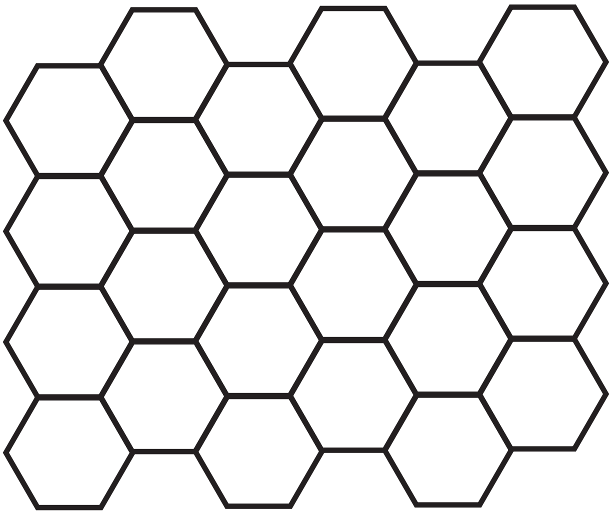
Dave Witte

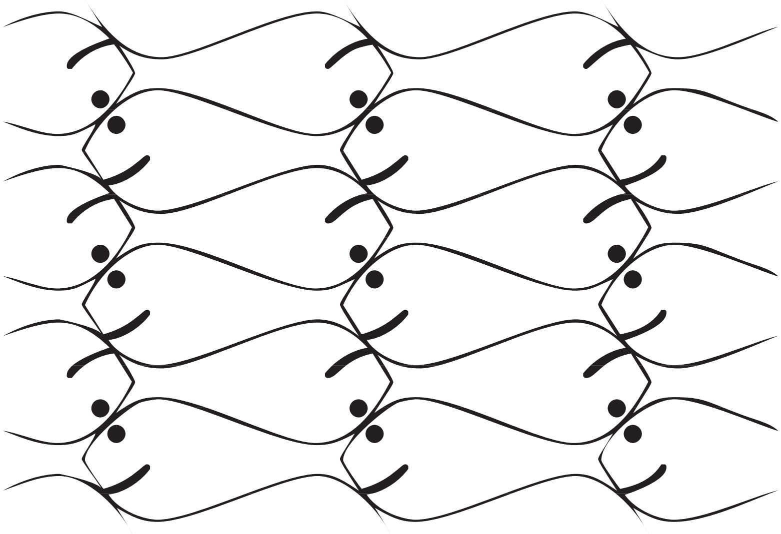
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078

dwitte@math.okstate.edu
http://www.math.okstate.edu/∼dwitte

Joint work with **Hee Oh**
heeoh@math.princeton.edu
and **Alessandra Iozzi**
iozzi@math.umd.edu

Classical example. Tilings of the Euclidean plane.





Which homogeneous spaces have a tessellation?

H = closed, connected subgroup of G,
so G/H is a "homogeneous space."

Question. *Does G/H have a tessellation?*
I.e., is there a discrete subgroup Γ of G, such that

- $\Gamma\backslash G/H$ *is compact*

and

- Γ *acts properly discontinuously on G/H?*

Defn. Γ acts properly discontinuously on X:
$\forall$ compact $F \subset X$,

$$\{\, \gamma \in \Gamma \mid \gamma F \cap F \neq \emptyset \,\} \text{ is finite.}$$

(In particular, all orbits are discrete.)

$G = \mathrm{SL}(n, \mathbb{R})$

$= $ (Zariski) connected, almost simple Lie grp

- $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{H})$
- $\mathrm{SO}(m, n)$, $\mathrm{SU}(m, n)$, $\mathrm{Sp}(n, \mathbb{R})$
- $\mathrm{Sp}(m, n)$, $\mathrm{SO}(n, \mathbb{C})$, $\mathrm{Sp}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{H})$
- finitely many exceptional groups

Classical examples.

If G/H is compact: let $\Gamma = e$.

If H is compact: let Γ be a lattice in G.

Defn. Γ is a (cocompact) lattice in G:

- Γ is discrete
- $\Gamma \backslash G$ is compact.

There is a lattice in every simple G [Borel].

(Idea: $G \cap \mathrm{GL}(n, \mathbb{Z})$ is a lattice in G.)

Assumption. Neither H nor G/H is compact.

