Using recurrence to study symmetries of the real line

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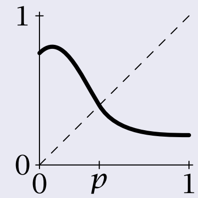
The real line has very few symmetries (just translations and a reflection). However, if we allow the line to be stretched (as is usual in topology), then a large group of symmetries arises. The study of this group leads naturally to the theory of left-invariant orderings. (A linear ordering of the elements of a group G is "left-invariant" if $ab \prec ac$ whenever $b \prec c$.) Orderings that are "nearly" right-invariant are sometimes provided by the Poincaré Recurrence Theorem (a fundamental result in the theory of dynamical systems), and this leads to short proofs of theorems that would be difficult or impossible to prove by algebraic methods.

Recurrence

Calculus (Intermediate Value Theorem)

Suppose f is a continuous function from $[0,1]$ to $[0,1]$.
Then $\exists p$, s.t. $f(p) = p$.
p is a *fixed point* of f.
Every continuous transformation of $[0,1]$ has a fixed point.



Brouwer Fixed-Point Theorem

Same true in n dimensions: (square, cube, hypercube, ...)
Every cont transformation of $[0,1]^n$ has a fixed pt.

A cont transf of a circle might not have a fixed pt.

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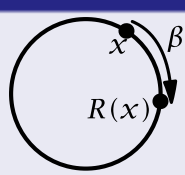
Example

Choose $\beta = \sqrt{2}$ (irrational).
R: rotate circle β degrees (clockwise).

No point is fixed.



Exercise

Points are recurrent — almost fixed by a power of R: the distance from $R^n(p)$ to p is less than ϵ.

Fact

Every cont transf of a circle has a recurrent pt.

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Proposition (Poincaré Recurrence Theorem)

If f is a cont transformation of a set X (closed & bounded) then f has recurrent points.

Idea of proof.

Subset $C \neq \emptyset$ of X is *invariant* if $f(C) \subseteq C$. E.g., X.
Choose C to be *minimal* inv't closed subset.
Exercise: Every point in C is recurrent.

Hint: $\overline{\{ f^n(p) \mid n \in \mathbb{N} \}}$ is closed and inv't, so all of C.) □

Typically: some points recurrent, others not.

Symmetry (Group Theory)

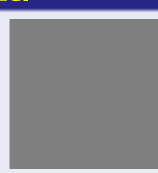
Definition

Symmetry of an object: transformation that returns the object to the space it originally occupied.

Only consider *orientation-preserving* symms. **Ignore** mirror symmetry.

The symmetries of a square in the plane.

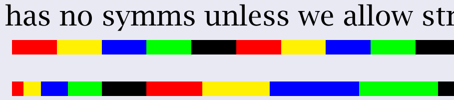
- quarter rotation order 4
- half rotation order 2
- 3/4 rotation order 4
- full rotation order 1 "trivial"



Note: Symmetries of any object form a group.

Symmetries of a line segment

Line seg has no symms unless we allow stretching.



A line segment has infinitely many symmetries.

Proposition

Line seg has no nontrivial *symmetry of finite order.*

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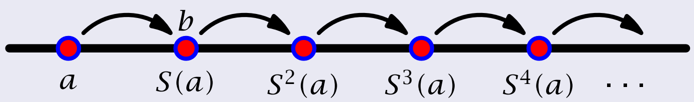
Proof.



Consider a nontrivial symmetry. Call it S.

Choose a point that is moved by S. Call it a.
Let's assume a moves (to the right) to some point $S(a)$.

Every time we perform S, pt a moves farther right.
No matter how many times we perform S,
we don't get back to where we started.

So S has infinite order. □

Problem

Understand the group of symmetries of $\mathbb{R}$. Symm($\mathbb{R}$)
What are its subgroups?
Find all G, *s.t.* $\exists$ *(nontrivial) homo* $\varphi\colon G \to \mathrm{Symm}(\mathbb{R})$.
Say G *acts on* $\mathbb{R}$.

Example

$\mathbb{Z}$ acts on $\mathbb{R}$. $(\ \varphi(n) = S_n,\quad S_n(x) = x + n\ )$
verify $\varphi(m+n) = \varphi(m)\varphi(n)$

Corollary

$\exists$ *homo* $G \longrightarrow$ *abelian* (no el'ts of finite order) $\implies$ G *acts on* $\mathbb{R}$.

