Some ideas in the proof of Ratner's Theorem

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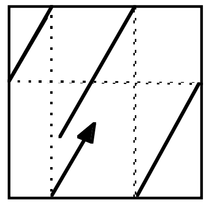
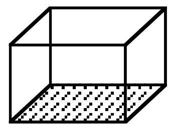
Abstract. Unipotent flows are very well-behaved dynamical systems. In particular, Marina Ratner has shown that every invariant measure for such a flow is of a nice algebraic form. This talk will present some consequences of this important theorem, and explain a few of the ideas of the proof. In general, algebraic technicalities will be pushed to the background as much as possible.

Eg. Let $X =$ torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.



Any $v \in \mathbb{R}^2$ defines a flow on $\mathbb{T}^2$: $\varphi_t(x) = x + tv$

If the slope of v is irrational, it is classical that Lebesgue is the only invariant measure.



Example. $v = (a, b, 0)$ defines a flow φ_t on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$
and $\mathbb{T}^2 \times \{0\}$ is invariant.
Lebesgue on $\mathbb{T}^2 \times \{0\}$ is an invariant measure.

For every v, any ergodic invariant probability measure for φ_t on $\mathbb{T}^3$ is the Lebesgue measure on some invariant subtorus $\mathbb{T}^k$ $\quad (0 \le k \le 3)$.

Note that $\mathbb{R}^2$ is a Lie group.

I.e., it is a group (under vector addition) and a manifold, and the group operations are smooth.

The subgroup $\mathbb{Z}^2$ is closed and discrete.

The quotient space $\mathbb{Z}^2 \backslash \mathbb{R}^2 = \mathbb{T}^2$ is a manifold.

Let G be any Lie group,
and let Γ be a closed, discrete subgroup.

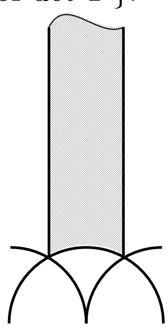
Then $\Gamma \backslash G$ is a manifold,
a *homogeneous space.*

It is best if $\Gamma \backslash G$ is compact.

(More generally, we can allow $\Gamma \backslash G$ to have finite volume. We say Γ is a *lattice.*)

Eg. $G = \mathrm{SL}_2(\mathbb{R}) = \{\ 2 \times 2$ real mat's of det $1\ \}$.
Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.
Then $X = \Gamma \backslash G$ has finite vol.

Other choices of Γ can make $\Gamma \backslash G$ cpct.



Define $u, a \colon \mathbb{R} \to \mathrm{SL}_2(\mathbb{R})$ by

$$u^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } a^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Each of these is a homomorphism.

u^t is a **unipotent** one-parameter subgroup.

Define $\varphi_t(x) = x u^t$.

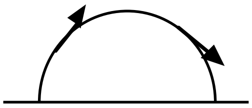
$u^t =$ "horocycle flow"
$a^t =$ "geodesic flow"

The volume on $\Gamma \backslash G$ is a finite, invariant measure.

Geometric description of the horocycle flow.

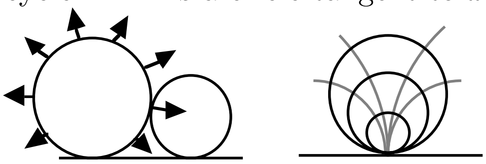
The geodesic flow on $T^1\mathbb{H}^2$.

$\gamma_t\colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$:



Given unit vector $\vec{v} \in T^1\mathbb{H}^2$, move it a distance t along the geodesic it determines.
Geodesic in $\mathbb{H}^2$ is a semicircle $\perp$ x-axis.

Horocycle in $\mathbb{H}^2$ is a circle tangent to x-axis.



$\vec{v} \in T^1\mathbb{H}^2$ is tangent to a unique geodesic.

$\vec{v}$ $\perp$ two horocycles: expanding and contracting.
Move it a distance t along the (contracting) horocycle it determines.

Factors through to flow on unit tangent bundle T^1M of any surface of constant neg curvature.

Why study unipotent flows?

- Flows on homogeneous spaces are "easy."

Algebraic tools can probe the structure.
This is particularly effective for unipotent flows.

- Study actions of $\mathrm{SL}_n(\mathbb{R})$ $(n > 2)$.

Only known C^∞ actions are on homog spaces ($\approx$).

$\mathrm{SL}_n(\mathbb{R})$ has many unipotent elements.

If a single unipotent element acts by translation, then the whole group acts by translations [Witte].

- **Applications in Number Theory.**

Oppenheim Criterion [Margulis].

Let Q be real, indefinite, non-degenerate quadratic form in ≥ 3 variables (e.g., $x^2 - \sqrt{2}xy + \sqrt{3}z^2$).

Then the $\mathbb{Z}$-values of Q are dense in $\mathbb{R}$
unless $Q \approx \mathbb{Z}$-coefficients.

