Introduction to Ratner's Theorems on Unipotent Flows

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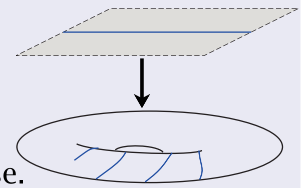
Abstract. Let f be the obvious covering map from Euclidean n-space to the n-torus. It is well known that if L is any straight line in n-space, then the closure of $f(L)$ is a very nice submanifold of the n-torus. In 1990, Marina Ratner proved a beautiful generalization of this observation that replaces Euclidean space with any Lie group G, and allows L to be any subgroup of G that is "unipotent." We will discuss the statement of this theorem and related results, some of the ideas in the proofs, and a few of the important consequences.

Elementary example

Let M = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
- covering map $f\colon \mathbb{R}^2 \to M$
- H = line in $\mathbb{R}^2$.

If the slope of H is irrational, it is classical that $f(H)$ is dense.



Exercise. Let M = n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$
- covering map $f\colon \mathbb{R}^n \to M$
- H = vector subspace of $\mathbb{R}^n$.

Closure $\overline{f(H)} = f(S)$ is a torus $\mathbb{T}^k$ ($\exists$ subspace S of $\mathbb{R}^n$).

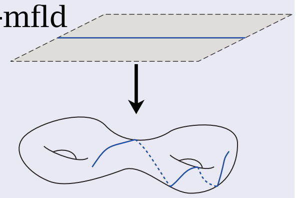
The closure of $f(H)$ is a very nice submanifold of M.

Example in Riemannian geometry

Example

Let M = compact, hyperbolic n-mfld
- covering map $f\colon \mathbb{H}^n \to M$
- line $\mathbb{H}^1 \hookrightarrow \mathbb{H}^n$

Closure $\overline{f(\mathbb{H}^1)}$ can be a fractal.



Consequence of Ratner's Thm [Shah, Payne]

$\mathbb{H}^2 \subset \mathbb{H}^n \implies \overline{f(\mathbb{H}^2)} = f(\mathbb{H}^k)$ is a submfld of M
(immersed, maybe not embedded).

(Similar for other locally symmetric spaces.)

Recall

- $f\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ • H = vector subspace of $\mathbb{R}^n$

$\implies \overline{f(H)} = f(S)$, $\exists$ vector subspace S of $\mathbb{R}^n$.

- $\mathbb{R}^n$ is a **Lie group** (group & manifold)
- subgroup $\mathbb{Z}^n$ is a **lattice**
  (discrete and $\mathbb{R}^n/\mathbb{Z}^n$ has finite volume)

Generalization (Ratner's Theorem) [1991]

Replace:
- $\mathbb{R}^n$ with any Lie group G
- $\mathbb{Z}^n$ with any lattice Γ in G
- H with any subgroup of G
  that is generated by "unipotent" elements
- S with a closed subgroup of G

Homogeneous Dynamics

study of dynamical systems on homogeneous spaces

finite-volume homogeneous space G/Γ:
- G = Lie group = closed subgroup of $\mathrm{SL}(n,\mathbb{R})$
  $\{\, n \times n$ mats with $\mathbb{R}$ entries, $\det = 1 \,\}$
  = group & manifold
- Γ = ~~closed subgroup of G~~ lattice in G

Coset space G/Γ is a manifold of finite volume.

"dynamical system" = action of subgroup H of G
$h\colon G/\Gamma \to G/\Gamma \qquad h(x\Gamma) = hx\Gamma$

E.g., understand the orbit $Hx\Gamma$ in G/Γ

Ratner's Theorem [1991]

- finite-volume homogeneous space G/Γ
- subgroup H gen'd by **unipotent** elements

$\Rightarrow \overline{Hx\Gamma} = Sx\Gamma$ for some closed subgroup S of G.

Also: $H \subseteq S$ and $(x\Gamma x^{-1}) \cap S$ is latt in S if H conn.

