Hamiltonian cycles in Cayley graphs

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Abstract

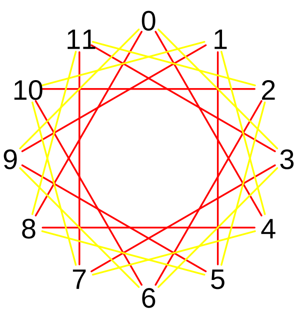
Circulant graphs are constructed from the cyclic group of integers modulo n. Replacing the cyclic group with a more interesting finite group leads to the class of Cayley graphs. These graphs have been studied both because of their symmetry properties and because of their important applications. It was conjectured more than 30 years ago that every Cayley graph has a hamiltonian cycle, but we are still nowhere near a proof. The talk will describe some of the results that are known and some of the basic questions that are still unanswered.

Eg. Circulant graph $\mathrm{Circ}(12; 3, 4)$

vertices: elements of $\mathbb{Z}_{12}$
edges: $v — v \pm 3$ & $v — v \pm 4$

Notation. $\mathrm{Circ}(n; s_1, s_2, \ldots, s_r)$.
Assume *connected*:
$\gcd(s_1, s_2, \ldots, s_r, n) = 1$.

Exer. Every circulant graph has a *hamiltonian cycle*.



Note. $\mathrm{Circ}(12; 3, 4) \cong$ *Cartesian product* $C_3 \,\square\, C_4$.

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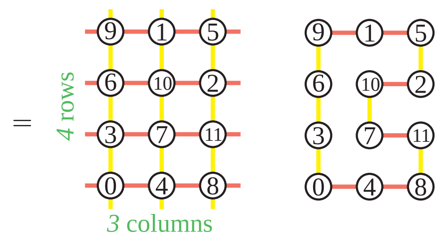
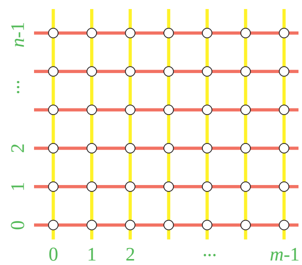


Exer (easy). $C_m \,\square\, C_n$ has a hamiltonian cycle.

Vertices of $C_m \,\square\, C_n$
= ordered pairs
= elements of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

Edges
$(x, y) — (x \pm 1, y)$
$(x, y) — (x, y \pm 1)$.



I.e., $v — v \pm e_1$ and $v — v \pm e_2$,
where $e_1 = (1, 0)$ and $e_2 = (0, 1)$
are the natural generators of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

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Defn. G = finite abelian group (e.g., $\mathbb{Z}_m \oplus \mathbb{Z}_n$)
S = generating set of G (e.g., $\{e_1, e_2\}$)
Cayley graph $\mathrm{Cay}(G; S)$:
vertices = elements of G
edge $v — v \pm s$ for $v \in G$ and $s \in S$

Eg. $\mathrm{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n; e_1, e_2) \cong C_m \,\square\, C_n$.

Exer. $\mathrm{Cay}(G; S)$ has a ham cycle (for any G and S).

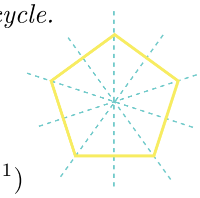
Defn.
G = finite [redacted] group
S = generating set of G
Cayley graph $\mathrm{Cay}(G; S)$:
vertices = elements of G
edge $v — v s^{\pm 1}$ for $v \in G$ and $s \in S$

Conj.
[redacted] $\mathrm{Cay}(G; S)$ has a ham cycle. (True if G abel.)
Very little progress (especially since 1990).

Conj.
- (Babai) *The conjecture is false.*
- $\mathrm{Cay}(G; S)$ *has a hamiltonian path.*
- $\mathrm{Cay}(G; S)$ *has a path of length* $\epsilon \, \#G$.