Therefore Γ must be infinite

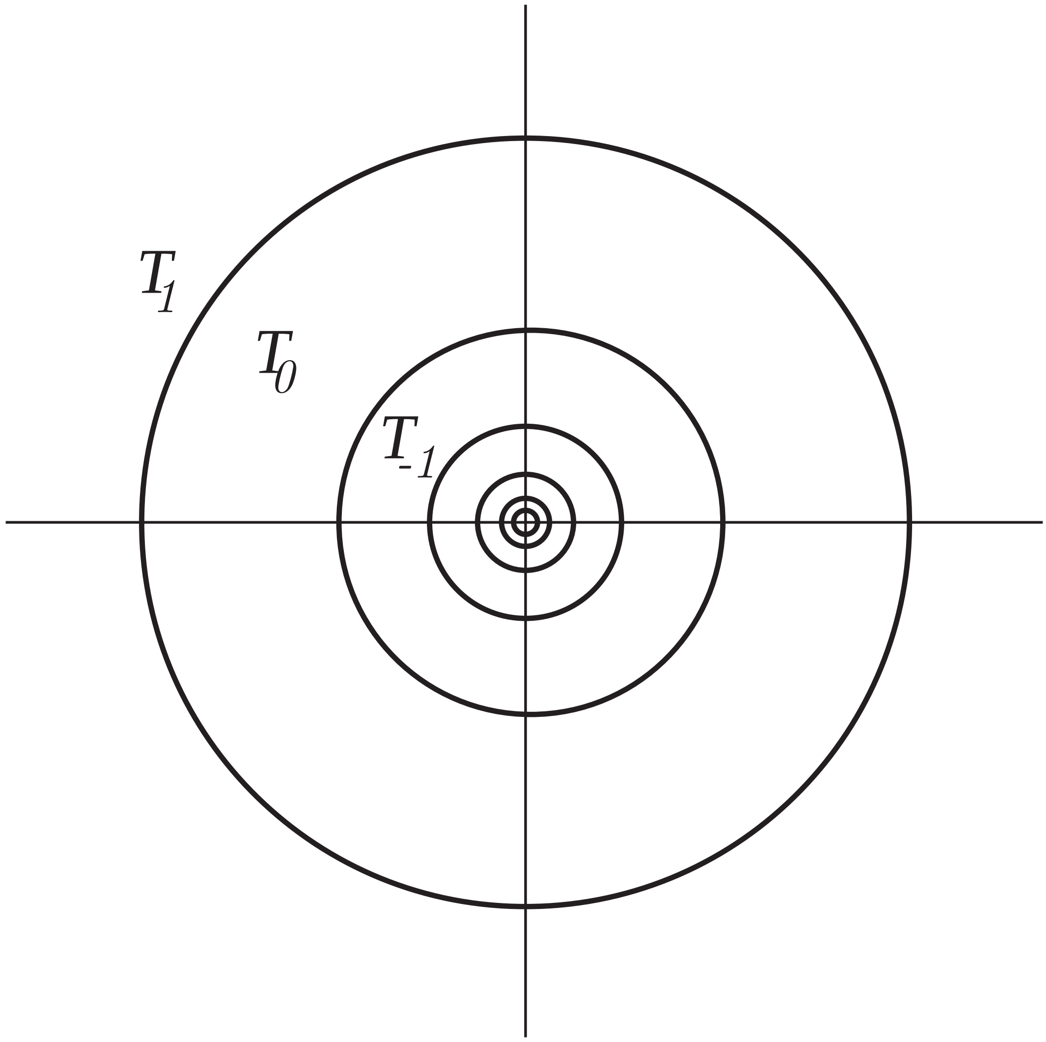
and Γ cannot be a lattice in G.

Eg. $G = \mathrm{GL}(2, \mathbb{R})$ is transitive on $X = \mathbb{R}^2 - \{0\}$.

So $X \cong G/H$,

where $H = \mathrm{Stab}_G(\overrightarrow{e_1}) = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$.

Let $T_i = \{\, v \in \mathbb{R}^2 \mid 2^i \leq \|v\| \leq 2^{i+1} \,\}$.



We have $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^j T_i = T_{i+j}$,

so each T_i is a fund dom for $\Gamma = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$.