More careful analysis shows (except in some small cases) that values of Q are uniformly distributed:

Thm [Eskin-Margulis-Mozes]. *Let Q be real, non-degenerate quadratic form of signature (p,q) with $p \geq 3$ and $q \geq 1$*
and assume $Q \not\approx \mathbb{Z}$-coefficients.
Then $\exists \lambda$, $\forall a < b$,

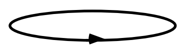
$$\#\left\{ v \in \mathbb{Z}^{p+q} \;\middle|\; \begin{array}{c} a < Q(v) < b, \\ \|v\| \leq T \end{array} \right\} \sim \lambda(b-a)T^{p-q-2} \qquad \text{as } T \to \infty.$$

In a similar vein, the theorem can also be applied to obtain asymptotic formulas for the number of integer points on certain homogeneous varieties [Eskin-Mozes-Shah].

Thm (Furstenberg). $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ *cpct* $\Rightarrow$ *the only invariant probability measure for u^t is the ordinary volume.*

Hence, every u^t-orbit is uniformly distributed.
(In contrast, a^t has *many* invariant measures.)

$\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ noncpct $\Rightarrow$ other possibility is the measure supported on a closed orbit [Dani].



Main Theorem (Ratner). *For unipotent flow on any homogeneous space G/Γ:*
every ergodic invariant probability measure is the volume on some closed orbit of some subgroup containing u^t.

Eg. $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}) \hookrightarrow \mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})$

Cor. (Ratner). *For unipotent flow on G/Γ, with Γ a lattice:*
the closure of every orbit is a closed orbit of some subgroup L containing u^t.
Also: the L-orbit has finite L-inv meas μ, and the u^t-orbit is unif distrib w.r.t. μ.

The same is true (except uniform distribution) if $\{u^t\}$ is replaced with any subgroup generated by unipotent elements.

Proof of Oppenheim Criterion.

- $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$.
- $Q_0(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3^2$.
- $H = \mathrm{SO}(2,1) = \mathrm{SO}(Q_0)$.

Assume (wolog) signature of Q is $(2,1)$,
so $\exists\, g \in \mathrm{SL}(3,\mathbb{R})$, $Q = \lambda\, Q_0 \circ g$.

Algebra: there are 2 possibilities for L in the conclusion of Ratner's Theorem: H or G.

Case 1. The orbit $\Gamma\backslash\Gamma g H$ has finite measure.
Then $\Gamma \cap gHg^{-1}$ is a lattice in $gHg^{-1} = \mathrm{SO}(Q)$,
so $Q \approx \mathbb{Z}$-coefficients.

Case 2. $\Gamma g H$ is dense in G.

$$
\begin{aligned}
Q(\mathbb{Z}^3) &= Q_0(\mathbb{Z}^3 g) && \text{(defn of } g\text{)} \\
&= Q_0(\mathbb{Z}^3 \Gamma g) && (\mathbb{Z}^3\Gamma = \mathbb{Z}^3) \\
&= Q_0(\mathbb{Z}^3 \Gamma g H) && \text{(defn of } H\text{)} \\
&\approx Q_0(\mathbb{R}^3) && \text{(assumption of this case)} \\
&= \mathbb{R}. \quad \square
\end{aligned}
$$

Cor (Ratner Rigidity Theorem). *Let φ_t, φ'_t be unip flows on finite-vol $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$, $\Gamma'\backslash\mathrm{SL}_2(\mathbb{R})$.*
If $\varphi_t \cong \varphi'_t$ measurably, then $\Gamma = \Gamma'$ (up to conj).

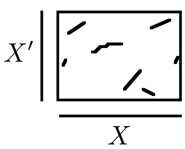
I.e., measurably isomorphic $\Rightarrow$ obvious even to an algebraist that they are isomorphic.

Cor (Witte). *True for any Lie group G in the place of $\mathrm{SL}_2(\mathbb{R})$.*
(In fact, even for $\Gamma\backslash G$ and $\Gamma'\backslash G'$.)

(Contrast: all geodesic flows are isomorphic.)

Proof. Let $\psi\colon (\varphi_t, X) \to (\varphi'_t, X')$ be an iso.

graph(ψ) $\subset X \times X'$
supports $\varphi_t \times \varphi'_t$-inv measure.



$\Rightarrow$ graph(ψ) is a subgroup ($\approx$).

$\Rightarrow$ ψ is a group homomorphism (a.e.). $\square$

Quotients. Let (φ_t, X) and (φ'_t, X') be flows.

If $\exists\, \psi\colon X \to X'$ with $\psi\big(\varphi_t(x)\big) = \varphi'_t\big(\psi(x)\big)$, then (φ'_t, X') is a **quotient** of (φ_t, X).