Unipotent matrices are conjugate to an element of
$$\left\{ \begin{bmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix} \right\} \subset \mathrm{SL}(n,\mathbb{R}).$$

Exer. u unip $\Leftrightarrow (u - I)^n = 0 \Leftrightarrow$ char poly $(x-1)^n$
$\Leftrightarrow$ only eigenvalue is 1 $\Rightarrow$ *not* diag'ble (unless $u = I$).

Applications of Ratner's Theorem

Example (Shah [1991], Payne [1999])

$M = \mathbb{H}^n/\Gamma, \quad f\colon \mathbb{H}^n \to M, \quad \mathbb{H}^2 \subset \mathbb{H}^n$

$\implies \overline{f(\mathbb{H}^2)}$ is (immersed) submanifold of M.

Idea of proof.

- $\mathbb{H}^n = K\backslash \mathrm{SO}(1,n)^\circ = K\backslash G \quad \Rightarrow \pi\colon G/\Gamma \to M$
- $f(\mathbb{H}^2) = \pi(\mathrm{SO}(1,2)^\circ x\Gamma) = \pi(Hx\Gamma)$

$\overline{f(\mathbb{H}^2)} = \overline{\pi(Hx\Gamma)} = \pi(\overline{Hx\Gamma}) \overset{\text{Ratner}}{=} \pi(Sx\Gamma)$

$=$ immersed submanifold. □

$H = \mathrm{SO}(1,2)^\circ \cong \mathrm{SL}(2,\mathbb{R}) = \left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right] = \left\langle \left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right] \right\rangle$

is generated by unipotent elements

"Oppenheim Conjecture" (Margulis [1987])

Let Q be a real quadratic form in $n \ge 3$ variables

(e.g., $x^2 - \sqrt{2}xy + \sqrt{3}z^2$).

Then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$

unless $\approx$ $\mathbb{Z}$-coefficients, or definite, or degenerate.

Proof for $n = 3$.

Let $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, and

$H = \mathrm{SO}(Q) = \{\, h \in \mathrm{SL}(3,\mathbb{R}) \mid Q(h\vec{x}) = Q(\vec{x}) \,\}$.

Ratner: $\overline{H\Gamma} = S\Gamma$, for some subgroup $S \supseteq H$.

Algebra: H is maximal in G, so $S = H$ or G.

$S = H \implies Q$ has $\mathbb{Z}$-coefficients $(\approx)$

So $H\Gamma$ is dense in G. Therefore

$\overline{Q(\mathbb{Z}^3)} \supset Q(\overline{H\Gamma}\mathbb{Z}^3) = Q(G\mathbb{Z}^3) = Q(\mathbb{R}^3) = \mathbb{R}$. □

Example (Shah [1998])

Γ, Λ lattices in $G = \mathrm{SL}(n,\mathbb{R})$ (any simple Lie group)

$\Rightarrow$ every Λ-orbit on G/Γ is either finite or dense.

Proof. Ratner: $\overline{\Lambda x\Gamma} = Sx\Gamma$, and $S \supseteq \Lambda$.

Borel Density Theorem: $\Lambda \not\subset$ conn, proper subgrp.

$\therefore \Lambda$ normalizes H connected $\Rightarrow N_G(H) = G$

$\Rightarrow$ (bcs G simple) $H = \{e\}$ or G.

$S^\circ = \{e\} \Rightarrow S/\Lambda$ finite. $\quad S^\circ = G \Rightarrow S/\Lambda = G/\Lambda$. □

Gap: Ratner's Thm requires Λ to be gen'd by unips.

Exer. Fix by using fact that G is gen'd by unips.

Hint: Look at orbit of $\{(g,g)\}$ on $(G \times G)/(\Gamma \times \Lambda)$.

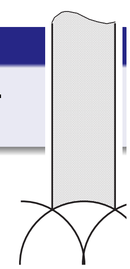
G simple $\Rightarrow \{(g,g)\}$ is max'l conn subgrp.

A Key Idea in the Proof

Example

$G = \mathrm{SL}(2,\mathbb{R}) = \{\, 2 \times 2$ real mat's of det $1 \,\}$.

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Then Γ is a lattice in G.



Other choices of Γ can make G/Γ compact.

