Some arithmetic groups that do not act on the circle

Dave Witte Morris

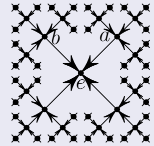
University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Lecture 3
What is an amenable group?
(used to prove actions have a fixed point)

Amenability: fundamental notion in group theory.
Definition: dozens of choices (all equivalent).

Example

Free group $F_2 = \langle a, b \rangle$. Every el't starts with \$1:
$f_0(g) = 1, \quad \forall g \in F_2$.
Everyone passes their dollar to the person next to them who is closer to the identity:
$f_1(g) = \$3$ (except $f_1(e) = \$5$).
Everyone ≥ \$2, & money only moved bdd distance.



Terminology

This is a *Ponzi scheme* on F_2.

Example

$\exists$ *Ponzi scheme* on F_2:
Everyone ≥ \$2, & money only moved bdd distance.

Exercise

On $\mathbb{Z}^n$, $\nexists$ Ponzi scheme.
($\exists$ Ponzi scheme $\implies$ exponential growth.)

Solvable grps of exp'l growth do *not* have a Ponzi:

Theorem (Gromov)

$\nexists$ Ponzi scheme on G $\iff$ G is "amenable".

Corollary

Amenability is a geometric notion (inv't under quasi-isom).

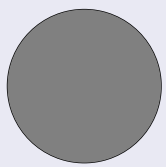
What is amenable really?

Answer

G is amenable $\iff$ G has *almost-invariant* subsets.

Example

G = abelian group (f.g.) $= \mathbb{Z}^2 = \langle a, b \rangle$.
G acts on itself by left translation.
F = G-inv't subset of G, $\left(\begin{smallmatrix} aF = F,\ bF = F, \\ \text{nonempty} \end{smallmatrix}\right)$
$\implies$ F is infinite.
$\nexists$ finite, invariant subset.
F = big ball $\implies$ F is 99.99% invariant ("almost inv't"):
$\#(F \cap aF) > (1 - \epsilon)\,\#F$



Definition

F is *almost invariant* (F is a "Følner set"):
$\#(F \cap aF) > (1 - \epsilon)\#F \quad \forall a \in S$

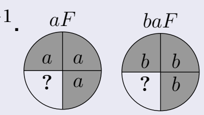
Definition

G *amenable* $\iff$ G has almost-inv't finite subsets
($\forall$ finite S, $\forall \epsilon > 0$)

Exercise

Free group F_2 is not amenable.

Idea. $\frac{3}{4}$ of F does not start with a^{-1}.
$\implies$ $\frac{3}{4}$ of aF starts with a.
$\implies$ $\frac{3}{4}$ of baF starts with b.
$aF \approx F \approx baF \implies \approx\frac{3}{4}$ of F starts with a and b. $\rightarrow\leftarrow$



Proposition

G amen $\iff$ every bdd func on G has an avg value.
I.e., $\exists$ A: $\ell^\infty(G) \to \mathbb{R}$, s.t.

- $A(1) = 1$,
- $A(a\varphi + b\psi) = a\,A(\varphi) + b\,A(\psi)$,
- $A(\geq 0) \geq 0$,
- $A(\varphi^g) = A(\varphi)$. (translation invariant)

Proof.

Choose sequence of almost-inv't sets F_n ($\epsilon = 1/n$).
Let $A_n(\varphi) = \frac{1}{\#F_n} \sum_{x \in F_n} \varphi(x)$.
Pass to subsequence, so $A_n(\varphi) \to A(\varphi)$.
Can make a consistent choice of $A(\varphi)$ for all φ.
[Ultrafilter, Hahn-Banach, Zorn's Lemma, Tychonoff, Axiom of Choice] □

Bounded cohomology

Define group cohomology as usual, except that all cochains are assumed to be bounded functions.

Theorem (B. E. Johnson)

G amenable
$\iff$ $H^n_{\mathrm{bdd}}(G; V) = 0$, $\forall$ G-module V $\left(\begin{smallmatrix} \text{such that } V \text{ is} \\ \text{a Banach space} \end{smallmatrix}\right)$.

Proof of ($\Rightarrow$). If G is finite, and $|G|$ is invertible, one proves $H^n(G; V) = 0$ by averaging:
$\overline{\alpha}(g_1, \ldots, g_n) = \frac{1}{|G|} \sum_{g \in G} \alpha(g, g_1, \ldots, g_n)$.
Since G is amenable, we can do exactly this kind of averaging for any *bounded* cocycle.

Proposition

G amen $\iff$ every bdd func on G has an avg value.

Average vals of characteristic funcs of subsets of G:

Corollary (von Neumann's original definition)

G amen $\iff$ $\exists$ finitely additive probability measure.

Corollary ($\iff$)

- *G amenable,*
- *G acts on compact metric space X* (by homeos)

$\implies$ every continuous function on X has an avg val
$\implies$ $\exists$ G-inv't probability measure μ on X. ($\mu(X) = 1$)

Corollary ($\iff$)

G amenable, acts on cpct metric space X (by homeos)
$\implies$ $\exists$ G-inv't probability measure μ on X. ($\mu(X) = 1$)

Corollary

G amenable, acts on S^1 (orient-preserving) *$\implies$ either:*
$\exists$ finite orbit or abelianization of G is infinite.

Fact: G amenable, acts on $\mathbb{R}$, finitely generated $\implies$ abelianization is ∞.

