Simple groups of real rank at least two have Kazhdan's property

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Let $G = \mathrm{SL}_3(\mathbb{R})$

(conn, simple, finite center, $\mathbb{R}$-rank$(G) \geq 2$)

Thm (Kazhdan). *G has Kazhdan's property.*

Rem. $\mathrm{Sp}(1, \mathrm{n})$ and F_4 are also Kazhdan.
But $\mathrm{SO}(1, n)$ and $\mathrm{SU}(1, n)$ are not.

Rem. Semisimple groups: $G \times H$ is Kazhdan iff both G and H are.

Rem. $\mathrm{SL}_3(\mathbb{R}) \times \mathbb{R}^3$ is not Kazhdan,
but $\mathrm{SL}_3(\mathbb{R}) \ltimes \mathbb{R}^3$ is Kazhdan.

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Defn. G has *Kazhdan's property T*:

For every unitary rep'n σ of G,

$$1 \prec \sigma \Rightarrow 1 \leq \sigma.$$

$(\pi, V) \leq (\sigma, W)$:
for every orthonormal basis $\phi_1, \phi_2, \ldots$ of V,
there are orthonormal vectors $\psi_1, \psi_2, \ldots$ in W,
such that $\langle g\psi_i, \psi_j \rangle = \langle g\phi_i, \phi_j \rangle, \ \forall g \in G, \ \forall i, j.$

Defn. $(\pi, V) \prec (\sigma, W)$:
for every cpct set K in G, every $\epsilon > 0$,
and for all orthonormal vectors $\phi_1, \ldots, \phi_n$ in V,
there are orthonormal vectors $\psi_1, \ldots, \psi_n$ in W,
such that

$$|\langle g\psi_i, \psi_j \rangle - \langle g\phi_i, \phi_j \rangle| < \epsilon, \ \forall g \in K, \ \forall i, j.$$

Thm (Kazhdan). *G has Kazhdan's property.*

Outline of proof. Assume $1 \prec \sigma$ and $1 \not\leq \sigma$.

Let $H = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \cong \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

Lem (Howe-Moore, "decay of matrix coeffs").
$1 \not\leq \sigma \Rightarrow \forall v \in V$, $\forall \epsilon > 0$, $\exists$ *cpct* K,
$\forall g \notin K$, $\langle gv, v \rangle < \epsilon$.

Therefore $1 \not\leq \sigma|_{\mathbb{R}^2}$.

Key Lemma. $1 \not\leq \sigma|_{\mathbb{R}^2} \Rightarrow \sigma|_H \prec \infty \cdot L^2(H)$.

Therefore $1 \prec \sigma|_H \prec \infty \cdot L^2(H)$, so $1 \prec L^2(H)$.

Therefore (by definition), H is amenable.
But H is not amenable, because $H/\operatorname{Rad} H = \mathrm{SL}_2(\mathbb{R})$ is not compact. $\rightarrow\leftarrow$

Prop. *A connected Lie group H is amenable iff $H/\operatorname{Rad} H$ is cpct.*

Defn. H *amenable* if $1 \prec L^2(H)$.

Lem. *σ a unitary repn of $H = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$.*

$$1 \not\prec \sigma|_{\mathbb{R}^2} \Rightarrow \sigma \prec \infty \cdot L^2(H).$$

We may assume σ is irreducible:

$$\sigma = \int_X \pi_x \, d\mu(x) \prec \int_X \big(\infty \cdot L^2(H)\big) \, d\mu(x)$$
$$= \infty \cdot L^2(H)$$

Thm (Mackey). $\sigma|_{\mathbb{R}^2} \neq 1 \Rightarrow$
$\exists$ conn proper subgrp $S \supset \mathbb{R}^2$, and a repn π of S, such that $\sigma = \mathrm{Ind}_S^H(\pi)$.

It suffices to show $\pi \prec \infty \cdot L^2(S)$, because then

$$\sigma = \mathrm{Ind}_S^H(\pi) \prec \mathrm{Ind}_S^H(\infty \cdot L^2(S)) = \infty \cdot L^2(H)$$

Note that S is solvable, hence amenable.

Lem. *π a repn of an amenable group S. Then $\pi \prec \infty \cdot L^2(S)$.*

Proof. S amenable $\Rightarrow 1 \prec L^2(S)$.
So $\pi \prec \pi \otimes L^2(S) \cong (\dim \pi) L^2(S) \leq \infty \cdot L^2(S)$.

Thm (Kazhdan). $G = \mathrm{SL}_3(\mathbb{R})$ *is Kazhdan.*

Proof. Assume $1 \prec \sigma$ and $1 \not\leq \sigma$.

Let $H \cong \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \subset G$.

Write $\sigma|_H = \int_X \sigma_x \, dx$.

$\sigma_x|_{\mathbb{R}^2} \neq 1 \Rightarrow \sigma_x = \mathrm{Ind}_{S_x}^H(\pi_x)$.

S_x amenable $\Rightarrow \pi_x \prec \infty \cdot L^2(S_x)$

$\Rightarrow \sigma_x = \mathrm{Ind}_{S_x}^H(\pi_x) \prec \infty \cdot L^2(H)$

$1 \prec \sigma|_H = \int_X \sigma_x \, dx \prec \infty \cdot L^2(H)$.

So H is amenable. $\rightarrow\leftarrow$

References

- A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Birkhäuser, 1994 (Chapter 3).
- R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, 1984 (Chap. 7).