Cor (Ratner). *Every quotient of $\big(u^t, \Gamma\backslash\mathrm{SL}_2(\mathbb{R})\big)$ is isomorphic to a unipotent flow $\big(u^t, \Gamma'\backslash\mathrm{SL}_2(\mathbb{R})\big)$.*

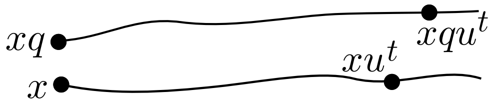
Cor (Witte). *Every quotient of any unipotent flow on any $\Gamma\backslash G$ is a double-coset space $\Gamma'\backslash G/K$.*

Geodesic flow: $a^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

$$a^{-t} q a^t = \begin{pmatrix} \alpha & \beta e^{-2t} \\ \gamma e^{2t} & \delta \end{pmatrix}$$

Points can move apart at exponential speed
transverse to the orbits.

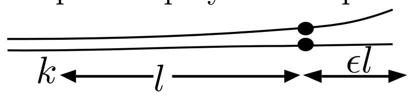
$$u^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a^t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$



$d(x, xq) = ||q||$. $\qquad d(xu^t, xqu^t) = ||u^{-t} q u^t||$.

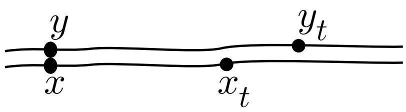
$$u^{-t} q u^t = \begin{pmatrix} \alpha - \gamma t & \beta + (\alpha - \delta)t - \gamma t^2 \\ \gamma & \delta + \gamma t \end{pmatrix}$$

Points move apart at polynomial speed.



$\exists \epsilon > 0$ such that if $d(x_t, y_t) < 1$ for $t \in [k, k+l]$, then $d(x_t, y_t) < 1.1$ for $t \in \big[k, k + (1+\epsilon)l\big]$.

Shearing. Fastest motion is parallel to the orbits.



Choose $\underline{u} \in \mathfrak{g}$ with $\exp(t\underline{u}) = u^t$,
$\quad \underline{q} \in \mathfrak{g}$ with $\exp \underline{q} = q$.

$$\begin{aligned} \underline{u^{-t} q u^t} &= (\mathrm{Ad}\, u^t)\underline{q} \\ &= \exp\big(\mathrm{ad}(t\underline{u})\big)\underline{q} \\ &= \underline{q} \pm t[\underline{q}, \underline{u}] \pm \tfrac{1}{2} t^2 [\underline{q}, \underline{u}, \underline{u}] \pm \tfrac{1}{6} t^3 [\underline{q}, \underline{u}, \underline{u}] \pm \cdots \end{aligned}$$

The largest term has the highest power of t:
$\quad$ the last nonzero term $(\mathrm{ad}\, \underline{u})^k \underline{q}$.
Then $[(\mathrm{ad}\, \underline{u})^k \underline{q}, \underline{u}] = 0$
$\quad$ (because the next term does not appear)
so $(\mathrm{ad}\, \underline{u})^k \underline{q}$ is in the centralizer of u^t.

The direction of fastest relative motion is along the centralizer of u^t.

Similarly: the fastest *transverse* motion is along the normalizer.

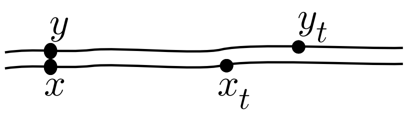
Fastest transverse motion: last term *not in* $\mathfrak{u}$,
$\quad$ so $[(\mathrm{ad}\, \underline{u})^k \underline{q}, \underline{u}] \in \mathfrak{u}$.
Therefore $(\mathrm{ad}\, \underline{u})^k \underline{q}$ normalizes $\mathfrak{u}$.

Thm (Ratner). *Every nontrivial quotient of* $\big(u^t, \Gamma\backslash \mathrm{SL}_2(\mathbb{R})\big)$ *has finite fibers.*

I.e., $\psi \colon \Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to X'$ with $\psi(xu^t) = \varphi'_t(\psi(x))$
$\quad \Rightarrow \psi^{-1}(z)$ is finite *a.e.*

Show: if $\psi^{-1}(x')$ infinite, then φ'_1 has a fixed pt.

Suppose $\quad x \approx y \quad$ with $\quad \psi(x) = \psi(y)$.
Assume ψ uniformly continuous [Lusin's Thm].



Flow along the orbits until $d(x_t, y_t) = 1$.

$\quad$ Then $y_t \approx \varphi_1(x_t)$.

So $\psi(y_t) \approx \psi(\varphi_1(x_t)) = \varphi'_1(\psi(x_t)) = \varphi'_1(\psi(y_t))$.

$\quad d(\varphi'_1(z_k), z_k) < 1/k$.