Corollary ($\iff$)

- *G amenable,*
- *acts by* (cont) *linear maps on vector space* $\left(\begin{smallmatrix} \text{locally} \\ \text{convex} \end{smallmatrix}\right)$,
- *C is a compact, convex, G-invariant subset* ($\neq \emptyset$)

$\implies$ $\exists$ fixed point in C.

Corollary (Furstenberg)

$\Gamma \doteq \mathrm{SL}(3,\mathbb{Z}) \subset \mathrm{SL}(3,\mathbb{R}) = G, \;\; P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$ (amenable).

Γ acts on S^1 $\implies$ $\exists$ Γ-equivariant
$\psi\colon G/P \to \mathrm{Prob}(S^1)$. { *probability measures on S^1* }

Theorem (Ghys)

ψ is constant (a.e.) *ψ is measurable*
$\therefore \exists$ Γ-inv't point in $\mathrm{Prob}(S^1)$
$\therefore \exists$ finite orbit (*since $\Gamma/[\Gamma,\Gamma]$ is finite*).

Proof of Corollary.

$\{\Gamma$-equivariant $\psi\colon G \to \mathrm{Prob}(S^1)\}$ is convex, cpct.
P acts by translation (on domain).
$\implies$ P has fixed pt, which factors through G/P. □

Another definition of amenability

Notation

G f.g. $\implies$ $\exists \phi\colon F_n \twoheadrightarrow G$.
Let $B_r = \{$ words of length $\le r \}$ in F_n.
(Note: $\#B_r \approx (2n-1)^r$.)

Example

$G = F_n \implies \#(B_r \cap \ker\phi) = 1 < (\#B_r)^{\epsilon}$.

$G = \mathbb{Z}^n \implies \#(B_r \cap \ker\phi) \approx \dfrac{\#B_r}{(2r+1)^n} = (\#B_r)^{1-\epsilon}$.

Theorem (R. I. Grigorchuk, J. M. Cohen)

G amenable $\iff$ $\#(B_r \cap \ker\phi) \ge (\#B_r)^{1-\epsilon}$.
I.e., amenable groups have maximal cogrowth.

Exercises

1) *Examples of amenable groups:*
   a) finite groups are amenable ($S = G = F_n$)
   b) $\mathbb{Z}$ is amenable ($S = \{1\}$, $F_n = \{1,2,3,\cdots,n\}$)
   c) amenable $\times$ amenable is amenable
   d) abelian groups are amenable
   e) $N \triangleleft G$ with $N, G/N$ amen $\implies$ G amen
   f) solvable groups are amenable (!!!)
   g) subgrps, quotients of amen grps are amen
   h) grps of subexp'l growth are amenable
2) Grps with a nonabel free subgrp are not amen.
   Remark: (difficult) There exist nonamenable groups that do not have nonabelian free subgrps. In fact, torsion groups can be nonamen.

Optional exercises

3) locally amenable $\implies$ amenable
4) $\exists$ Følner sets $\implies$
   a) $\nexists$ Ponzi scheme.
   b) every bdd func on G has an avg value.
5) amenable $\implies$ $\nexists$ *paradoxical decomposition*.
   (If $G = \left(\coprod_{i=1}^m A_i\right) \coprod \left(\coprod_{j=1}^n B_j\right)$ (disjoint unions)
   and $g_1,\ldots,g_m,h_1,\ldots,h_n \in G$,
   show either $G \ne \bigcup_{i=1}^m g_i A_i$ or $G \ne \bigcup_{j=1}^n h_j B_j$.
6) Find an *explicit* paradoxical decomp of a free grp.
7) G acts on S^1, $\exists$ G-inv't probability measure
   $\implies$ $\exists$ finite orbit or $G/[G,G]$ is infinite.
8) G_1 has Ponzi, G_1 quasi-isom to G_2 $\Rightarrow$ G_2 has Ponzi.

Related reading

- D. Morris: *Introduction to Arithmetic Groups* (preprint). (Has chapter on amenable groups.) `http://people.uleth.ca/~dave.morris/books/IntroArithGroups.html`
- É. Ghys: Groups acting on the circle. *L'Enseignement Mathématique* 47 (2001) 329–407. `http://retro.seals.ch/cntmng;?type=pdf&rid=ensmat-001:2001:47::210`
- D. W. Morris: Can lattices in $\mathrm{SL}(n,\mathbb{R})$ act on the circle?, in *Geometry, Rigidity, and Group Actions*, University of Chicago Press, Chicago, 2011. `http://arxiv.org/abs/0811.0051`

Further reading

Amenability:

- A. L. T. Paterson: *Amenability.* American Mathematical Society, Providence, RI, 1988.
- J.-P. Pier: *Amenable Locally Compact Groups.* Wiley, New York, 1984.
- S. Wagon: *The Banach-Tarski Paradox.* Cambridge U. Press, Cambridge, 1993.

Ponzi schemes:

- M. Gromov: *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, Boston, 1999. (See Lemma 6.17 and Exercise $6.17\frac{1}{2}$ on p. 328.)

Ghys' proof:

- É. Ghys: Actions de réseaux sur le cercle. *Invent. Math.* 137 (1999) 199–231.

A different way to show ψ is constant:

- U. Bader, A. Furman, A. Shaker: Superrigidity, Weyl groups, and actions on the circle (preprint). `http://arxiv.org/abs/math/0605276`