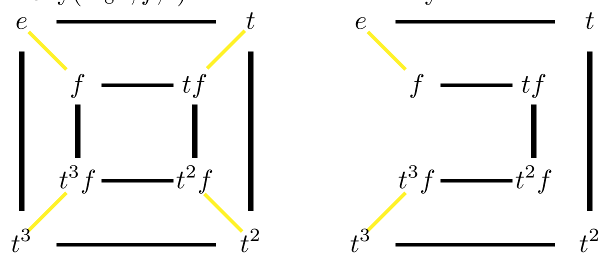
Conj. $\mathrm{Cay}(G; S)$ *has a hamiltonian cycle.*

Eg. Dihedral group D_{2n} of order $2n$
= symmetries of a regular n-gon
- n rotations $(t^0, t^1, t^2, \ldots, t^{n-1})$
- n reflections $(f, ft^1, ft^2, \ldots, ft^{n-1})$
- $D_{2n} = \langle\, t, f \mid t^n = e,\ f^2 = e,\ ftf = t^{-1} \,\rangle$



Rem. $\mathrm{Cay}(D_{2n}; f, t)$ has a hamiltonian cycle.

Rem. Cay(D_8 ; f, t) has a hamiltonian cycle.

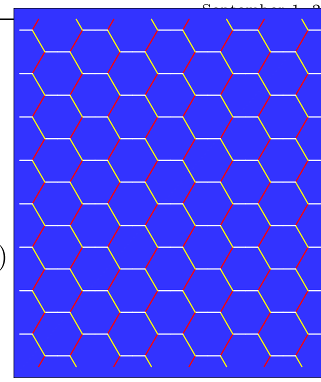


Exer. If $\gcd(a, b, n) = 1$, then $\langle f, ft^a, ft^b \rangle = D_{2n}$,

so Cay(D_{2n}; f, ft^a, ft^b) has valence 3.

(Embeds on torus, with every face a hexagon.)

Cay(D_{2n}; f, ft^a, ft^b):



Thm (Alspach-Zhang).
Cay(D_{2n}; f, ft^a, ft^b)
has a hamiltonian cycle.

Conj. Cay(D_{2n}; {*reflections*})
has a ham cyc.
(Then every Cay(D_{2n}; S)
has a ham cyc.)

Conj. Cay(G; S) *has a hamiltonian cycle.*

True when G is "*almost*" abelian.

Defn. commutator subgroup of $G = [G, G]$
$= \langle g^{-1}h^{-1}gh \mid g, h \in G \rangle$.

Rem. G is abelian $\Leftrightarrow$ $[G, G] = \{e\}$.

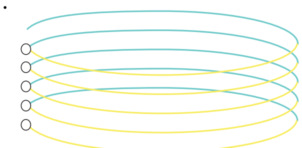
Thm (Durnberger, Marušič, Keating-Witte).
Cay(G; S) *has a ham cycle if* $[G, G]$ *has prime order or, more generally, is cyclic of prime-power order.*

Problem. *Find hamiltonian cycle if* $[G, G] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Thm (Durnberger, Marušič, Keating-Witte).
Cay(G; S) *has a ham cycle if* $[G, G]$ *has prime order.*

Idea of proof. $\overline{G} = G/[G, G]$ is abelian
$\Rightarrow$ Cay($\overline{G}$; $\overline{S}$) has a ham cyc $\overline{C}$.

Lift $\overline{C}$ to a *path* P in Cay(G; S).



Assume P is *not* a cycle.
["*Marušič's method*"]
Then we construct ham cyc
in Cay(G; S)
by concatenating translates of P.

Thm (Witte). Cay(G; S) *has a hamiltonian cycle if* $\#G$ *is a prime power* p^n.

Problem. *Find hamiltonian cycle if* $\#G = 2p^n$.

Problem. *Find hamiltonian cycle if* $G = P \times Q$
where $\#P$ *and* $\#Q$ *are prime powers.*
(G is "nilpotent.")

