Introduction to Kazhdan's property T

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Eg. G cpct $\Rightarrow 1 \in L^2(G)$

$\Rightarrow L^2(G)$ has an invariant vector

I.e., $\phi \equiv 1 \Rightarrow g\phi = \phi \qquad (g\phi)(x) = \phi(g^{-1}x)$

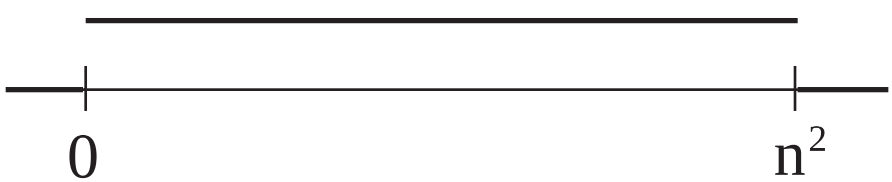
(and conversely)

Eg. $G = \mathbb{R}$

$L^2(G)$ does not have an invariant vector

but it has **almost invariant** vectors

Let $\phi_n = \dfrac{1}{n}\big(\text{characteristic function of } [0, n^2]\ \big)$



Note that ϕ_n is a unit vector.

For $g \in [-n, n]$, we have

$$||g\phi_n - \phi_n||^2 \leq \int_{-n}^{n} \frac{1}{n^2}\, dx + \int_{n^2-n}^{n^2+n} \frac{1}{n^2}\, dx = \frac{4}{n}$$

For every compact $K \subset G$ and every $\epsilon > 0$,

$\exists \phi \in L^2(G)_{(1)}$, $\forall g \in K$, $||g\phi - \phi|| < \epsilon$

Defn. G **amenable**: $L^2(G)$ has alm inv vectors.

For every compact $K \subset G$ and every $\epsilon > 0$,

$$\exists \phi \in L^2(G)_{(1)}, \forall g \in K, ||g\phi - \phi|| < \epsilon$$

Prop. *The following groups are amenable:*

- *abelian groups*
- *solvable groups*
- *compact groups*

Prop. $\mathrm{SL}_2(\mathbb{R})$ *is not amenable*

Cor. *Noncpct semisimple Lie grp is not amenable*

Cor. *A connected Lie group* G *is amenable iff* $G/\operatorname{Rad} G$ *is compact.*

Kazhdan's property T: "opposite" of amenability

Defn. G **amenable**: $L^2(G)$ has alm inv vectors.

Defn. G has **Kazhdan's property** T:
no unitary rep of G has almost invariant vectors
(unless it has invariant vectors)

Prop. *G has Kazhdan's property*
$\Rightarrow G$ is not amenable
(unless G is cpct)

Proof. Suppose G is amenable and Kazhdan.

G amenable $\Rightarrow L^2(G)$ has alm inv vectors,
so Kazhdan $\Rightarrow L^2(G)$ has inv vectors.

Therefore, G is compact. ■

Cor. *Γ Kazhdan $\Rightarrow \Gamma/[\Gamma,\Gamma]$ is cpct.*

Proof. Being abelian, $\Gamma/[\Gamma,\Gamma]$ is amenable.
Γ Kazhdan $\Rightarrow \Gamma/[\Gamma,\Gamma]$ is also Kazhdan. ■

Eg. Free groups: neither Kazhdan nor amenable.

Defn. G has **Kazhdan's property** T:
no unitary rep of G has almost invariant vectors
(unless it has invariant vectors)

Thm (Kazhdan). *G conn, simple, finite center.*
$\mathbb{R}\text{-rank}(G) \geq 2 \;\Rightarrow$ *G has Kazhdan's property.*

(Proof next week.)

Prop (Kazhdan). *G Kazhdan, Γ a lattice in G*
$\Rightarrow \Gamma$ *Kazhdan.*

Cor (Kazhdan). *Γ a lattice in G.*
G conn, simple, finite center with $\mathbb{R}\text{-rank}(G) \geq 2$.
$\Rightarrow \Gamma$ *is finitely generated*
and $\Gamma/[\Gamma,\Gamma]$ is finite.

