Arithmetic subgroups whose representations all map into $\mathrm{GL}_n(\mathbb{Z})$

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Suppose Γ is an arithmetic subgroup of a semisimple Lie group G. For any finite-dimensional representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$, a classical paper of J. Tits determines whether $\rho(\Gamma)$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Combining this with a well-known surjectivity result in Galois cohomology provides a short proof of the known fact that every G has an arithmetic subgroup Γ, such that the containment is true for *every* representation ρ. We will not assume the audience is acquainted with Galois cohomology or the theorem of Tits.

G = simple Lie grp = $\mathrm{SL}_k(\mathbb{R})$ or $\mathrm{SO}(k,1)^\circ$ (conn, linear, noncompact)

Γ = arithmetic subgroup

(~) $\exists n, \exists \rho\colon G \hookrightarrow \mathrm{GL}_n(\mathbb{R}),\ \rho(\Gamma) \doteq \mathrm{GL}_n(\mathbb{Z}) \cap \rho(G)$

$\overset{?}{\Leftrightarrow}$ $\exists n, \exists \rho\colon G \to \mathrm{GL}_n(\mathbb{R}),\ \rho(\Gamma) \subseteq \mathrm{GL}_n(\mathbb{Z})$.

Definition

Γ **universally arithmetic** $\Leftrightarrow$ $\forall n, \forall \rho\colon G \to \mathrm{GL}_n(\mathbb{R})$,
$\exists M \in \mathrm{GL}_n(\mathbb{R}),\ M^{-1}\rho(\Gamma)M \subseteq \mathrm{GL}_n(\mathbb{Z})$ (and Γ is arithmetic).

Proposition (Morris [2004])

Every G has a universally arithmetic subgroup Γ.

1. My original proof was tedious (and explicit).
2. Better pf (conceptual) by **Prasad-Rapinchuk** [2006].
3. Today: idea in another proof (direct, natural) [2015].

Defn. Γ universally arith $\Leftrightarrow$ $\forall n, \forall \rho\colon G \to \mathrm{GL}_n(\mathbb{R})$,
$\exists M \in \mathrm{GL}_n(\mathbb{R}),\ M^{-1}\rho(\Gamma)M \subseteq \mathrm{GL}_n(\mathbb{Z})$ (and Γ is arithmetic).

Eg. $G = \mathrm{SL}_k(\mathbb{R}) \Rightarrow \mathrm{SL}_k(\mathbb{Z})$ is univ arith. ($\mathbb{Q}$-split$\Rightarrow$univ arith)

Converse: $\mathrm{SL}_k(\mathbb{Z})$ is the only one up to commensurability.

Proof.

$\rho\colon G \xrightarrow{=} \mathrm{SL}_k(\mathbb{R})$.

Γ is univ arith
$\Rightarrow M^{-1}\rho(\Gamma)M \subseteq \mathrm{SL}_k(\mathbb{Z})$
$\Rightarrow M^{-1}\Gamma M \doteq \mathrm{SL}_k(\mathbb{Z})$. □

Example

$G = \mathrm{SO}(k,1)$ and k even $\Rightarrow \Gamma$ is univ arith.

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$\exists M \in \mathrm{GL}_n(\mathbb{R}),\ M^{-1}\rho(\Gamma)M \subseteq \mathrm{GL}_n(\mathbb{Z})$ (and Γ is arithmetic).

Eg. $G = \mathrm{SO}(k,1)$ and k even $\Rightarrow \Gamma$ is univ arith.

Proof. k even $\Rightarrow$

1. $Z(\mathrm{SO}(k,1)_{\mathbb{C}})$ is trivial,
2. $\mathrm{SO}(k,1)_{\mathbb{C}}$ has no outer automorphisms. □

There are two obstructions to Γ being univ arith:

1. a cohomology class in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{Q}); Z(G_{\mathbb{C}}))$,
2. an **outer** automorphism of $G_{\mathbb{C}}$.

Defn. G **$\mathbb{C}$-universal** $\Leftrightarrow$ $\forall n, \forall \rho\colon G \to \mathrm{GL}_n(\mathbb{C})$,
$\exists M \in \mathrm{GL}_n(\mathbb{C}),\ M^{-1}\rho(G)M \subseteq \mathrm{GL}_n(\mathbb{R})$.