Summary. Cay(G; S) has a hamiltonian cycle if

- G is abelian, or
- $[G, G]$ is cyclic of prime-power order, or
- G is of prime-power order.

Important. True for *all* generating sets S of G.
(Not just particular generating sets of G.)

Most groups are **not** covered.
But conjecture is true for small groups.

Thm (**?** Friedman-Jungreis, Kutnar-Marušič-Morris^2-Šparl).
Cay(G; S) *has a hamiltonian cycle if* $\#G =$

- $k\,p$, *with* $k \le 23$ ($\ne 16$),
- $k\,p^3$, *with* $k \le 2$,
- pqr,
- $k\,p^2$, *with* $k \le 4$,
- $k\,pq$, *with* $k \le 4$,
- p^n.

For $\#G \le 100$, missing

- $48 = 2^4 \cdot 3 = 16p$,
- $80 = 2^4 \cdot 5 = 16p$,
- $72 = 2^3 \cdot 3^2 = 8p^2$,
- $96 = 2^5 \cdot 3 = 32p$.

Should do $16p$ and $8p^2$. **(Help???)**
$32p$ will be *too much work.*

There are many results on *specific* generating sets.

Thm (Ruskey-Savage). *$\mathrm{Cay}(S_n; S)$ has a ham cyc if S consists of transpositions (i, j).*

Thm (Alspach). *$\mathrm{Cay}(G; s, t)$ has a ham cyc if $\langle s \rangle$ is a normal subgroup of G.*

Thm (Rankin). *$\overrightarrow{\mathrm{Cay}}(G; s, t)$ has a ham cyc if st^{-1} has order 2 (i.e., $(st^{-1})^2 = e$).*

Cor. *G simple $\Rightarrow \exists\ s, t \in G$, $\overrightarrow{\mathrm{Cay}}(G; s, t)$ has ham cyc.*

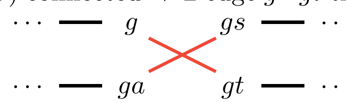
Cor (Pak). *$\forall G$, $\exists S$, $\overrightarrow{\mathrm{Cay}}(G; S)$ has a ham cyc, and $\#S \le \log_2 \#G$.*

Thm (Rankin). *$\overrightarrow{\mathrm{Cay}}(G; s, t)$ has a ham cyc if $(st^{-1})^2 = e$.*

Proof. $\forall g$, $g - gs - gs^2 - \cdots - gs^n = g$ (cycle).

Vertices of $\mathrm{Cay}(G; S)$ covered by disjoint cycles.

$\mathrm{Cay}(G; S)$ connected $\Rightarrow \exists$ edge $g - gt$ that joins two cycles.



Let $a = st^{-1}$, so $a^2 = e$ and $s = at$.

Then $gas = ga(at) = ga^2t = gt$ and $gat = gs$.

Remove edges $g - gs$ and $ga - gt$: 2 cycles $\to$ 1.

Continue: combine *all* cycles into one (ham).

Conj. *$\mathrm{Cay}(G; S)$ has a hamiltonian cycle.*

Problem. *Find ham cyc in prism $\mathrm{Cay}(G; S) \,\square\, P_2$.*

Done when valence is three:

Thm (Rosenfeld).
The prism over X has a hamiltonian cycle if X is cubic and 3-connected.

Survey articles.

B. Alspach,
The search for long paths and cycles in vertex-transitive graphs and digraphs.
Combinatorial mathematics, VIII (*Geelong, 1980*), pp. 14–22, Lecture Notes in Math., 884, Springer, Berlin-New York, 1981.
MR 83b:05080

D. Witte and J.A. Gallian,
A survey: Hamiltonian cycles in Cayley graphs,
Discrete Math. 51 (1984), no. 3, 293–304.
MR 86a:05084

S.J. Curran and J.A. Gallian,
Hamiltonian cycles and paths in Cayley graphs and digraphs–a survey,
Discrete Math. 156 (1996), no. 1–3, 1–18.
MR 97f:05083