Strictly convex norms on amenable groups

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Abstract. It is easy to see that any group with a left-invariant total order has a strictly convex norm. We use elementary measure theory and dynamics to show that the converse is true for amenable groups. (A similar argument had previously been used to show that left-orderable, amenable groups are "locally indicable.") This is joint work with Peter Linnell of Virginia Tech.

Question: ¿ ∃ a *strictly convex* norm $\rho \colon G \to \mathbb{R}$?

I.e., balls are strictly convex:

$\rho(g) < \rho(gh)$ or $\rho(g) < \rho(gh^{-1})$ (if $h \neq e$)

Summary of 1st lecture

act on $\mathbb{R}$ ⇔ left-inv't total order

⇒ strictly convex norm ⇔ locally inv't partial ord ⇔ extreme points

⇒ unique product $\not\Leftarrow$ ⇒ no zero divisors ⇒ torsion free

Question: Which implications can be reversed?

Theorem (Linnell-Morris)

∃ *strictly convex norm on G, and G is amenable*
⟹ *G acts on* $\mathbb{R}$ (*orientation-preserving, faithfully*).

Theorem (Linnell-Morris)

∃ *locally invariant partial order, and G is amenable*
⟹ *G has a left-invariant total order.*

Proof gets left-inv't total order by using *recurrence*.

1. amenability: G has a locally inv't partial order $\prec$ that is *recurrent*.
2. Positive cone of $\prec$ is $P = \{ g \mid g \succ e \}$. This is pos cone of a left-inv't total order.

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1) Recurrence

Defn. Topology of ptwise conv. on {loc inv't orders}:

$\prec_n \to \prec$ means $\forall g,h \in G$, $\forall$ large n, $g \prec_n h \iff g \prec h$.

Loc(G) is a compact metric space $\subset 2^{G\times G}$.

G acts (continuously) on Loc(G) by translation on right:

$a \prec^g b \iff ag^{-1} \prec bg^{-1}$

Poincaré Recurrence Thm

For each $g \in G$, ∃ *recurrent* $\prec \in$ Loc(G):

$\prec$ is an accumulation point of $\{ \prec^{g^n} \mid n \in \mathbb{Z}^+ \}$.

G *amenable* (& countable) ⟹ can reverse quantifiers:

$\exists \prec \in$ Loc(G) that is recurrent for all $g \in G$.

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1) $\exists \prec \in$ Loc(G) that is recurrent for all $h \in G$.

2) Positive cone $P = \{ g \mid g \succ e \}$.

Define $g \gg h \iff h^{-1}g \gg e \iff h^{-1}g \in P$.

- left-invariant: by definition ✓
- antireflexive: $e \notin P$ ✓
- total: $\forall g \neq e$, either $g \in P$ or $g^{-1} \in P$. ✓
- transitive: P is closed under multiplication.

Spse $g,h \succ e$, with $e \succ gh$.

Then $g \succ e \overset{\text{recurrence}}{\Longrightarrow} \exists n,\ gh^n \succ eh^n \succ e \succ gh$

$\overset{\text{recurrence}}{\Longrightarrow} \exists M,\ gh^{n-M} \succ gh^{1-M}$.

However, $g \succ gh$, so $g \prec gh^{-1} \prec gh^{-2} \prec gh^{-3}$

$\prec \cdots \prec gh^{n-M} \prec \cdots \prec gh^{1-M} \prec \cdots$ →←

Theorem (Linnell-Morris)

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Theorem (Linnell-Morris)

∃ *locally invariant partial order, and G is amenable*
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Exercise (Morris)

G has a *recurrent* left-invariant total order.

Proposition (P. Conrad, 1959)

G has a recurrent (Conradian) left-invariant total order
⟹ *G is locally indicable.*

Every f.g. subgrp of G has quotient isomorphic to $\mathbb{Z}$.

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bi-invariant total order $\not\Leftarrow$ ⇒ recurrent left-invariant total order $\not\Leftarrow$ ⇒ locally indicable

$\not\Leftarrow$ ⇒ act on $\mathbb{R}$ ⇔ left-inv't total order ⇒ strictly convex norm ⇔ locally inv't partial ord

⇒ extreme points ⇒ unique product $\not\Leftarrow$ ⇒ no zero divisors ⇒ torsion free

For amenable groups:

- "recurrent" ⇔ ⋯ ⇔ "strictly convex".
- other two non-reversibles still do not reverse
- what about the others?

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