Hamiltonian paths in projective checkerboards

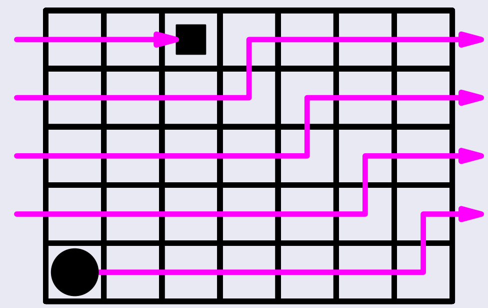
Dave Witte Morris

University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Abstract. Place a checker in some square of an $m \times n$ rectangular checkerboard, and glue opposite edges of the checkerboard to make a projective plane. We determine whether the checker can visit all the squares of the checkerboard (without repeating any squares), by moving only north and east. This is joint work with Dallan McCarthy, and no advanced mathematical training will be needed to understand most of the talk.

A checker is in the Southwest corner of an $m \times n$ checkerboard (with $m, n \geq 3$)

Can the checker tour the board?



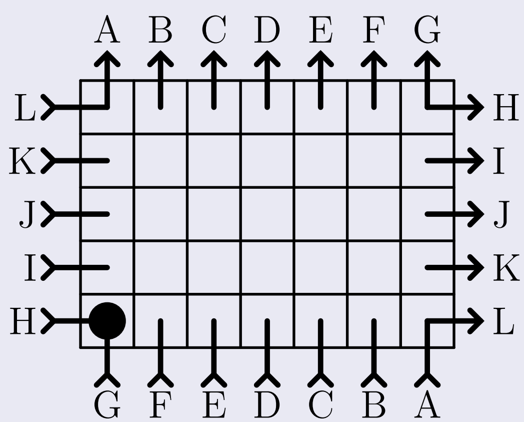
- A tour must visit each square exactly once: **hamiltonian path**
- The checker can only move North and East
- *Allow the checker to step off the edge of the board.* (The board is now toroidal, rather than flat.)

Answer: yes. *Too easy!*

Every $m \times n$ toroidal checkerboard has a ham path. (easy)

Change the topology

Instead of gluing the edges of the board to make a torus, we could glue with a twist, making a **projective plane**.



Proposition

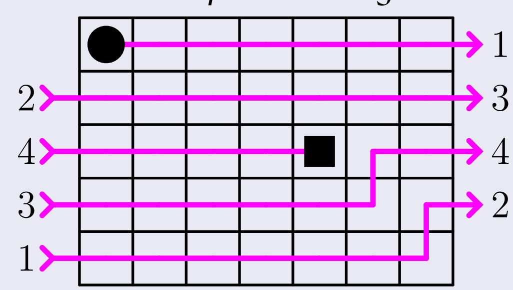
∄ hamiltonian path (if $m, n \geq 3$).

Proposition

$m \times n$ projective checkerboard has no hamiltonian path that starts in the southwest corner

Observation

There does exist a ham path starting in the **north***west corner*

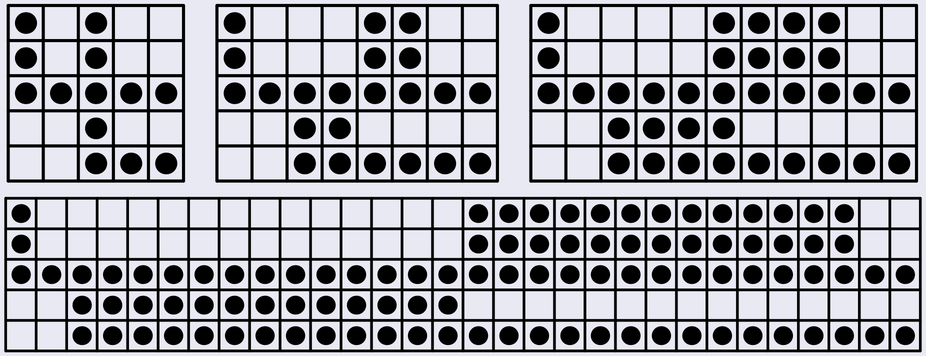


Problem

Which squares are the starting square of a ham path on an $m \times n$ projective checkerboard?

Problem. *Which squares are the starting square of a ham path?*

Answer



- For $m = n$: undergrads at Williams College in 1992
- General case: **Dallan McCarthy** (ULeth math major) & me

Moreover, for each possible starting square, determined where the ham paths can **end.**

Problem. *Which squares are the starting square of a ham path?*

Sample arguments:

Lemma

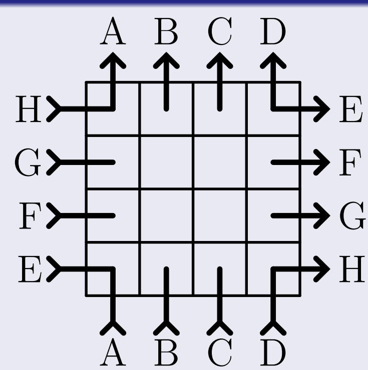
Either the start or the end must be on an edge of the board.

No hamiltonian path starts in the southwest corner

1. must end in northeast corner
2. direction-forcing diagonals
3. the path must be a palindrome
4. can only step off edge once

Change the topology again: Klein bottle

- glue top to bottom without a twist (like torus),
- glue left to right with a twist (like projective plane).



Proposition (Williams undergrads in 1992 [unpublished])

On an $n \times n$ (square) Klein checkerboard, with n even, there is a hamiltonian path starting at every square, because there is a hamiltonian ***cycle.***

¿Starting squares for $m \times n$? ¿End for start in SW corner?

- M. H. Forbush, E. Hanson, S. Kim, A. Mauer, R. Merris, S. Oldham, J. O. Sargent, K. Sharkey, and D. Witte: Hamiltonian paths in projective checkerboards, *Ars Combin.* 56 (2000), 147–160. MR1768611
- J. A. Gallian and D. Witte: Hamiltonian checkerboards. *Math. Mag.* 57 (1984) 291–294. MR0765645 `http://dx.doi.org/10.2307/2689603`
- D. McCarthy and D. W. Morris: Hamiltonian paths in $m \times n$ projective checkerboards (preprint, 28+ pages). `http://arxiv.org/abs/1607.04001`
- J. J. Watkins, *Across the board: the mathematics of chessboard problems,* Princeton University Press, 2004. MR2041306