Converse true? (Assume G countable — finitely generated.)

Converse?

G acts on $\mathbb{R}$ $\overset{?}{\implies}$ $\exists$ homo $G \longrightarrow$ abelian (no el'ts of finite order)

Proposition (Hölder, 1901)

G acts on $\mathbb{R}$, no el't has a fixed pt $\implies$ *G abelian.*

Proof.

Fix $a(0) > 0$. May assume $a(x) = x + 1$ (no f.p.).
$m \le g(0) < m + 1 \Rightarrow a^m(x) \le g(x) < a^{m+1}(x)$
So $gh(0) \ge a^m(h(0)) = m + h(0) > g(0) + h(0) - 1$.
Conclude $gh(0) = g(0) + h(0) \pm 1$. ("nearly additive")
Define $\displaystyle \varphi(g) = \lim_{k\to\infty} \frac{g^k(0)}{k}$.
Exercise. φ is a homomorphism $G \hookrightarrow \mathbb{R}$. □

Proposition (Hölder, 1901)

G acts on $\mathbb{R}$, no el't has a fixed pt $\implies$ *G abelian.*

Proposition

G acts on $\mathbb{R}$ (faithful) $\implies$ *G has a left-inv't order.*

Definition

$a \prec b \iff a(0) < b(0)$ or … (break ties)

Exercise

1. $\prec$ *is a total order on G.*
2. $\prec$ *is left-invariant.* $(a \prec b \implies ca \prec cb)$

Hint: orientation-pres, so increasing, so $x < y \Rightarrow c(x) < c(y)$.

Question

¿ *G acts on $\mathbb{R}$ $\overset{?}{\implies}$ $\exists$ homo $\varphi\colon G \to$ abelian ?*

Proposition (Hölder, 1901)

G acts on $\mathbb{R}$, no el't has a fixed pt $\implies$ *G abelian.*

Proposition

G acts on $\mathbb{R}$ (faithful) $\implies$ *G has a left-inv't order.*

Exercise

G acts on $\mathbb{R}$, no el't has a fixed pt $\left(\begin{matrix}\forall x, g(x) > x\\ \text{or}\\ \forall x, g(x) < x\end{matrix}\right)$
$\iff$ *G has bi-invariant order.*
$a \prec b \implies ca \prec cb$ and $ac \prec bc$

Want to get a bi-inv't order from a left-inv't order.

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Definition

Assume: $\prec$ is left-invariant, $g \in G$.
Define $a \prec^g b \iff ag \prec bg$.
Then $\prec^g$ is left-invariant.
G acts on the set of left-invariant orders.

Want $\prec^g = \prec$. I.e., $\prec$ is a *fixed point*.
Unfortunately, g might not have a fixed point.
Poincaré Recurrence Thm: g has a *recurrent* point.

Theorem

$\prec$ left-inv't order on G, recurrent for all $g \in G$
$\implies$ *$\exists$ homo $\varphi\colon G \to$ abelian*

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$\implies \exists$ homo $\varphi: G \to$ abelian

Poincaré Recur Thm: $\forall g, \exists \prec$, $\prec$ is recurrent for g.

Orders that are recurrent for g_1 may not be recurrent for g_2.

To apply thm, need to *reverse the quantifiers*.

Exercise (hard?)

El'ts of an abelian group all agree on a recurrent pt:
G ~~abelian~~ $\implies \exists p$, s.t. p is recurrent for all $g \in G$.
~~*solvable*~~ *amenable* *(¿ G has no free subgroups ?)*

Corollary

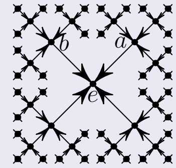
G amenable, acts on $\mathbb{R}$ $\implies \exists$ homo $\varphi: G \to$ abelian.

What is *amenable* ?

Example

Free group $F_2 = \langle a, b \rangle$. Every el't starts with \$1:

$f_0(g) = 1, \quad \forall g \in F_2$.



Everyone passes their dollar to the person next to them who is closer to the identity:

$f_1(g) = \$3$ (except $f_1(e) = \$5$).

Everyone richer, & money only moved bdd distance.

Definition. This is a *Ponzi scheme* on F_2.

Definition

G is *amenable* $\iff \nexists$ Ponzi scheme on G.