Defn. G **$\mathbb{C}$-universal** $\Leftrightarrow$ $\forall n, \forall \rho\colon G \to \mathrm{GL}_n(\mathbb{C})$,
$\exists M \in \mathrm{GL}_n(\mathbb{C}),\ M^{-1}\rho(G)M \subseteq \mathrm{GL}_n(\mathbb{R})$.

Assume $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ irred. $\rho\colon G_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$.

If $\rho(G) \subseteq \mathrm{GL}_n(\mathbb{R})$, then $\overline{\rho(g_{\mathbb{C}})} = \rho(\overline{g_{\mathbb{C}}})$.

In general: $\overline{\rho(g_{\mathbb{C}})} = \rho(\varphi(\overline{g_{\mathbb{C}}}))$, $\exists \varphi \in \mathrm{Aut}(G_{\mathbb{C}})$

$\Rightarrow \overline{\rho(g)} = \rho(\epsilon^{-1} g \epsilon) = \rho(\epsilon)^{-1}\rho(g)\rho(\epsilon)$ if $\nexists$ outer aut.

$M^{-1}\rho(g)M \overset{?}{=} \overline{M^{-1}\rho(g)M} = \overline{M}^{-1}\rho(\epsilon)^{-1}\rho(g)\,\rho(\epsilon)\,\overline{M}$.

$\rho(\epsilon) \overset{?}{=} M\overline{M}^{-1}$, $\exists M \in \mathrm{GL}_n(\mathbb{C})$.

Obstruction: need $\rho(\epsilon)\,\overline{\rho(\epsilon)} = I$. $(\Leftrightarrow)$

Cor. G is $\mathbb{C}$-universal if $\epsilon\overline{\epsilon} = e$.

Exer. $\epsilon\overline{\epsilon} \in Z(G_{\mathbb{C}})$. (because $\overline{\overline{\rho(g)}} = \rho(g)$)

Abstract reformulation

Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $G_{\mathbb{C}}$:

$H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \underline{G_{\mathbb{C}}}) = \{\, \epsilon \in \underline{G_{\mathbb{C}}} \mid \epsilon\overline{\epsilon} = e \,\}/\sim$ $(\eta\overline{\eta}^{-1} \sim e)$
$= \{\, \epsilon \in G_{\mathbb{C}} \mid \epsilon\overline{\epsilon} \in Z(G_{\mathbb{C}}) \,\}/\sim$

Choice of $[\epsilon]$ $\leftrightarrow$ choice of G in $G_{\mathbb{C}}$.

E.g., $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \mathrm{Aut}\,\mathrm{SO}(5,\mathbb{C})) \longleftrightarrow \{\, \mathrm{SO}(5), \mathrm{SO}(4,1), \mathrm{SO}(3,2) \,\}$.

Short exact sequence $e \to Z(G_{\mathbb{C}}) \to G_{\mathbb{C}} \to \underline{G_{\mathbb{C}}} \to e$

yields $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \underline{G_{\mathbb{C}}}) \to H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); Z(G_{\mathbb{C}}))$.
$[\epsilon] \mapsto \epsilon\overline{\epsilon} =: \tau \in Z(G_{\mathbb{C}})$

$H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); Z(G_{\mathbb{C}})) = \{\, \tau \in Z(G_{\mathbb{C}}) \mid \overline{\tau} = \tau \,\}/\{z\overline{z}\}$

$G \rightsquigarrow \tau \in Z(G)$ (cohomology class in H^2)

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Let $\rho \colon G \to \mathrm{GL}_n(\mathbb{C})$ irreducible.
Schur's Lemma: $\tau \in Z(G) \Rightarrow \rho(\tau)$ is a scalar.
$\rho(\tau)$ is a root of unity, and $\overline{\tau} = \tau$, so $\rho(\tau) = \pm 1$.

Proposition (Tits)

Assume $G_{\mathbb{C}}$ has no outer auts. *(G is an inner form)*

- $\rho(\tau) = 1 \Leftrightarrow M^{-1}\rho(G)\,M \subseteq \mathrm{GL}_n(\mathbb{R})$.
- $\rho(\tau) = -1 \Leftrightarrow M^{-1}\rho(G)M \nsubseteq \mathrm{GL}_n(\mathbb{R})$.

Cor. G is $\mathbb{C}$-universal (i.e., $\forall \rho$, $\exists M$)
$\Leftrightarrow \rho(\tau) = 1$ for all ρ $\Leftrightarrow \tau = e$.