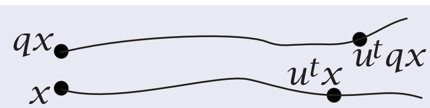
Definition

Define $\quad u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$.

Each is a homomorphism from $\mathbb{R}$ to $\mathrm{SL}(2,\mathbb{R})$.

u^t is a **unipotent** one-parameter subgroup.

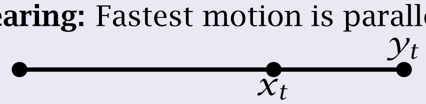
$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$



$d(x, qx) = \|q\|. \qquad d(u^t x, u^t qx) = \|u^t q u^{-t}\|.$

$$u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

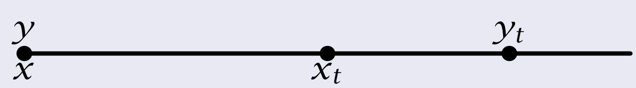
Shearing: Fastest motion is parallel to the orbits.



Cor. If $x \approx y$, then $\exists t$, $y_t \approx x_{t+1}$.

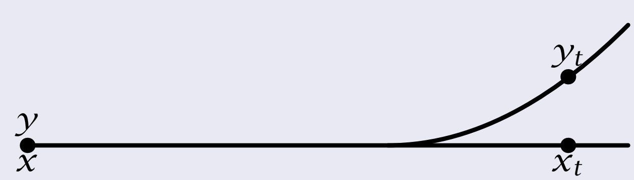
Shearing

Fastest motion is parallel to the orbits.



Contrast: $a^t q a^{-t} = \begin{bmatrix} \alpha & \beta e^{2t} \\ \gamma e^{-2t} & \delta \end{bmatrix}$

Fastest motion is transverse to the orbits.



Further reading (and references to primary sources)

chapter of **forthcoming book on arithmetic grps**
free PDF file on my web page (or the arxiv)
http://people.uleth.ca/~dave.morris
/books/IntroArithGroups.html

my book: ***Ratner's Theorems on Unipotent Flows***
free PDF file on my web page (or the arxiv)
http://people.uleth.ca/~dave.morris
/books/Ratner.html
published by University of Chicago Press (2005)
$\ast\ast\ast$ available from Amazon $\ast\ast\ast$

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Part 2: Variations of Ratner's Theorem and Additional Ideas of the Proof

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Ratner's Theorem [1991]

- G = Lie group = closed subgroup of $\mathrm{SL}(n,\mathbb{R})$
- Γ = lattice in G = discrete subgroup with $\mathrm{vol}(G/\Gamma) < \infty$
- $X = G/\Gamma$
- H = subgroup of G **gen'd by unips** $\subseteq \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\Rightarrow \overline{Hx} = Sx$ for some subgroup S of G.

Conjecture (Shah [1998])

Suffices to assume ***Zariski closure*** $\overline{\overline{H}}$ *gen'd by unips*

$\approx$ *smallest* almost *conn subgrp of* G *that* $\supseteq H$.

Progress by Benoist-Quint [2013]

True if $\overline{\overline{H}} = \mathrm{SL}(n,\mathbb{R})$ (or semisimple) and H/H° is finitely gen'd

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Ratner's Theorem

- G = Lie group = closed subgroup of $\mathrm{SL}(n,\mathbb{R})$
- Γ = lattice in G = discrete subgroup with $\mathrm{vol}(G/\Gamma) < \infty$
- $X = G/\Gamma$
- H = subgroup of G **gen'd by unips** $\subseteq \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\Rightarrow \overline{Hx} = Sx$ for some subgroup S of G.

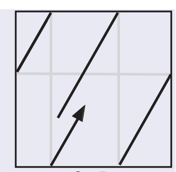
Derived from special case where $H = \{u^t\}$ is 1-dim'l

Ratner's Theorem

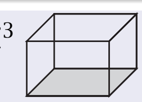
$\{u^t\}$ unipotent $\Rightarrow \overline{\{u^t\}x} = Sx$, $\exists$ subgrp S of G.

Also: $S \supseteq \{u^t\}$ and Sx has finite S-inv't volume.

