Does every Cayley graph have a hamiltonian cycle?

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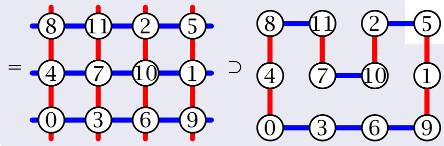
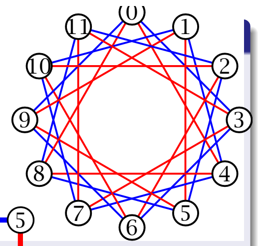
Abstract. It was conjectured 40 years ago that every connected Cayley graph has a hamiltonian cycle, but there is very little evidence for such a broad claim. The talk will describe some of the progress that has been made, and present a few of the many open problems. Almost all of the talk will be understandable to anyone familiar with the fundamentals of graph theory and group theory.

Example

Cayley graph $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 4)$

vertices: elements of $\mathbb{Z}_{12}$
edges: v — $v \pm 3$ & v — $v \pm 4$





hamiltonian cycle

Exercise

G abelian $\Rightarrow \forall S$, $\mathrm{Cay}(G;S)$ has a hamiltonian cycle
(*if connected, i.e., if $\langle S \rangle = G$*).

Conjecture

$\forall G$, $\forall S$, $\mathrm{Cay}(G;S)$ *has a hamiltonian cycle.*

(We always assume $\langle S \rangle = G$ and $S \neq \emptyset$.)

Conjecture is known to be true if:

- G is abelian [easy by induction on $\#S$]
- $G \cong D_{2n}$ is dihedral, and $4 \mid |G|$ [Alspach-Chen-Dean 2010]
- $[G,G] \cong \mathbb{Z}_{p^k}$ (p prime) [Marušič / Durnberger, Keating-Witte 1985]
- $|G|$ is "small", e.g., $p, 2p, 3p, \ldots, 31p$ (not $24p$) [Kutnar-Marušič-Morris²-Šparl (Curran, Morris², Ghaderpour) 2011]
- $|G| = p^n$, i.e., G is a p-group [Witte 1986]

Open problem

Show $\mathrm{Cay}(G;S)$ has a hamiltonian cycle if $[G,G] \cong \mathbb{Z}_{pq}$ (with p, q distinct primes).

Conjecture is known to be true if

- $|G| = p^n$, i.e., G is a p-group.

Open problem

Show $\mathrm{Cay}(G;S)$ has a hamiltonian cycle if
$G = G_1 \times G_2$, with $|G_i| = p_i^{n_i}$. (G is "nilpotent")

Remark

Cayley graphs are *vertex-transitive*.

Open problem

Assume X is a vertex-transitive graph with p^n vertices.
Show X has a hamiltonian cycle.

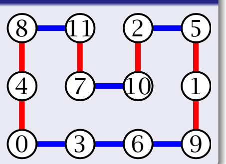
Two inductive constructions of ham cycles (via group theory):

- Factor Group Lemma (e.g., $[G,G] = \mathbb{Z}_{p^k}$),
- Skewed-generators argument (e.g., $|G| = p^n$).

Notation

Ham cyc in $\mathrm{Cay}(\mathbb{Z}_{12}; 3, 4)$:

$(3, 3, 3, 4, 4, -3, -4, -3, 4, -3, -4, -4)$



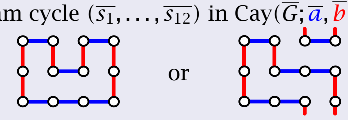
Ham cycle in $\mathrm{Cay}(G;S)$: List $(s_1, \ldots, s_n)$ of el'ts of $S \cup S^{-1}$, s.t.

- $\{ s_1, s_1s_2, s_1s_2s_3, \ldots, s_1s_2\cdots s_n \} = G$ (without repeats)
- $s_1s_2\cdots s_n = e$

Example

Suppose $S = \{a, b\}$, $[G,G] = \mathbb{Z}_p = N$, and
$G/N = \overline{G} \cong \mathbb{Z}_4 \times \mathbb{Z}_3 = \langle \overline{a} \rangle \times \langle \overline{b} \rangle$.

Ham cycle $(\overline{s_1}, \ldots, \overline{s_{12}})$ in $\mathrm{Cay}(\overline{G}; \overline{a}, \overline{b})$:

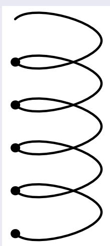


So $s_1, s_1s_2, \ldots, s_1s_2\cdots s_{12}$ are coset reps of N in G.

Let $x = s_1s_2\cdots s_{12} = a^3b^2a^{-1}b^{-1}a^{-1}ba^{-1}b^{-2}$
or $a^2b^{-1}ab^2a^{-2}ba^{-1}b^{-2}$.

