Hamiltonian paths in solvable Cayley digraphs

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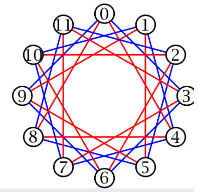
Abstract. Cayley graphs are very nice graphs that are constructed from finite groups. If the group is abelian, then it is easy to show that the graph has a hamiltonian cycle. It is conjectured that the nonabelian Cayley graphs also have hamiltonian cycles.

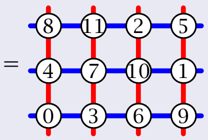
We will discuss a few recent results (both positive and negative) on the related problem where the graph is replaced by a directed graph, and the finite group is assumed to be solvable (which means it is not too far from being abelian).

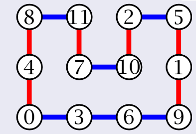
Example

Cayley graph $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 4)$



vertices: elements of $\mathbb{Z}_{12}$
edges: v — $v \pm 3$ & v — $v \pm 4$





hamiltonian cycle

Exercise

G abelian $\Rightarrow \forall S$, $\mathrm{Cay}(G; S)$ has a hamiltonian cycle
(if connected, i.e., if $\langle S \rangle = G$).

Conjecture

$\forall G, \forall S$, $\mathrm{Cay}(G; S)$ *has a hamiltonian cycle.*

(We always assume $\langle S \rangle = G$ and $S \neq \emptyset$.)

Exercise: True if G is abelian (induct on $\#S$).

My work in combinatorics proves other special cases.

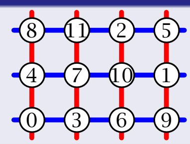
Conjecture is true if $|G|$ is "small".
E.g., $p, 2p, 3p, \ldots, 31p$, (not $24p$) $p^2, 2p^2, 3p^2, 4p^2$
[Kutnar-Marušič-Morris²-Šparl (Curran, Morris², Ghaderpour) 2011]

Work in progress

Show $\mathrm{Cay}(G; S)$ has a ham cyc if $[G, G] \cong \mathbb{Z}_{pq}$
(with p, q prime).
[Marušič, Durnberger, Keating-Witte ~1982 — Ghaderpour, Morris 2012]

Recall

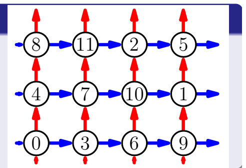
Cayley graph $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 4)$
vertices: elements of $\mathbb{Z}_{12}$
edges: v — $v \pm 3$ & v — $v \pm 4$



Conjecture: $\mathrm{Cay}(G; S)$ has ham cyc. (True if G abel.)

Definition

Cayley digraph $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12}; 3, 4)$
vertices: elements of $\mathbb{Z}_{12}$
directed edges:
$v \to v + 3$ & $v \to v + 4$



Fact: $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12}; 3, 4)$ does *not* have ham cyc.

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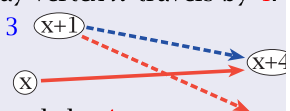
Proof.

Suppose H is a ham cycle. Say vertex x travels by 4.
Then $x + 1$ cannot travel by 3
(collision at $x + 4$).
So $x + 1$ must travel by 4.
By induction, every vertex travels by 4. $\to\leftarrow$
So no vertex travels by 4. All travel by 3. $\to\leftarrow$ □



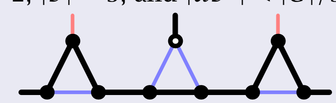
Problem: When does $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; S)$ have ham cyc?

Exercise: G abelian $\implies \overrightarrow{\mathrm{Cay}}(G; S)$ has a ham *path*.

Unfortunately, $\exists\ \overrightarrow{\mathrm{Cay}}(G; S)$ with no ham path. [Milnor]

Exercise (J. Milnor?, ≈1975?)

$\overrightarrow{\mathrm{Cay}}(G; a, b)$ no ham path
if $|a| = 2$, $|b| = 3$, and $|ab^2| < |G|/9$.



Example

No ham path in $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_6 \ltimes \mathbb{Z}_p;\ (3,0), (2,1))$
if $p > 9$ and $p \equiv 1 \pmod 6$

Remark: These are on *solvable* groups,
so solvable Cayley digs need not have ham paths.

I constructed other examples on $\mathbb{Z}_k \ltimes \mathbb{Z}_p$. (in prep)

- $\nexists$ ham path in $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_k \ltimes \mathbb{Z}_p;\ a, b)$ if ...
- G solvable $\not\Rightarrow \exists$ ham path in $\overrightarrow{\mathrm{Cay}}(G; S)$
- G abelian $\implies \exists$ ham path in $\overrightarrow{\mathrm{Cay}}(G; S)$

abelian $\subset$ nilpotent $\subset$ solvable
Nilpotent: $G = P_1 \times P_2 \times \cdots \times P_r$, $|P_i| = {p_i}^{n_i}$

Question: G nilpotent $\overset{?}{\implies} \exists$ ham path in $\overrightarrow{\mathrm{Cay}}(G; S)$

Theorem (Morris)

Assume G nilpotent.

- [1986] $|G| = p^n \implies \exists$ *ham cycle in* $\overrightarrow{\mathrm{Cay}}(G; S)$.
- [2011] $\exists$ *ham path in* $\overrightarrow{\mathrm{Cay}}(G; a, b)$

Theorem (Morris, 2011)

G nilpotent $\implies \exists$ *ham path in* $\overrightarrow{\mathrm{Cay}}(G; a, b)$.

Idea of proof.

Let $H = \langle a^{-1} b \rangle$. ("arc-forcing subgroup")
G nilpotent $\implies H \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_n \triangleleft G$.
Assume $H_n = H$. I.e., $H \triangleleft G$.
G/H gen'd by a, so (a^k) is ham cyc in $\overrightarrow{\mathrm{Cay}}(G/H; S)$.
Then $H = \langle a^k, a^k(a^{-1}b) \rangle = \langle a^{k-1} S \rangle$.
H abel $\implies \overrightarrow{\mathrm{Cay}}(H; a^{k-1} S)$ has ham path $(a^{k-1} s_i)_{i=1}^{m}$.
Ham path in $\overrightarrow{\mathrm{Cay}}(G; S)$ is $(a, a, \ldots, a, s_i)_{i=1}^{m+1}\sharp$. □

K. Kutnar et al.: Hamiltonian cycles in Cayley graphs whose order has few prime factors, *Ars Math. Contemp.* 5 (2012), no. 1, 27–71. http://amc.imfm.si/index.php/amc/article/view/177

D. W. Morris: 2-generated Cayley digraphs on nilpotent groups have hamiltonian paths, *Contrib. Discrete Math.* 7 (2012), no. 1, 41–47. http://cdm.ucalgary.ca/cdm/index.php/cdm/article/view/299

D. W. Morris: Odd-order Cayley graphs with commutator subgroup of order pq are hamiltonian (preprint). http://arxiv.org/abs/1205.0087