Example



Let X = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
Any $v \in \mathbb{R}^2$ defines a flow on $\mathbb{T}^2$:
$x_t = x + tv \doteq u^t x$
If the slope of v is irrational,
it is classical that every orbit is dense (& unif dist).
Also: Lebesgue is the only inv't probability measure.



$v = (a,b,0)$ defines a flow on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$
and $\mathbb{T}^2 \times \{0\}$ is invariant.
Lebesgue on $\mathbb{T}^2 \times \{0\}$ is an invariant measure.

$\forall v$, any ergodic inv't probability meas for flow on $\mathbb{T}^3$ is Lebesgue meas on some subtorus $\mathbb{T}^k$ $(0 \le k \le 3)$.

Ratner's Theorem on Orbit Closures

$\{u^t\}$ unipotent $\Rightarrow \overline{\{u^t\}x} = Sx$, $\exists$ subgrp S of G.

Ratner's Equidistribution Theorem

$\{u^t\}x$ is ~~dense~~ **equidistributed** in Sx:

$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\,\mathrm{vol} \quad \text{for } f \in C_c(G/\Gamma)$$

where vol = S-invariant volume form on Sx.

Ratner's Measure-Classification Theorem

Any ergodic u^t-invariant probability measure on G/Γ is S-invariant volume form on some Sx.

Generalized to p-adic groups. ¿characteristic p?
[Ratner, Margulis-Tomanov] [Einsiedler, Ghosh, Mohammadi]

measure-classification $\Rightarrow$ equidist $\Rightarrow$ orbit-closure

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measure-classification $\Rightarrow$ equidistribution

$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\,\mathrm{vol}$$

Easy case

μ = **unique** u^t-inv't probability meas on X (compact)
$\Rightarrow$ every u^t-orbit is equidistributed.

Proof. $M_T(f) := \frac{1}{T}\int_0^T f(u^t x)\,dt$
$M_T \colon C_c(X) \to \mathbb{C}$ positive linear functional

Riesz Rep Thm: $M_T \in \mathrm{Meas}(X) = \{$prob meas on $X\}$.

Banach-Alaoglu: $\mathrm{Meas}(X)$ weak* compact
$\Rightarrow$ subsequence $M_{T_n} \to M_\infty$ u^t-invariant.

Therefore $M_\infty = \mu$. So $M_T(f) \to \mu(f) = \int f\,d\mu$. □

Example: $G = SL(2,\mathbb{R})$

Theorem (Furstenberg [1973])

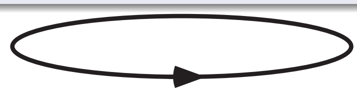
$SL(2,\mathbb{R})/\Gamma$ compact ⇒ the only invariant probability measure for $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ is the ordinary volume.

Corollary

Every u^t-orbit is equidistributed ($\therefore$ *dense*).

Theorem (Dani [1978])

$SL(2,\mathbb{R})/\Gamma$ noncompact ⇒ other ergodic u^t-invariant measure is the measure supported on a closed orbit.

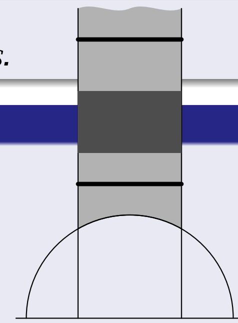


Theorem (Dani [1978])

$SL(2,\mathbb{R})/\Gamma$ noncompact ⇒ other ergodic u^t-invariant measure is the measure supported on a closed orbit.

General fact

Every invariant measure is a combination of ergodic measures.



Corollary

Every u^t-inv't probability meas on $SL(2,\mathbb{R})/\Gamma$ is a combination of Lebesgue measure with measures on cylinders of closed orbits.

Ratner's Measure-Classification Theorem

Ergodic u^t-inv't prob meas is S-inv't vol on Sx $(\exists S, x)$

Fix x_0. Sx_0 has finite S-inv't volume
⇔ gSx_0 has finite gSg^{-1}-inv't volume.
If $u^t \subseteq gSg^{-1}$, this provides a u^t-invariant measure.
- gSx_0 is analogue of a closed orbit.

