Which flows are sums of hamiltonian cycles in abelian Cayley graphs?

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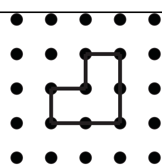
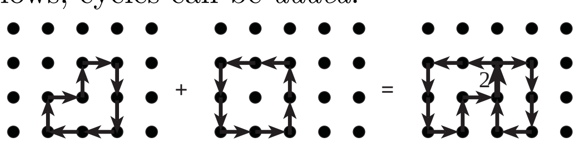
Abstract

If X is any connected Cayley graph on any finite abelian group, we determine precisely which flows on X can be written as a sum of hamiltonian cycles. In particular, if the degree of X is at least 5, and X has an even number of vertices, then it is precisely the even flows, that is, the flows f, such that $\sum_{\alpha \in E(X)} f(\alpha)$ is divisible by 2. On the other hand, there are infinitely many examples of degree 4 in which not all even flows can be written as a sum of hamiltonian cycles. Analogous results were already known 10 years ago, from work of Brian Alspach, Stephen Locke, and Dave Witte, for the case where X is cubic, or has an odd number of vertices.



Any cycle in a graph X,
when given an orientation
determines a **flow** on X.

As flows, cycles can be *added*.



Easy. Every flow on X is a sum of cycles.

Ques (Alspach).
Which flows are a sum of **hamiltonian** *cycles?*
For X = cartesian product of cycles.

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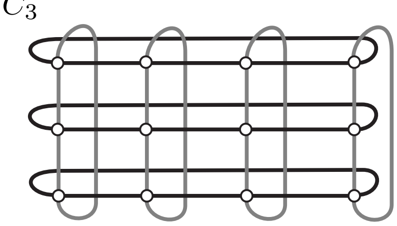
Which flows are a sum of **hamiltonian** *cycles?*
For X = cartesian product of cycles.

Eg. $C_4 \,\square\, C_3$



In general, $C_{n_1} \,\square \cdots \square\, C_{n_r}$
$\cong \mathrm{Cay}\big(\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}; \{e_1, \ldots, e_r\}\big)$

Alspach actually asked the question for
all (connected) Cayley graphs on abelian groups.

Ques. *Which flows are a sum of ham cycles H?*

Constraint. $|X|$ even

$\Rightarrow$ H has even length (even # edges)

$\Rightarrow$ H is an even flow

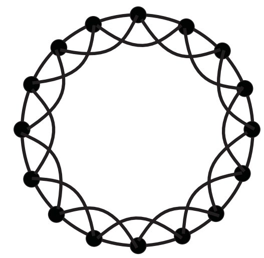
i.e., $\sum_{e \in E(X)} f(e)$ is even.

$\Rightarrow$ sum of ham cycs is always an even flow.

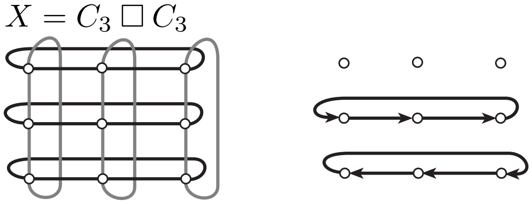
Ans (Alspach-Locke=Witte, Morris[2]-Moulton).

- if $|X|$ is odd: all flows unless $X \cong C_3 \square C_3$.
- if $|X|$ is even: all even flows unless
  - X is cubic or
  - $\deg X = 4$ and $X \cong \mathrm{Cay}(\mathbb{Z}_n; \pm 1, \pm 2)$ or $|X| \equiv 2 \pmod 4$ and X is not bipartite.

$\mathrm{Cay}(\mathbb{Z}_n; \pm 1, \pm 2)$ is the square of a cycle.



Eg. . $X = C_3 \square C_3$



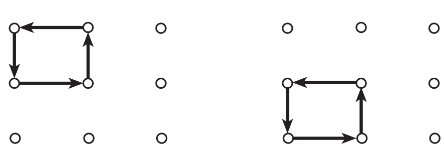
Put weight 1 on the edges of two parallel cycles with opposite orientation.

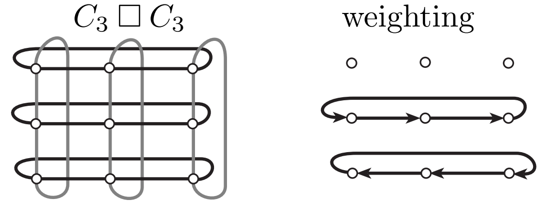
Weight 0 on all the other edges.

