

Strictly convex norms on amenable groups

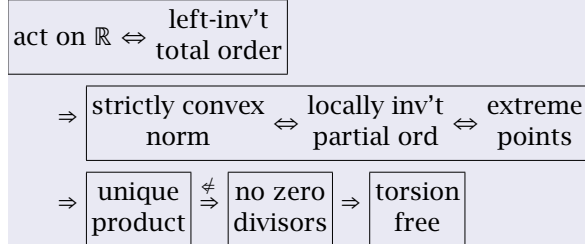
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Abstract. It is easy to see that any group with a left-invariant total order has a strictly convex norm. We use elementary measure theory and dynamics to show that the converse is true for amenable groups. (A similar argument had previously been used to show that left-orderable, amenable groups are "locally indicable.") This is joint work with Peter Linnell of Virginia Tech.

Question: \exists a *strictly convex* norm $\rho: G \rightarrow \mathbb{R}$?
 I.e., balls are strictly convex:
 $\rho(g) < \rho(gh)$ or $\rho(g) < \rho(gh^{-1})$ (if $h \neq e$)

Summary of 1st lecture



Question: Which implications can be reversed?

Theorem (Linnell-Morris)

\exists strictly convex norm on G , and G is amenable
 $\Rightarrow G$ acts on \mathbb{R} (orientation-preserving, faithfully).

Theorem (Linnell-Morris)

\exists locally invariant partial order, and G is amenable
 $\Rightarrow G$ has a left-invariant total order.

Proof gets left-inv't total order by using *recurrence*.

- **amenability:** G has a locally inv't partial order $<$ that is *recurrent*.
- **Positive cone** of $<$ is $P = \{g \mid g > e\}$. This is pos cone of a left-inv't total order.

1) Recurrence

Defn. Topology of ptwise conv. on $\{\text{loc inv't orders}\}$:

$$\langle_n \rightarrow \langle \text{ means } \forall g, h \in G, \forall \text{ large } n, \\ g \langle_n h \Leftrightarrow g \langle h.$$

$\text{Loc}(G)$ is a compact metric space $\subset 2^{G \times G}$.

G acts (continuously) on $\text{Loc}(G)$ by translation on right:

$$a \langle^g b \Leftrightarrow ag^{-1} \langle bg^{-1}$$

Poincaré Recurrence Thm

For each $g \in G$, \exists *recurrent* $\langle \in \text{Loc}(G)$:

\langle is an accumulation point of $\{\langle^{g^n} \mid n \in \mathbb{Z}^+\}$.

G *amenable* (& countable) \Rightarrow can reverse quantifiers:

$\exists \langle \in \text{Loc}(G)$ that is recurrent for all $g \in G$.

1) $\exists \langle \in \text{Loc}(G)$ that is recurrent for all $h \in G$.

2) **Positive cone** $P = \{g \mid g > e\}$.

Define $g \gg h \Leftrightarrow h^{-1}g \gg e \Leftrightarrow h^{-1}g \in P$.

- left-invariant: by definition \checkmark
- antireflexive: $e \notin P$ \checkmark
- total: $\forall g \neq e$, either $g \in P$ or $g^{-1} \in P$. \checkmark
- transitive: P is closed under multiplication.

Spse $g, h > e$, with $e > gh$.

Then $g > e \xrightarrow{\text{recurrence}} \exists n, gh^n > eh^n > e > gh$
 $\xrightarrow{\text{recurrence}} \exists M, gh^{n-M} > gh^{1-M}$.

However, $g > gh$, so $g < gh^{-1} < gh^{-2} < gh^{-3} < \dots < gh^{n-M} < \dots < gh^{1-M} < \dots \rightarrow$

Theorem (Linnell-Morris)

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Theorem (Linnell-Morris)

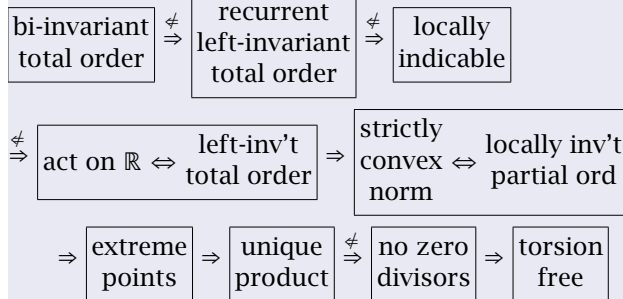
\exists locally invariant partial order, and G is amenable
 $\Rightarrow G$ has a left-invariant total order.

Exercise (Morris)

G has a *recurrent* left-invariant total order.

Proposition (P. Conrad, 1959)

G has a recurrent (Conradian) left-invariant total order
 $\Rightarrow G$ is *locally indicable*.
 Every f.g. subgrp of G has quotient isomorphic to \mathbb{Z} .



For amenable groups:

- "recurrent" $\Leftrightarrow \dots \Leftrightarrow$ "strictly convex".
- other two non-reversibles still do not reverse
- what about the others?

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