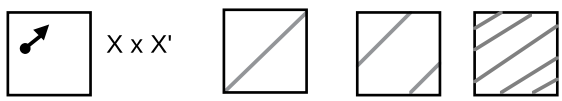
Subsequence converges to a fixed point.

Thm (Ratner). *Every nontrivial quotient of* $\big(u^t, \Gamma\backslash \mathrm{SL}_2(\mathbb{R})\big)$ *has finite fibers.*

Cor (Ratner). *Many horocycle flows are prime. (I.e., they have no nontrivial, proper quotients.)*

Thm (Ratner). *Many horocycle flows have minimal self joinings.*

Joining: An invariant measure on $X \times X'$ that projects to the given measures on X and X'.



Thm (Ratner). *Non-product, ergodic joinings of horocycle flows have finite fibers.*

(μ is concentrated on a set with only finitely many points from each horizontal or vertical line.)

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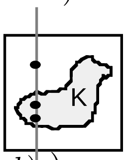
Take $(x, a),\ (x, b) \in X \times X$.



Suppose $a \approx b$. $\qquad a_t \approx \phi_1(b_t)$.

$(x, a)_t = (x_t, a_t) \approx \big(x_t, \phi_1(b_t)\big) = v_1\big((x, b)_t\big)$.

If μ is v_t-invariant, then μ = product measure.
So $(v_1)_*(\mu) \neq \mu$, but $(v_1)_*(\mu)$ is ϕ_t-invariant.
Therefore $(v_1)_*(\mu) \perp \mu$.

$\exists K \subset X \times X$ with $\mu(K) > 0.9$, $d\big(K, v_1(K)\big) > 0$.

Claim. K intersects each vert line in a finite set.

If not, $\exists (x, a), (x, b) \in K$ very close together.
$(x, a)_t,\ (x, b)_t \in K \Rightarrow v_1(x, b)_t \in v_1(K)$.
Since $(x, a)_t \approx v_1\big((x, b)_t\big)$, then $d\big(K, v_1(K)\big) = 0$.

Direct proof of the rigidity theorem.

Defn. Suppose $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$, $g \in G$, $\tilde{g} \in H$.
If $\psi(\Gamma x g) = \psi(\Gamma x) \cdot \tilde{g}$ for a.e. $x \in G$,
then ψ is *affine for g (via $\tilde{g}$)*.

Thm. *$u, \tilde{u}$ ergodic unipotent on $\Gamma\backslash G$ and $\Lambda\backslash H$,
$\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is measure-preserving.
ψ affine for u via $\tilde{u} \Rightarrow \psi$ affine for each $g \in G$.*
[This implies ψ is an affine map (a.e.).]

Suppose G and H are abelian (and $\Gamma = \Lambda = e$).

Fix $g \in G$. Define $\Delta_g\colon G \to H$ by
$\psi(x + g) = \psi(x) + \Delta_g(s)$.

We wish to show Δ_g is constant (a.e.):
set $\tilde{g} = \Delta_g$.

Since u is ergodic, suffices to show Δ_g is constant:
$\Delta_g(x + u) = \Delta_g(x)$.

$$
\begin{aligned}
\psi(x) + \Delta_g(x) &= \psi(x + g) \\
&= \psi(x + u + g - u) \\
&= \psi\big((x + u) + g\big) - \tilde{u} \\
&= \psi(x + u) + \Delta_g(x + u) - \tilde{u} \\
&= \psi(x) + \tilde{u} + \Delta_g(x + u) - \tilde{u} \\
&= \psi(x) + \Delta_g(x + u).
\end{aligned}
$$

Same argument works in general if:

- $g \in C_G(u)$,
- $\Delta_g(x) \in C_G(\tilde{u})$,
- $C_G(\tilde{u})$ acts freely on $\Lambda\backslash H$.

Key Fact (Ratner). ψ maps $C_G(u)^\circ$-orbits into $C_H(\tilde{u})^\circ$-orbits.

Proof. Polynomial divergence of orbits (+ Lusin).

Proof of special case of Ratner's Theorem.
(Similar to proof of joinings theorem.)

Assume $U = \mathrm{Stab}_G(\mu)$ is unipotent.

We show μ is supported on a single U-orbit.
By ergodicity, it suffices to find a U-orbit of positive measure.

So suppose all U-orbits have measure 0.

For a.e. $x \in \Gamma\backslash G$, $\exists\ y \approx x$, such that

- $y \notin xU$, and
- y is in the support of μ.

$y_{t'} \approx x_t c$, for some $c \in N_G(U) \ominus U$.

We may assume $x_t, y_{t'} \in K$, where $K \cap Kc = \emptyset$.

This contradicts the fact that $d(K, Kc) > 0$. $\square$

[Actually, we only showed μ is supported on an orbit of $N_G(U)$ (because $c = e$ if $y \in xN_G(U)$), but it is easy to finish from there.]

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