τ has been calculated for every G.
E.g., $\mathrm{Spin}(k,1)$ is $\mathbb{C}$-univ iff $k \equiv 0, 1, 2 \pmod 8$.

$G \rightsquigarrow \tau \in Z(G)$ (cohomology class in H^2)

Same idea

Arith subgrp $\Gamma \rightsquigarrow \tau_{\mathbb{Q}} \in H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{Q}); Z(G_{\mathbb{C}}))$
$M^{-1}\,\rho(\Gamma)\,M \subseteq \mathrm{GL}_n(\mathbb{Z}) \Leftrightarrow \rho_*\tau_{\mathbb{Q}}$ trivial
in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{Q}); \mu_n)$.

Proof that G has universally arith subgroup.

$G \rightsquigarrow \epsilon \in H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \underline{G_{\mathbb{C}}}) \rightsquigarrow \tau = \epsilon\overline{\epsilon} \in Z(G_{\mathbb{C}})$.
Note that ϵ lifts to $\tilde{\epsilon} \in H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); G_{\mathbb{C}}/\langle\tau\rangle)$.

Lem. $\tilde{\epsilon}$ lifts to $\tilde{\epsilon}_{\mathbb{Q}} \in H^1(\mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q}); G_{\mathbb{Q}[i]}/\langle\tau\rangle)$.

$\epsilon_{\mathbb{Q}} \rightsquigarrow$ arithmetic subgroup Γ of G with $\tau_{\mathbb{Q}} \in \langle\tau\rangle$.
If $\rho_*\tau$ is trivial in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \mu_n)$,
then $\tau_{\mathbb{Q}} \in \langle\tau\rangle \subseteq \ker\rho$,
so $\rho_*\tau_{\mathbb{Q}}$ is trivial in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{Q}); \mu_n)$. □

Real representations

Irred reps of G over $\mathbb{C}$ are well known. ("highest weight")

Let W = vector space over $\mathbb{C}$,
$\rho_{\mathbb{C}} \colon G \to \mathrm{GL}(W) \cong \mathrm{GL}_n(\mathbb{C})$ irreducible.
Thinking of W as a vector space over $\mathbb{R}$ yields
$\rho \colon G \to \mathrm{GL}(W_{\mathbb{R}}) \cong \mathrm{GL}_{2n}(\mathbb{R})$.

Exer. $W_{\mathbb{R}}$ is irred or $W_{\mathbb{R}} = V \oplus V$ (2 copies of an irred)

Proposition

This provides a one-to-one correspondence
$\{$ *irred reps over* $\mathbb{C}$ $\}/\!\sim\; \longleftrightarrow \{$ *irred reps over* $\mathbb{R}$ $\}$
where $V \sim \overline{V}$ (complex conjugate).

Lemma (Schur's Lemma)

$\rho \colon G \to \mathrm{End}_{\mathbb{R}}(V) \cong \mathrm{Mat}_{n\times n}(\mathbb{R})$ ***irreducible.***
$C = \mathrm{End}_G(\mathbb{R}^n)$
$= \{\, T \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid T\,\rho(g) = \rho(g)\,T,\ \forall g \,\}$
$\Rightarrow$ *C is a division algebra (every nonzero el't has inverse)*
$\Rightarrow$ *$C = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Exercise

$\mathrm{End}_G(V \otimes \mathbb{C}) \cong \mathrm{End}_G(V) \otimes \mathbb{C}$.

Corollary

$V \otimes \mathbb{C} =$

1. *irreducible if $C = \mathbb{R}$* $(\rho(\tau) = 1)$
2. *$W \oplus W$ if $C = \mathbb{H}$* $(\rho(\tau) = -1)$
3. *$W \oplus \overline{W}$ (not $\cong$) if $C = \mathbb{C}$* *(outer automorphism)*

A list of references is in the bibliography of:

D. W. Morris:
A cohomological proof that real representations of semisimple Lie algebras have $\mathbb{Q}$-forms,
SIGMA (Symmetry Integrability Geom. Meth. Appl.) 11 (2015) 034, 12 pages. MR 3339733,
http://dx.doi.org/10.3842/SIGMA.2015.034

See especially:
J. Tits: Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque,
J. Reine Angew. Math. 247 (1971) 196–220.
MR 0277536,
http://dx.doi.org/10.1515/crll.1971.247.196

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