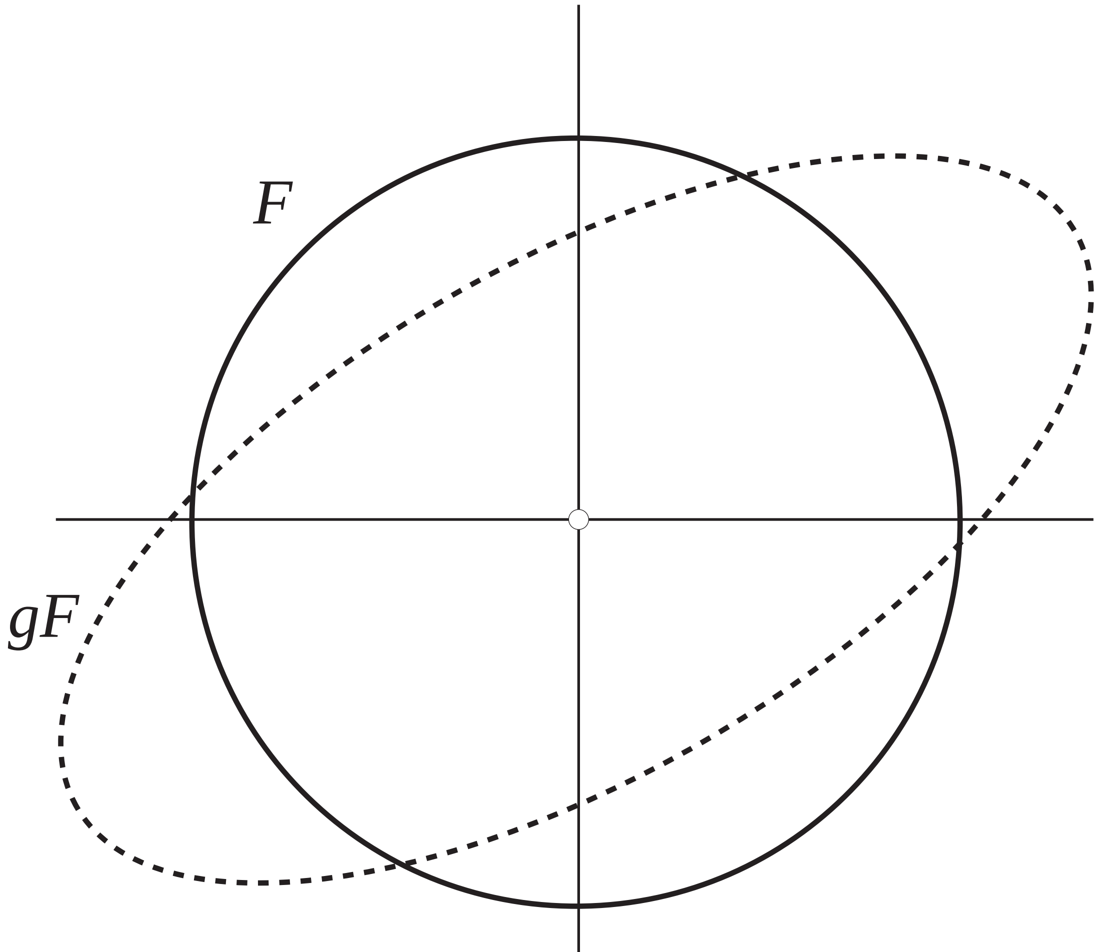
So X has a tessellation.

Eg. $G = \mathrm{SL}(2,\mathbb{R})$ is transitive on $X = \mathbb{R}^2 - \{0\}$.

So $X \cong G/H$,

where $H = \mathrm{Stab}_G(\overrightarrow{e_1}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Let $F =$ unit circle (compact).



$\forall g \in G, \;\; gF \cap F \neq \emptyset$

$\Rightarrow$ no infinite subgrp acts properly disc'ly

$\Rightarrow \Gamma$ is finite

$\Rightarrow X$ does not have a tessellation.

Prop (Calabi-Markus phenomenon).

$\exists$ *cpct* $F \subset G/H$, *s.t.* $\forall g \in G,\ gF \cap F \neq \emptyset$

$\Rightarrow G/H$ *does not have a tessellation.*

Group-theoretic restatement.

$\exists$ cpct sub**set** $C \subset G$, such that $G = CHC$

$\Rightarrow G/H$ does not have a tessellation.