Defn. The *weight* of a flow f is the weighted sum of its edge-flows:

$$\mathrm{wt}(f) = \sum_{e \in E(X)} w(e)\, f(e).$$

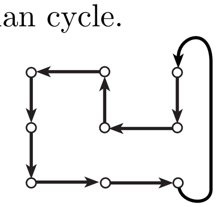
Eg. wt(4-cycle) $= \pm 1$ or ± 2.





Eg. wt(4-cycle) $= \pm 1$ or ± 2.

Eg. A hamiltonian cycle.



$\mathrm{wt}(H) = (-1) + (-1) + (-1) = -3 \equiv 0 \pmod 3$.

Any hamiltonian cycle looks similar to this:

$\forall$ ham cyc H, $\mathrm{wt}(H) \equiv 0 \pmod 3$.

f = sum of ham cycs

$\Rightarrow \mathrm{wt}(f) \equiv 0 \pmod 3$.

Converse is true.

Eg. For $X = C_3 \,\square\, C_3$:

f = sum of ham cycs $\Rightarrow \mathrm{wt}(f) \equiv 0 \pmod 3$

and converse is true.

Eg. X is cubic:

- prism (over a cycle)
- Möbius ladder

There are very few hamiltonian cycles.

Easy to find weighting of edges, such that

f = sum of ham cycs $\Rightarrow \mathrm{wt}(f) \equiv 0 \pmod k$

and converse is true.

Example. $X = \mathrm{Cay}(\mathbb{Z}_n; \pm 1, \pm 2)$

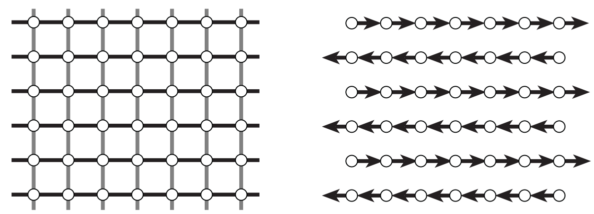
Same story.

Remaining exception:

- $\deg X = 4$,
- $|X| \equiv 2 \pmod 4$, and
- X is not bipartite.

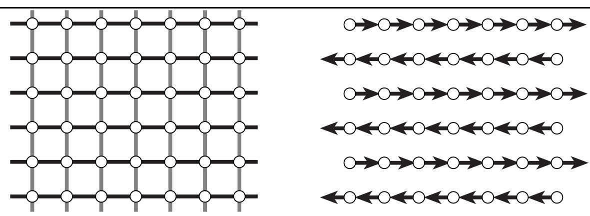
Main example. $X = C_m \,\square\, C_n$

- m odd
- $n \equiv 2 \pmod 4$.



Give the C_m-cycles alternating orientations
(with weight 1 on each edge)
and put weight 0 on each edge of the C_n-cycles.

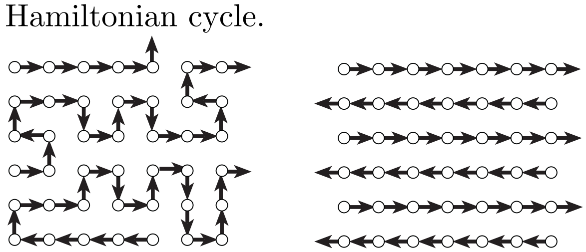
Eg. $\mathrm{wt}(\text{4-cycle}) = 1 + 0 + 1 + 0 = \pm 2$.



Eg. $\mathrm{wt}(\text{4-cycle}) = \pm 2$.

Exer. wt(even flow) is even.

Eg. Hamiltonian cycle.



$$\mathrm{wt}(H) = 2 - 1 + 1 - 1 - 1 - 2 - 1 - 2 + 1 - 1 + 2 + 1 + 6 + 4 = 8 \equiv 0 \pmod 4.$$

Thm. $\mathrm{wt}(\text{sum of ham cycs}) \equiv 0 \pmod 4$
and converse is true.

Main example. $X = C_m \,\square\, C_n$

- m odd
- $n \equiv 2 \pmod 4$.

Thm. $\mathrm{wt}(\text{ham cyc}) \equiv 0 \pmod 4$.