$N(u^t, S) := \{ g \in G \mid u^t \subseteq gSg^{-1} \}$ (~submanifold).
$N(u^t, S)x_0 \subseteq G/\Gamma$ ~ cylinder of closed orbits:
each gSx_0 has an u^t-invariant prob meas.

Lem. $\exists$ only countably many possible subgroups S.

Cor. Every u^t-inv't prob meas on G/Γ is a combo of measures on these countably many "tubes."

What does "ergodic" mean?

Defn. μ **ergodic** u^t-inv't meas:
every u^t-inv't meas'ble func is constant (a.e.)

Pointwise Ergodic Theorem

μ ergodic ⇔ a.e. u^t-orbit is μ-equidistributed.

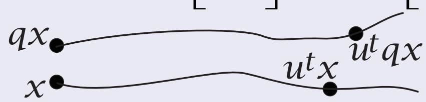
$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_X f\,d\mu \quad \text{a.e.}$$

Exer. $\mu_1 \neq \mu_2$, both ergodic
⇒ μ_1 and μ_2 are mutually singular.
$\mu_1(C_1) = 1,\ \mu_2(C_2) = 1,\ C_1 \cap C_2 = \varnothing$

Hint. $\mu_1 = f\mu_2 + \mu_{sing}$ and f is u^t-inv't.

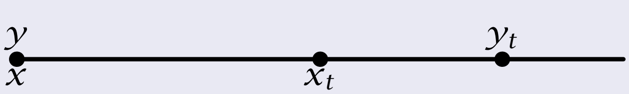
Proof of Measure-Classification

$$G = SL(2,\mathbb{R}), \quad u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad a^s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}.$$



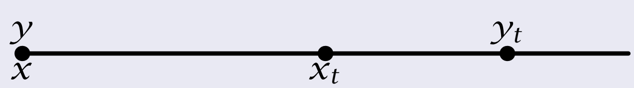
$$u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Shearing: Fastest motion is parallel to the orbits.



Shearing

Fastest motion is parallel to the orbits.

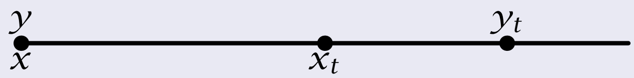


Contrast: $a^s q a^{-s} = \begin{bmatrix} \alpha & \beta e^{2s} \\ \gamma e^{-2s} & \delta \end{bmatrix}$

Fastest motion is transverse to the orbits.

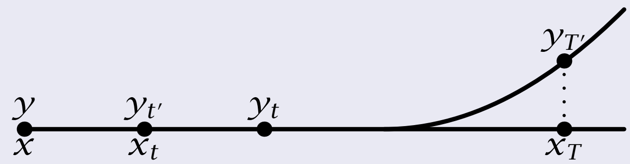
Shearing

Fastest motion is parallel to the orbits.



Key idea in proof of Measure-Classification

Ignore motion along the orbit, and look at the *transverse* motion perpendicular to the orbit.



Example

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \; u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Fastest motion is along $\{u^t\}$.

Ignoring this, largest terms are diagonal (in $\{a^s\}$)

Observation

$a^s \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} a^{-s} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$: a^s **normalizes** $\{u^t\}$.

Proposition

For action of a unipotent subgroup, the fastest transverse divergence is along the ***normalizer****.*

Prop. Fastest transverse div is along **normalizer**.

Corollary (Step 1 of Ratner's Proof)

μ is u^t-inv't and ergodic (*and* …) $\Rightarrow$ *μ is a^s-inv't.*

Proof.

a^s normalizes $u^t \Rightarrow u^t(a^s\mu) = a^s(u^{t'}\mu) = a^s\mu$.
μ and $a^s\mu$ are two *different* ergodic measures
$\Rightarrow$ live on disjoint u^t-invariant sets C and a^sC.
Assume $d(C, a^sC) > \epsilon$.
For $x \approx y$ in C: $C \ni u^t x \approx a^s u^{t'} y \in a^s C$ $(\exists t, t')$.
$\Rightarrow d(C, a^sC) < \epsilon$. $\to\leftarrow$ □

Step 2: entropy calculation $\Rightarrow \mu$ inv't under $\begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$.