Eg. $G = \mathrm{SL}(n, \mathbb{R})$

$K = \mathrm{SO}(n) = \{\text{rotations}\}$ (compact)

$$A = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$$ diagonal ("split torus")

$$N = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & \ddots & * \\ & & & 1 \end{pmatrix}$$ upper triangular

Cartan decomposition. $G = KAK$

Fact. $G = KNK$ [Kostant]

Cor. *If* $A \subset H$ *or* $N \subset H$, *then* $G = KHK$, *so* G/H *does not have a tessellation.*

Prop. $G = \mathrm{SL}(2,\mathbb{R})$
$\Rightarrow$ *G/H does not have a tessellation.*

Proof. $\dim A = 1$.

$\mu(e) = e, \qquad \lim_{h\to\infty} \mu(h) = \infty \qquad \Rightarrow \mu(H) = A^+$

I.e., $A^+ \subset KHK$.

So $G = KA^+K \subset KHK$.

Rem. Same proof whenever $\dim A = 1$.

- $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{SL}(2,\mathbb{H})$
- $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$, $F_{4,1}$

Next case. $\dim A = 2$.

- $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SL}(3,\mathbb{H})$
- $\mathrm{SO}(2,n)$, $\mathrm{SU}(2,n)$, $\mathrm{Sp}(2,n)$
- finitely many others

Thm (Benoist, Oh-Witte).
$G = \mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SL}(3,\mathbb{H})$
$\Rightarrow$ *G/H does not have a tessellation.*

Given $g \in G$.

$G = KAK \Rightarrow \exists a \in A$, s.t. $g \in KaK$.

But a is not unique

Let $A^+ = \left\{ \begin{pmatrix} t & \\ & 1/t \end{pmatrix} \;\middle|\; t \geq 1 \right\}$

$=$ "positive Weyl chamber."

Then $\exists! a \in A^+$, s.t. $g \in KaK$.

Defn (Cartan projection). $\mu \colon G \to A^+$

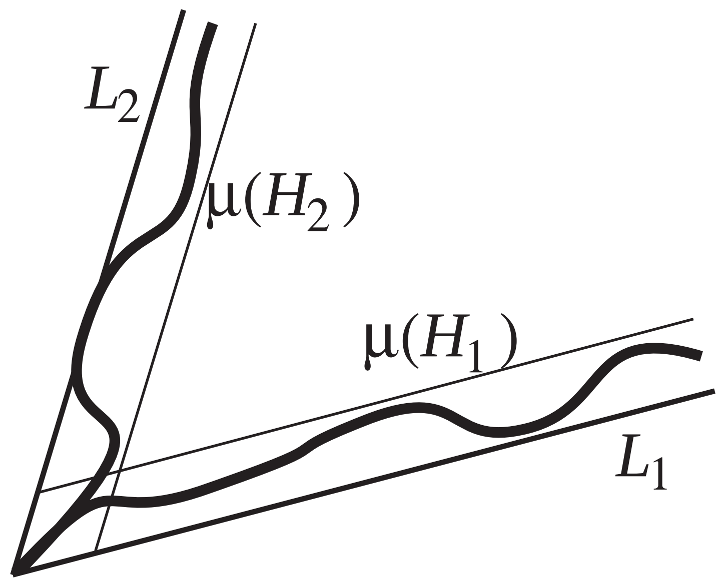
by $g \in K\,\mu(g)\,K$.

μ is continuous and proper.

Henceforth assume $H \subset AN$ (triangular)
($G = KAN$ "$\Rightarrow$" $KH = KH'$ with $H' \subset AN$.)

Thm (Oh-Witte). $G = \mathrm{SO}(2,n)$, *n even.*
G/H has a tessellation iff

- $\dim H = n$*; and*
- $\mu(H) \approx$ *wall of* A^+*.*



Thm (Iozzi-Witte). $G = \mathrm{SU}(2,n)$, *n even.*
G/H has a tessellation iff

- $\dim H = 2n$*; and*
- $\mu(H) \approx$ *wall of* A^+*.*

We explicitly describe the possible subgroups H. (In fact, we calculate $\mu(H)$ for every connected subgroup of AN.)

