Hamiltonian paths and cycles in vertex-transitive graphs and digraphs

Dave (Witte) Morris

Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge, AB T1K 3M4
dmorris@cs.uleth.ca

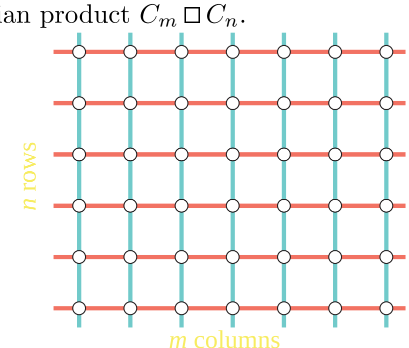
Dept of Math, Okla State U, Stillwater, OK 74078, USA
dwitte@math.okstate.edu

http://www.math.okstate.edu/∼dwitte

Abstract

It was conjectured, more than 30 years ago, that every vertex-transitive graph has a hamiltonian path, and that every Cayley graph has a hamiltonian cycle (unless the graph is disconnected). This talk will survey the progress that has been made on these problems, both of which remain very much open. For example, it is still not known whether every Cayley graph on every dihedral group has a hamiltonian cycle. (The cubic case was settled by Brian Alspach and C.–Q. Zhang.) Related questions on directed graphs will also be discussed; for example, it is easy to see that every (connected) circulant digraph has a hamiltonian path, but we do not know which circulant digraphs have hamiltonian cycles.

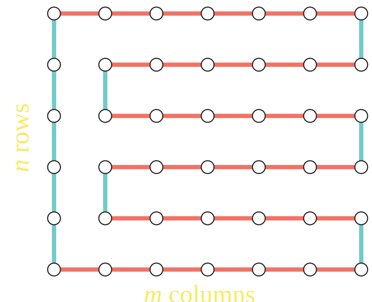
Eg. Cartesian product $C_m \square C_n$.



Exer (Easy). $C_m \square C_n$ has a Hamilton cycle.

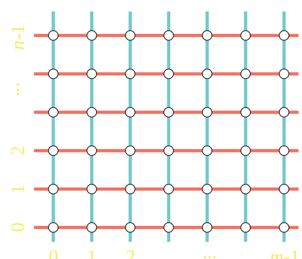
(omit)

Eg. Hamilton cycle in $C_m \square C_n$ when n is even.



Vertices of $C_m \square C_n$
 = ordered pairs
 = elements of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

Edges
 (x,y)—$(x\pm1, y)$
 (x,y)—$(x, y\pm1)$.



I.e., v—$v\pm e_1$ and v—$v\pm e_2$,
 where $e_1 = (1,0)$ and $e_2 = (0,1)$
 are the natural generators of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

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Defn. G = finite abelian group (e.g., $\mathbb{Z}_m \oplus \mathbb{Z}_n$)
 S = generating set of G (e.g., $\{e_1, e_2\}$)

Cayley graph Cay$(G;S)$:
 vertices = elements of G
 edge v—$v \pm s$ for $v \in G$ and $s \in S$

Eg. $\mathrm{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n;\{e_1,e_2\}) \cong C_m \square C_n$.

Exer. Cay$(G;S)$ has a Hamilton cycle (for any G and S).

In fact, Cay$(G;S)$ has *many* Hamilton cycles.

Thm (Chen-Quimpo). *Cay$(G;S)$ is Hamilton connected*
 (i.e., $\forall v,w$, $\exists$ Ham path from v to w)
unless valence ≤ 2 (or graph is bipartite).

Conj (Alspach). *Cay$(G;S)$ is Hamilton decomposable*
 (i.e., edge-disjoint union of Ham cycles [+ 1-factor?]).

Thm (Bermond-Favaron-Mahéo). *True if valence ≤ 5.*

In fact, Cay$(G;S)$ has *many* Hamilton cycles.

Recall. Any *flow* in any graph is a sum of *cycles*.

- A *flow* is a function $\phi\colon E^{\pm}(X) \to \mathbb{Z}$, s.t. ...
- Flows can be added. $(\phi+\psi)(e) = \phi(e) + \psi(e)$
- Any directed cycle defines a flow $\phi\colon E^{\pm}(X) \to \{0, \pm1\}$.

Thm (Alspach-Locke-Witte, Locke-Witte).
Every flow in Cay$(G;S)$ is a sum of **Hamilton** *cycles if $\#G$ is odd (and Cay$(G;S) \not\cong C_3 \square C_3$).*

$\#G$ even $\Rightarrow$ Ham cycs are *even* flows: $\displaystyle\sum_{e \in E^+(X)} f(e) \in 2\mathbb{Z}$.

$\#G$ even $\Rightarrow$ Ham cycs are *even* flows: $\displaystyle\sum_{e \in E^+(X)} f(e) \in 2\mathbb{Z}$.

Converse:

Thm (Morris, Morris, Moulton).
Every even flow in $\mathrm{Cay}(G;S)$ *is a sum of Hamilton cycles if valence* ≥ 5.

We have *almost* finished classifying the counterexamples, but there is still an open case of valence 4.