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Part 3: Completion of the Proof of an easy special case of Ratner's Measure-Classification Theorem

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Measure-Classification Theorem [Furstenberg, Dani]

$G = SL(2,\mathbb{R})$, Γ = lattice in G, $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$,
μ = ergodic u^t-inv't probability measure on G/Γ
$\implies$ μ is Lebesgue measure or on closed orbit.

Step 1 of the proof (yesterday)

Shearing: fastest motion is parallel to the orbits.

Prop. Fastest **transverse** motion is along norm'zer.

Cor. μ is inv't under $a^s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}$ unless supported on closed u^t-orbit.

Step 2: μ is inv't under $\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$. *entropy calculation*

Step 2: Entropy calculation

What is entropy?

Fix **partition** $\mathcal{P} = \{A_1, \ldots, A_n\}$ of X. Let $p_i = \mu(A_i)$.

Suppose x is an <u>unknown</u> point in X.
Learning $x \in A_i$ gives us information:
bits of info is $|\log(\mu(A_i))| = |\log p_i|$.

Expected # bits info is $\sum_i p_i |\log p_i|$
= the **entropy** of $\mathcal{P}$ $= h_\mu(\mathcal{P}) \geq 0$.

For $T\colon X \to X$, **entropy** $h_\mu(T)$
= growth rate from $x, Tx, T^2x, \ldots, T^n x$.
$\mathcal{P}_n = \{A_i\} \vee \{T^{-1}A_i\} \vee \cdots \vee \{T^{-n}A_i\}$
$h_\mu(T) := \lim_{n\to\infty} h_\mu(\mathcal{P}_n)/n$. (usually independent of $\mathcal{P}$)

$h_\mu(T) := \lim_{n\to\infty} h_\mu(\mathcal{P}_n)/n \geq 0$

Eg. $T(x) = x + \alpha \pmod 1$ (with α irrational).
$[0,1) = [0,1/2) \cup [1/2,1)$
$\Rightarrow \#\mathcal{P}_n \leq 2 + \#\mathcal{P}_{n-1} \leq n$
$\Rightarrow h_{leb}(\mathcal{P}_n) \leq \log 2n$
$h_{leb}(T) \leq \lim_{n\to\infty} \frac{1}{n} \log 2n = 0$.

More generally:

Prop. $T \in \mathrm{Isom}(X) \implies h_\mu(T) = 0$.

no distances are stretched $\implies$ entropy is 0

amount of stretching = entropy for diffeos

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amount of stretching = entropy for diffeos

Pesin Entropy Formula [1977]

- *T = vol-pres diffeo of manifold M* (cpct, smooth)
- *tangent bundle $\mathcal{T}M = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$* ($T$-inv't), $\forall \xi \in \mathcal{E}_i$, $\|T(\xi)\| = \tau_i \|\xi\|$.

Then $h_{\mathrm{vol}}(T) = \sum_{\tau_i > 1} (\dim \mathcal{E}_i) \log \tau_i$.

Example (entropy of geodesic flow)

Recall $a^s q a^{-s} = \begin{bmatrix} \alpha & \beta e^{2s} \\ \gamma e^{-2s} & \delta \end{bmatrix}$. $h_{\mathrm{vol}}(a^s) = 2|s|$

$\mathcal{T}M = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}$

Theorem (Ledrappier-Young [1985]**)**

- *T = meas-pres diffeo of manifold M*
- *tangent bundle $\mathcal{T}M = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$* ($T$-inv't), $\forall \xi \in \mathcal{E}_i$, $\|T(\xi)\| = \tau_i \|\xi\|$.
- *H-orbits are tangent to $\bigoplus_{\tau_i > 1} \mathcal{E}_i$*
- $\eta = \sum_{\tau_i > 1} (\dim \mathcal{E}_i) \log \tau_i$

Then $h_\mu(T) \leq \eta$. *Equality $\Leftrightarrow$ μ is H-inv't.*

"measure of maximal entropy is nice"

Idea. If supp μ misses directions that are stretched,
they do not contribute as much as they should.
To exploit all directions (along the H-orbits),
μ must be Lebesgue on every H-orbit.
So μ is H-invariant.