Thm (Oh-Witte). $G = \mathrm{SO}(2,n)$. *TFAE:*

- $H \sim \mathrm{SO}(1,n) \cap AN$
- $\dim H = n$ *and* $\mu(H) \approx L_1$
- $\dim H \geq n$ *and* $\mu(H) \not\supset L_2$

Prop (Oh-Witte). $G = \mathrm{SO}(2,2m)$,

$H_{\mathrm{SU}} = \mathrm{SU}(1,m) \cap AN$

$\Rightarrow \mu(H_{\mathrm{SU}}) \approx L_2$ *and* $\dim H_{\mathrm{SU}} = 2m = n$.

Thm (Oh-Witte). $G = \mathrm{SO}(2,2m)$. *TFAE:*

- $H \sim$ *certain deformations of* H_{SU}
- $\dim H = 2m$ *and* $\mu(H) \approx L_2$
- $\dim H \geq 2m$ *and* $\mu(H) \not\supset L_1$.

Thm (Iozzi-Witte). $G = \mathrm{SU}(2,n)$.

Similar conclusions, except:

- $\mathrm{SO}(1,n) \mapsto \mathrm{SU}(1,n)$
- $\mathrm{SU}(1,m) \mapsto \mathrm{Sp}(1,m)$
- $\dim n \mapsto \dim 2n$

Conj. *The homogeneous spaces*

$$\mathrm{SO}(2,2m+1)/\mathrm{SU}(1,m)$$

and

$$\mathrm{SU}(2,2m+1)/\mathrm{Sp}(1,m)$$

do not have tessellations.

Thm. $G = \mathrm{SO}(2,2m+1),\ \mathrm{SU}(2,2m+1)$. *Conjecture* $\Rightarrow$ G/H *does not have a tessellation.*

Similar methods should apply to $\mathrm{Sp}(2,n)$
and other cases with $\dim A = 2$.

Not if $\dim A \geq 3$.

Thm (Benoist, Kobayashi). Γ *proper on* G/H
$\Leftrightarrow \mu(\Gamma)$ *diverges from* $\mu(H)$ *in* A^+
i.e., $\forall$ *cpct* $C \subset A$, $\mu(\Gamma) \cap \mu(H)C$ *is finite.*

Cor. *Assume*

- $\dim A = 2$;
- L_1 *and* L_2 *are the two walls of* A^+;
- $\mu(H_i) \approx L_i$ *for* $i = 1, 2$;
- Γ *acts properly on* G/H.

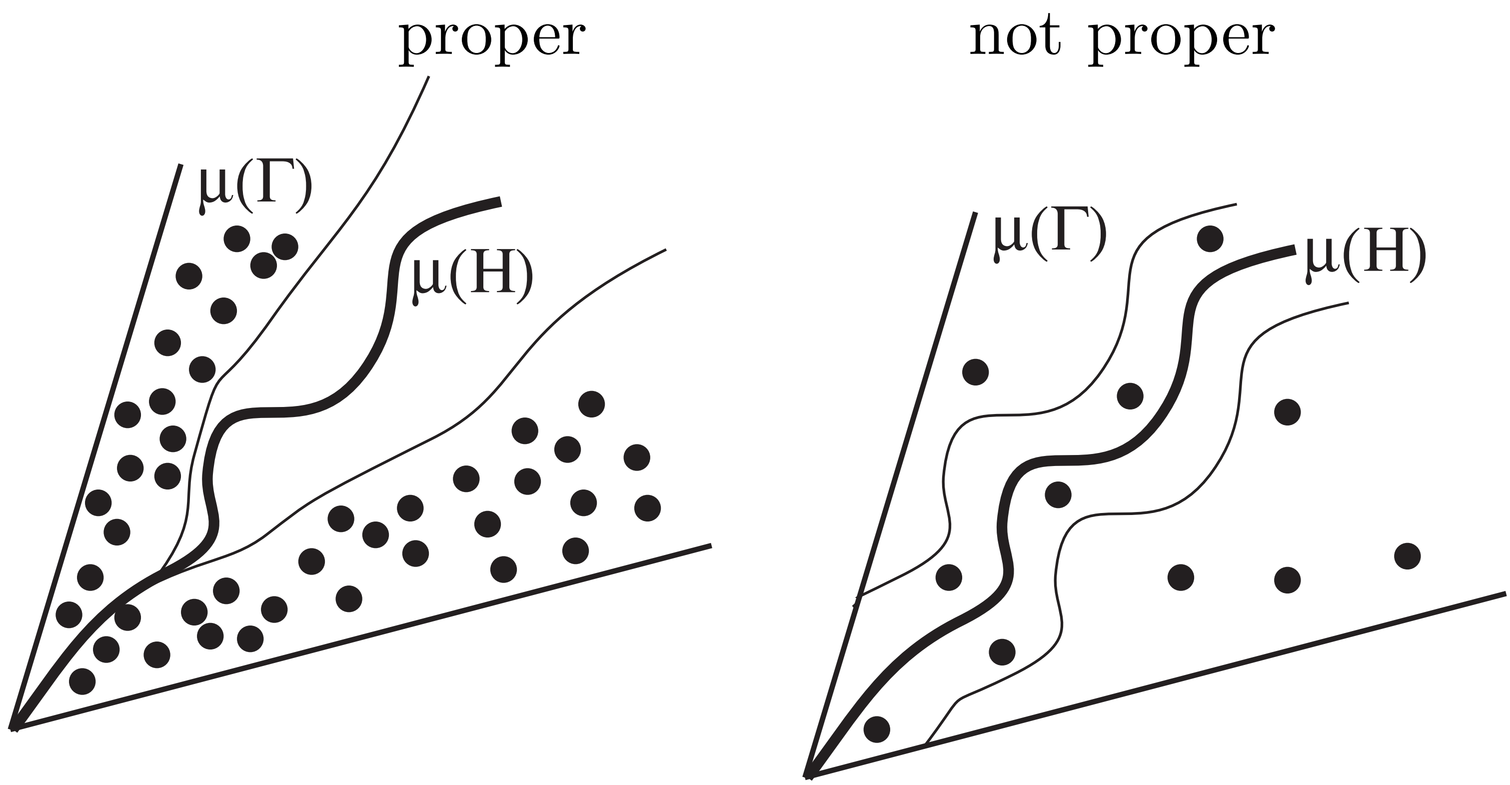
Then Γ *acts properly on either* G/H_1 *or* G/H_2.