Cor. $\mathrm{SL}_2(\mathbb{R})$ *not Kazhdan.*

Proof. Surface grp $\Gamma = \langle a, b, c, d \mid [a,b][c,d] = e \rangle$.
Then $\Gamma/[\Gamma,\Gamma]$ infinite, so Γ not Kazhdan.

Cor. $\mathrm{SL}_2(\mathbb{R})$ *not Kazhdan.*

Eg. $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ (free product)

$\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ (free prod with amalg)

Prop (Watatani). Γ *discrete, Kazhdan*

$\Rightarrow \Gamma \neq A *_C B$ *with* A, B, C *finite*

"Kazhdan's property $T \Rightarrow$ Serre's property (FA)"

Lem (Serre). Γ *discrete, Kazhdan.*

If Γ *acts by isometries on Hilbert space* H*,*

then Γ *has a fixed point.*

Weak containment (Fell topology)**.**

σ has alm inv vectors $\Leftrightarrow 1 \prec \sigma$

Defn. (π, V) is **weakly contained** in (σ, W) if:
for every cpct set K in G, every $\epsilon > 0$,
and all $\phi_1, \ldots, \phi_n \in V_{(1)}$,
$\exists \psi_1, \ldots, \psi_n \in W_{(1)}$, such that

$$|\langle g\psi_i, \psi_j\rangle - \langle g\phi_i, \phi_j\rangle| < \epsilon,\ \forall g \in K,\ \forall i, j.$$

Defn. G Kazhdan: $\forall \sigma$, $1 \prec \sigma \Rightarrow 1 \leq \sigma$.

Neighborhood of (π, V) in $\hat{G}$: fix K, ϵ, ϕ_i's
Let $U = \{(\sigma, W) \mid \exists \psi_1, \ldots, \psi_n,$ such that $\cdots\}$.

$1 \prec \sigma \Leftrightarrow$ every neighborhood of 1 contains σ
$\Leftrightarrow$ 1 is in the closure of $\{\sigma\}$.

So G Kazhdan iff $\forall$ rep σ, $1 \in \overline{\sigma} \Rightarrow 1 \leq \sigma$.

This is equivalent to: 1 is isolated in $\hat{G}$ [Wang].

Prop (Kazhdan). *G Kazhdan, Γ a lattice in G*
$\Rightarrow \Gamma$ Kazhdan.

Proof. Spse rep π of Γ has alm inv vectors.
Then $\pi \succ 1$, so

$$\mathrm{Ind}_\Gamma^G(\pi) \succ \mathrm{Ind}_\Gamma^G(1) = L^2(G/\Gamma) \geq 1.$$

Thus, Kazhdan implies $\mathrm{Ind}_\Gamma^G(\pi) \geq 1$.
So $\pi \geq 1$.

Prop. Γ *discrete, Kazhdan* $\Rightarrow \Gamma$ *finitely gen.*

Proof. Let $\{H_n\}$ = all fin gen subgrps of Γ, and define $\pi = L^2(\Gamma/H_1) \oplus L^2(\Gamma/H_2) \oplus \cdots$
where $(g\phi)(xH_n) = \phi(g^{-1}xH_n)$.

Any cpct set $K \subset \Gamma$ is finite, so $\exists H_n \supset K$.
Then K fixes the base point p in Γ/H_n,
so, for $\phi = \delta_p \in L^2(\Gamma/H_n) \subset \pi$,
we have $g\phi = \phi$ for all $g \in K$.
Thus, π has almost invariant vectors.
Therefore, π must have an invariant vector.

So some $L^2(\Gamma/H_n)$ has an invariant vector.
Since Γ is transitive on Γ/H_n,
an invariant function must be constant.
So a constant function is in $L^2(\Gamma/H_n)$,
which means Γ/H_n is finite.

H_n finitely generated $\Rightarrow \Gamma$ finitely generated.

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