Theorem (Ledrappier-Young [1985])

If H-orbits tangent to the expanding directions of T, then $h_\mu(T) \le \eta =$ *total stretching.*

Equality $\Leftrightarrow$ μ *is H-inv't.*

Cor. Suppose μ is a^s-inv't on $\mathrm{SL}(2,\mathbb{R})/\Gamma$.
Then $h_\mu(a^s) \le 2|s|$, with equality iff μ is u^t-inv't.

Step 2. μ inv't under u^t and $a^s \implies \mu =$ Lebesgue.

Proof.

μ is u^t-inv't $\implies h_\mu(a^s) = 2|s| \implies h_\mu(a^{-s}) = 2|s|$

$\implies \mu$ is invariant under $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = v^r$.

μ is invariant under $\langle u^t, a^s, v^r \rangle = \mathrm{SL}(2,\mathbb{R})$. □

Measure-classification ⇒ Equidistribution

Show $M_T(f) := \frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\mathrm{vol}$ $\quad \exists S$

Measure-Classification.

- Each ergodic measure is vol_{Sy}.
- Every inv't meas is $(\approx)$ $\sum_i \mathrm{vol}_{S_i y_i}$.

Recall that $M_T \in \mathrm{Meas}\,X$ and $\mathrm{Meas}(X)$ is compact.
Need to show only acc pt M_∞ of $\{M_T\}$ is some vol_{Sy}.

Key to Proof. Show $M_\infty(Sy) \ne 0 \Rightarrow \{u^t x\} \subseteq Sy$.

$\therefore \{u^t x\} \subseteq Sx$ and $\dim S$ minimal $\Rightarrow M_\infty = \mathrm{vol}_{Sx}$.

Claim. $d(u^t x, Sy)^2$ is polynomial function of t.

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Taylor series: $\log u = \sum_{k=1}^n (-1)^{k+1}\frac{1}{k}(u-I)^k$
So $u^t = \exp(t \log u) = \sum_{k=1}^n \frac{1}{k!}\, t^k\, (\log u)^k$.

Each matrix entry of u^t is polynomial function.

Linearization (Dani-Margulis [1993])

Can show $S \doteq \overline{\overline{S}}$, so

$\exists$ homo $\rho\colon G \to \mathrm{SL}(D,\mathbb{R})$, and $\vec{v} \in \mathbb{R}^D$,

such that $S = \mathrm{Stab}_G(\vec{v})$. (Chevalley's Theorem)

Write $x = g\Gamma$, and assume $y = e\Gamma$.

$d(u^t x, Sy)^2 \doteq d(u^t g, S)^2 \doteq d(u^t g v, v)^2$.

u^t is polynomial func of t $\Rightarrow u^t g v$ is polynomial
$\Rightarrow d(u^t g v, v)^2$ is polynomial.

Key to Proof. Show $M_\infty(Sy) \ne 0 \Rightarrow \{u^t x\} \subseteq Sy$.

We know $d(u^t x, Sy)^2$ is poly func of t of degree N.

$f \in C_c(X)_{\le 1}$, supported in δ-neigh of Sy,
such that $M_T(f) > 0.01$.

$$0.01 < M_T(f) = \frac{1}{T}\int_0^T f(u^t x)\,dt$$
$$\le \frac{1}{T}\ell(\{\, t \mid f(u^t x) \ne 0 \,\})$$
$$\le \frac{1}{T}\ell(\{\, t \mid d(u^t x, Sy) < \delta \,\})$$

$d(u^t x, Sy)^2$ is poly that is $< \delta$ on 1% of $[0,T]$
$\Rightarrow d(u^t x, Sy)^2 < \epsilon$ on $[0,T]$.
Let $T_k \to \infty$: $\quad d(u^t x, Sy) = 0$ for all t.
So $\{u^t x\} \subseteq Sy$. □