Proof. If $\mu(H)$ contains $(\approx)$ $\mu(H_1)$ or $\mu(H_2)$, the conclusion follows from Benoist-Kobayashi.

Thus, WMA $\mu(H)$ diverges from both walls.

Either $\mu(\Gamma)$ diverges from one of the walls (done); or Γ has at least two ends.

We will see below that this is impossible.


proper
μ(Γ)
μ(H)
not proper
μ(Γ)
μ(H)

Prop. $G/H \underset{\text{homeo}}{\cong} K \times \mathbb{R}^d$, $d = \dim(AN) - \dim H$.

Proof. $G = KAN$, and $AN/H \underset{\text{homeo}}{\cong} \mathbb{R}^d$ [Mostow]

Cor. $\Gamma\backslash G/H$ *tessellation* $\Rightarrow \Gamma$ *has only one end.*

Proof. We have $d > 1$ (else H contains A or N).

Thm (Kobayashi). *If Γ acts properly on G/H, then*

$$\operatorname{cohodim}_{\mathbb{R}}(\Gamma) \le \dim(AN) - \dim H$$

with equality iff $\Gamma\backslash G/H$ is compact.

Cor. *If $\Gamma\backslash G/H$ is a tessellation, then*

$$\dim H \ge \min\{\dim H_1, \dim H_2\}.$$

Proof. We may assume Γ acts properly on G/H_1.

$$\begin{aligned}\dim(AN) - \dim H &= \operatorname{cohodim}_{\mathbb{R}}(\Gamma) \\ &\le \dim(AN) - \dim H_1\end{aligned}$$

Proof. $H^p\big(\Gamma; H^q(K;A)\big) = H^p\big(\Gamma; H^q(G/H;A)\big)$

$\Rightarrow H^{p+q}(\Gamma\backslash G/H;A)$ (spectral sequence)

For $N = \operatorname{cohodim}_{\mathbb{R}}(\Gamma)$, we have

$$H^N(\Gamma;A) \cong H^{N+\dim K}(\Gamma\backslash G/H;A)$$

so $N \leq \dim(\Gamma\backslash G/H) - \dim K$

$= \dim(AN) - \dim H$

with equality iff $\Gamma\backslash G/H$ is compact.

References

• Y. Benoist, Actions propres sur les espaces homogènes réductifs, *Ann. Math.* 144 (1996) 315–347.