If $x \neq e$, then $\langle x \rangle = N$, so
$(s_1, \ldots, s_{12})^p$ is a ham cyc in G:

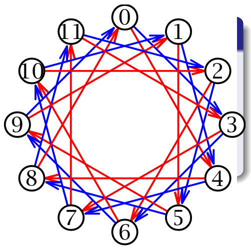
$$\begin{matrix} s_1 & s_1s_2 & \cdots & s_1s_2\cdots s_{12} \\ xs_1 & xs_1s_2 & \cdots & xs_1s_2\cdots s_{12} \\ \vdots & \vdots & \ddots & \vdots \\ x^{p-1}s_1 & x^{p-1}s_2 & \cdots & x^{p-1}s_1s_2\cdots s_{12} \end{matrix}$$



Example

Cayley digraph $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12}; 3, 4)$

vertices: elements of $\mathbb{Z}_{12}$
edges: $v \to v + 3$ & $v \to v + 4$



Exercise

G abelian $\Rightarrow \forall S$, $\mathrm{Cay}(G;S)$ has a hamiltonian cycle
(if connected, i.e., if $\langle S \rangle = G$).

Example

$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12}; 3, 4)$ does *not* have a hamiltonian cycle.

Example

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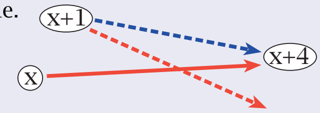
Proof.

Suppose H is a hamiltonian cycle.
Say vertex x travels by 4.
Then $x + 1$ cannot travel by 3
(collision at $x + 4$).
So $x + 1$ must travel by 4.
By induction, every vertex travels by 4. $\rightarrow\leftarrow$
So no vertex travels by 4. All travel by 3. $\rightarrow\leftarrow$ □



Open problem: When does $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; S)$ have a ham cycle?

Conjecture

$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; S)$ *has a ham cycle if* $\#S \geq 5$.

Theorem

If $|G| = p^n$, then $\overrightarrow{\mathrm{Cay}}(G;S)$ has a ham cycle.

Proof uses the Skewed-generators Argument
for Cayley *di*graphs.

Corollary

If $|G| = p^n$, then $\mathrm{Cay}(G;S)$ has a ham cycle.

Open problem

Find a direct proof. (Factor Group Lemma?)

Proposition

If $|G| = p^n$ and G is abelian, then $\overrightarrow{\mathrm{Cay}}(G;S)$ has a ham cycle.

Note: We may assume S is *minimal* generating set.

Recall: B a basis for vector space V, $\#B_1 < \#B \Rightarrow \langle B_1 \rangle \neq V$.

Similar fact for generating sets of p-groups

$|G| = p^n$, S min'l gen set, $\#T < \#S \Rightarrow \langle T \rangle \neq G$.

Choose $a \in S$. Let $T = a^{-1}S \smallsetminus \{e\}$, $H = \langle T \rangle$, $d = |G : H|$,
and $S' = a^d(a^{-1}S) = a^{d-1}S$.

$\langle S' \rangle = H = \langle T \rangle < G \overset{\text{induction}}{\Longrightarrow}$
$\overrightarrow{\mathrm{Cay}}(H; S')$ has ham cycle $(a^{d-1}s_1, a^{d-1}s_2, \ldots, a^{d-1}s_m)$.

Then $(a, a, a, \ldots, a, s_1, a, a, a, \ldots, a, s_2, \ldots, a, a, a, \ldots, a, s_m)$
is a ham cycle in $\overrightarrow{\mathrm{Cay}}(G;S)$.

Conjectures about $\mathrm{Cay}(G;S)$:

- $\exists$ hamiltonian cycle.
- $\exists$ hamiltonian path.
- $\exists$ path of length $\epsilon\,\#G$.
- $\mathrm{Cay}(G;S)$ has a hamiltonian cycle for *some* (minimal) S.
- [Babai] $\nexists$ cycle of length $(1-\epsilon)\,\#G$.

Proposition

- [Babai] $\exists$ *path (& cycle) of length* $\approx \sqrt{\#G}$.
- [Witte] $\forall S, \exists S'$, $\mathrm{Cay}(G;S')$ *has ham cyc, and* $\#S' \leq 2(\#S)^2$.
- [Pak] $\forall G, \exists S$, $\mathrm{Cay}(G;S)$ *has ham cyc, and* $\#S \leq \log_2 \#G$.
- [Rankin + CFSG] *G simple $\Rightarrow \exists$ $\overrightarrow{\mathrm{Cay}}(G; a, b)$ with ham cycle.*
  ($\overrightarrow{\mathrm{Cay}}(G; a, b)$ has ham cycle if $(a^{-1}b)^2 = e$.)

Some references

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