But there are *many* hamiltonian cycles;
not obvious that $\mathrm{wt}(H)$ is always divisible by 4.

Use a more geometric defn of wt(even cycle).

Define "imbalance" of C:

- 2-color complement $X \setminus C$.
- $\mathrm{imb}(C) = (\#\text{black}) - (\#\text{white}) \pmod 4$.

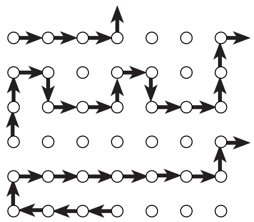
Key fact. $\mathrm{wt}(C) \equiv \mathrm{len}(C) + \mathrm{imb}(C) - 2 \pmod 4$

Proof of Thm.

- $\mathrm{len}(H) = |X| \equiv 2 \pmod 4$.
- $\mathrm{imb}(H) = 0$ ($\#\text{black} = 0 = \#\text{white}$).

$\Rightarrow \mathrm{wt}(H) \equiv 2 + 0 - 2 = 0$. $\square$

Definition of imb(C).



X is not bipartite, but $X \smallsetminus C$ can be 2-colored.

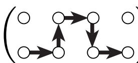
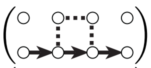
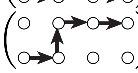
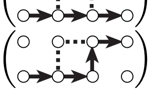
Always shift the color scheme when C is crossed.

Key fact. $\mathrm{wt}(C) \equiv \mathrm{len}(C) + \mathrm{imb}(C) - 2 \pmod 4$

Defn. $\mathrm{wli}(C) := \mathrm{wt}(C) + \mathrm{len}(C) + \mathrm{imb}(C) + 2$.

Want to show $\mathrm{wli}(C) \equiv 0 \pmod 4$.

Lem.

- wli () ≡ wli ().
- wli () ≡ wli ().

Lem. *C increasing*

$\Rightarrow \mathrm{imb}(C) \equiv 2 \times (\#\text{edges in even rows})$.

Exer. For $s_i = i + k_1 + \cdots k_i \quad (1 \le i \le 2n)$,
$\{\, i \mid s_i \text{ odd} \,\} - \{\, i \mid s_i \text{ even} \,\} \equiv 2 \sum k_{2i} \pmod 4$.

Open problems.

Problem. *Generalize to Cayley graphs on some nonabelian groups.*

Seems hopelessly difficult, even for dihedral grps.

Defn. $D_{2n} = \langle\, t, f \mid t^n = e, f^2 = e, ftf = t^{-1} \,\rangle$.

Rem. $\mathrm{Cay}\big(D_{2n}; \{t, f\}\big)$ has a hamiltonian cycle.

Thm (Alspach-Zhang).
$\mathrm{Cay}\big(D_{2n}, \{f, ft^a, ft^b\}\big)$ *has a hamiltonian cycle.*

Conj. $\mathrm{Cay}(D_{2n}, S)$ *has a ham cyc, for every S.*

Problem (Alspach).
Show $\mathrm{Cay}(D_{2n}, S)$ *is ham connected if $\#S = 3$.*
(or hamiltonian laceable, if bipartite)

This would prove the preceding conjecture.

For abelian groups:

Conj (Alspach).
$\mathrm{Cay}(G; S)$ *is hamiltonian decomposable*
(i.e., edge-disjoint union of Ham cycles
[+ 1-factor?]).

Thm (Bermond-Favaron-Mahéo).
True if degree ≤ 5.

References

B. Alspach, S. C. Locke, and D. Witte:
The Hamilton spaces of Cayley graphs on abelian groups, *Discrete Math.* 82 (1990) 113–126.

S. C. Locke and D. Witte:
Flows in circulant graphs of odd order are sums of Hamilton cycles, *Discrete Math.* 78 (1989) 105–114.

D. Morris, J. Morris, and D. P. Moulton:
Flows that are sums of hamiltonian cycles in abelian Cayley graphs (in preparation).

Surveys on hamiltonian cycles in Cayley graphs:

S.J. Curran and J.A. Gallian:
Hamiltonian cycles and paths in Cayley graphs and digraphs — a survey,
Discrete Math., **156** (1996), no. 1–3, 1–18.

D. Witte and J.A. Gallian:
A survey: Hamiltonian cycles in Cayley graphs,
Discrete Math. **51** (1984), no. 3, 293–304.