• A.Iozzi & D.Witte, Tessellations of homogeneous spaces of SU$(2, n)$ (in preparation).

• T. Kobayashi, Discontinuous groups and Clifford Klein forms of pseudo-Riemannian homogeneous manifolds, in: B. Ørsted & H. Schlichtkrull, eds., *Algebraic and Analytic Methods in Representation Theory*, Academic Press, New York, 1997, pp. 99–165.

• B. Kostant, On convexity, the Weyl group, and the Iwasawa decomposition, *Ann. Sc. ENS.* 6 (1973) 413–455.

• F. Labourie, Quelques résultats récents sur les espaces localement homogènes compacts, in: P. de Bartolomeis, F. Tricerri and E. Vesentini, eds., *Manifolds and Geometry*, Symposia Mathematica, v. XXXVI, Cambridge U. Press, 1996.

• H.Oh & D.Witte, Cartan-decomposition subgroups of SO$(2, n)$ (preprint).

• H.Oh & D.Witte, Compact Clifford-Klein forms of homogeneous spaces of SO$(2, n)$ (preprint).

• H.Oh & D.Witte, New examples of compact Clifford-Klein forms of homogeneous spaces of SO$(2, n)$. *Internat. Math. Res. Notices* (to appear).

$$\mathrm{SO}(2,n) = \mathrm{Isom}\left(v_1 v_{n+2} + v_2 v_{n+1} + \sum\nolimits_{i=3}^{n} v_i^2\right)$$

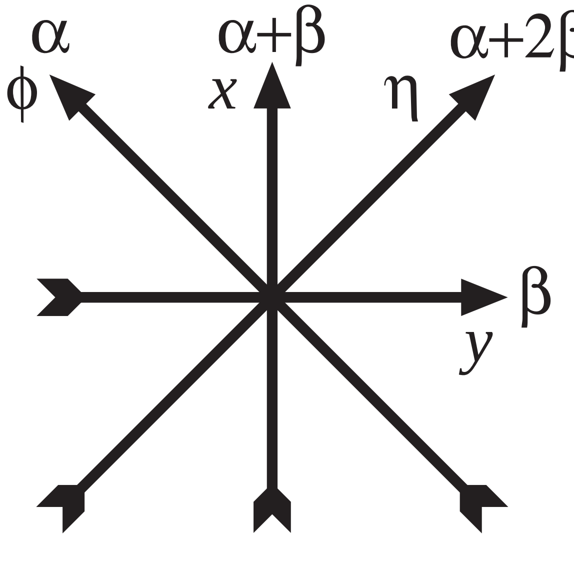
$$\mathrm{SU}(2,n) = \mathrm{Isom}\left(z_1 \overline{z_{n+2}} + z_2 \overline{z_{n+1}} + \sum\nolimits_{i=3}^{n} |z_i|^2\right)$$

$\mathfrak{su}(2,n)$:

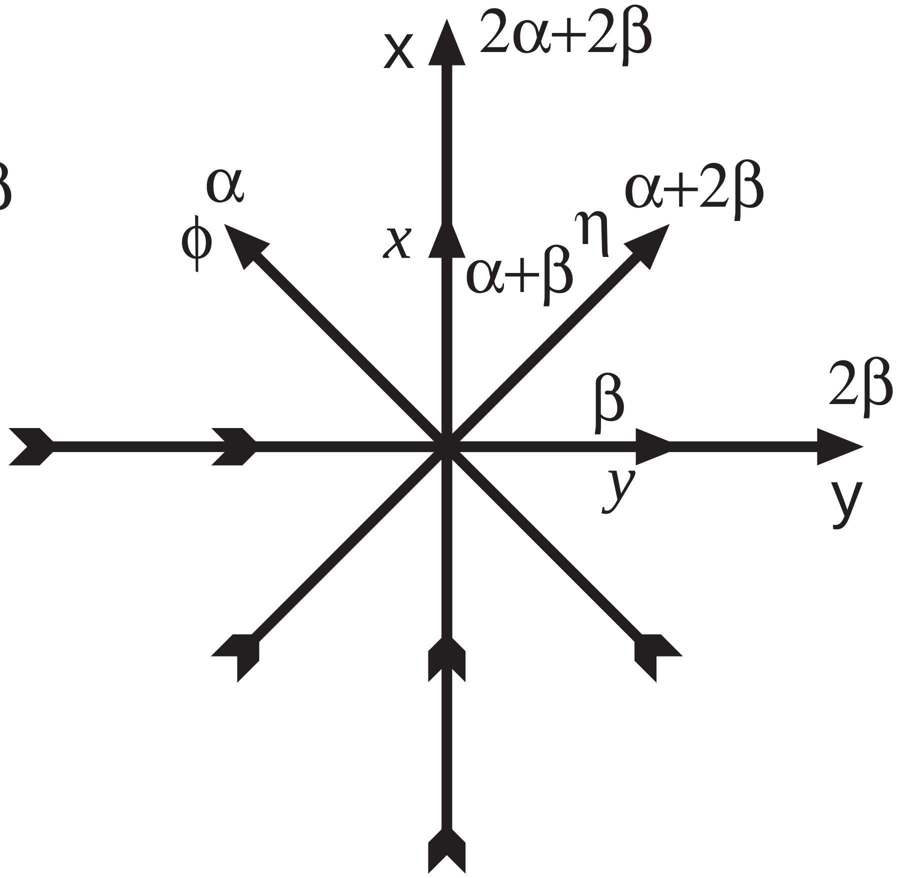
$$\mathfrak{a} + \mathfrak{n} = \left\{ \begin{pmatrix} t_1 & \phi & \overrightarrow{x} & \eta & \mathsf{x} \\ & t_2 & \overrightarrow{y} & \mathsf{y} & -\overline{\eta} \\ & & 0 & -\overrightarrow{y}^{*} & -\overrightarrow{x}^{*} \\ & & & -t_2 & -\phi \\ & & & & -t_1 \end{pmatrix} \right\}$$

$$t_1, t_2 \in \mathbb{R}, \qquad \overrightarrow{x}, \overrightarrow{y} \in \mathbb{C}^{n-2}, \qquad \phi, \eta \in \mathbb{C}, \qquad \mathsf{x}, \mathsf{y} \in i\mathbb{R}$$

$$\mathfrak{so}(2,n) = \mathfrak{su}(2,n) \cap \mathrm{Mat}_{n+2}(\mathbb{R})$$



Roots of $\mathfrak{so}(2,n)$



Roots of $\mathfrak{su}(2,n)$

Thm (Oh-Witte). $G = \mathrm{SO}(2,n)$, *n even.*
G/H has a tessellation iff

- $H \sim \mathrm{SO}(1,n) \cap AN$*; or*
- $H \sim H_B$

H_B is a generalization of $\mathrm{SU}(1,n/2) \cap AN$:

$$\mathfrak{h}_B = \left\{ \left(\begin{matrix} t & 0 & \overrightarrow{x} & \eta & 0 \\ & t & B(\overrightarrow{x}) & 0 & -\eta \\ & & \cdots & & \end{matrix} \right) \middle| \begin{matrix} \overrightarrow{x} \in \mathbb{R}^{n-2} \\ t, \eta \in \mathbb{R} \end{matrix} \right\}$$

$B \colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ has no real eigenvalues

Thm (Iozzi-Witte). $G = \mathrm{SU}(2,n)$, *n even.*
G/H has a tessellation iff

- $H \sim \mathrm{SU}(1,n) \cap AN$*; or*
- $H \sim \hat{H}_B$

$\hat{H}_B$ is a generalization of $\mathrm{Sp}(1,n/2) \cap AN$:

$$\hat{\mathfrak{h}}_B = \left\{ \left(\begin{matrix} t & 0 & \overrightarrow{x} & \eta & \mathsf{x} \\ & t & B(\overrightarrow{x}) & -\mathsf{x} & -\overline{\eta} \\ & & \cdots & & \end{matrix} \right) \middle| \begin{matrix} \overrightarrow{x} \in \mathbb{C}^{n-2} \\ t \in \mathbb{R} \\ \eta \in \mathbb{C} \\ \mathsf{x} \in i\mathbb{R} \end{matrix} \right\}$$

$B \colon \mathbb{C}^{n-2} \to \mathbb{C}^{n-2}$ anti-symplectic $\mathbb{R}$-linear, s.t.
$\{\overrightarrow{x}, B\overrightarrow{x}\}$ linearly independent over $\mathbb{C}$