Conj (Moulton).
Not every even flow is a sum of Ham cycs in $C_{\mathrm{odd}} \square C_{4k+2}$.

Conj. $\mathrm{Cay}(G;S)$ *has a Hamilton cycle.*

Problem. *Show* $\mathrm{Cay}(G;S)$ *has a Ham cyc if G is small (say,* $\#G < 100$*).*

Frank Ruskey: *cubic* of order < 100 may be done?
(Gordon Royle's web page has a list.)

Problem. *Find Ham cyc in prism* $\mathrm{Cay}(G;S) \square P_2$.

Done for cubic case:

Thm (Rosenfeld).
The prism over X has a Hamilton cycle if X is cubic and 3-connected.

Conj. $\mathrm{Cay}(G;S)$ *has a Hamilton cycle.*

Positive result when G is "*almost*" abelian.

Defn. commutator subgroup of $G = [G,G]$
$= \langle g^{-1}h^{-1}gh \mid g,h \in G \rangle$.

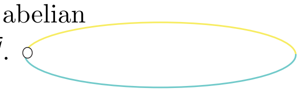
Rem. G is abelian $\Leftrightarrow [G,G] = \{e\}$.

Thm (Durnberger, Marušič, Keating-Witte).
$\mathrm{Cay}(G;S)$ *has a Hamilton cycle if* $[G,G]$ *has prime order or, more generally, is cyclic of prime-power order.*

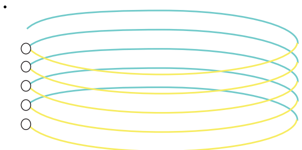
Problem. *Find Hamilton cycle if* $[G,G] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Thm (Durnberger, Marušič, Keating-Witte).
$\mathrm{Cay}(G;S)$ *has a Hamilton cycle if* $[G,G]$ *has prime order.*

Idea of proof. $\overline{G} = G/[G,G]$ is abelian
$\Rightarrow \mathrm{Cay}(\overline{G};\overline{S})$ has a Ham cyc $\overline{C}$.



Lift $\overline{C}$ to a *path* P in $\mathrm{Cay}(G;S)$.



Assume P is *not* a cycle.
["*Marušič's method*"]
Then we construct Ham cyc in $\mathrm{Cay}(G;S)$
by concatenating translates of P. $\square$

Thm (Witte). $\mathrm{Cay}(G;S)$ *has a Hamilton cycle if* $\#G$ *is a prime power* p^n.

Problem. *Find Hamilton cycle if* $\#G = 2p^n$.

Problem. *Find Hamilton cycle if* $G = P \times Q$ *where* $\#P$ *and* $\#Q$ *are prime powers.*
(G is "nilpotent.")

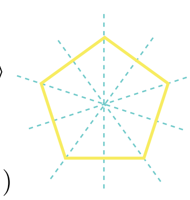
Problem. *Generalize to vertex-transitive graphs.*

Thm (Yu Qing Chen).
Vertex-transitive graphs of order p^4 *have Ham cycs.*

Summary. $\mathrm{Cay}(G;S)$ has a Hamilton cycle if
- $[G,G]$ is cyclic of prime-power order, or
- G is of prime-power order.

Most groups do not fall into these categories.

Eg. Dihedral group D_{2n} of order $2n$
$= \langle t, f \mid t^n = e, f^2 = e, ftf = t^{-1} \rangle$
= symmetries of a regular n-gon
- n rotations $(t^0, t^1, t^2, \ldots, t^{n-1})$
- n reflections $(f, ft^1, ft^2, \ldots, ft^{n-1})$



Rem. $\mathrm{Cay}\big(D_{2n}; \{t,f\}\big) \cong$ prism (over cycle $\langle t \rangle$) has a Hamilton cycle.

Eg. Dihedral group D_{2n} of order $2n$
$= \langle t, f \mid t^n = e, f^2 = e, ftf = t^{-1} \rangle$.

Rem. $\mathrm{Cay}\big(D_{2n}; \{t,f\}\big)$ has a Hamilton cycle.

Exer. If $\gcd(a,b,n) = 1$, then $\langle f, ft^a, ft^b \rangle = D_{2n}$,
so $\mathrm{Cay}\big(D_{2n}; \{f, ft^a, ft^b\}\big)$ is cubic.
(Embeds on torus, with every face a hexagon.)

Thm (Alspach-Zhang).
$\mathrm{Cay}\big(D_{2n}, \{f, ft^a, ft^b\}\big)$ *has a Hamilton cycle.*

Conj. $\mathrm{Cay}\big(D_{2n}, \{f, ft^{a_1}, ft^{a_2}, \ldots, ft^{a_r}\}\big)$ *has a Ham cyc.*
(Then $\mathrm{Cay}(D_{2n}, S)$ has a Ham cyc, for any S.)

Conj. $\mathrm{Cay}\big(D_{2n}, \{f, ft^{a_1}, ft^{a_2}, \ldots, ft^{a_r}\}\big)$ *has Ham cyc.*

Possible approach (stronger induction hypothesis):

Problem (Alspach).
Show $\mathrm{Cay}(D_{2n}, S)$ *is Hamilton connected if* $\#S = 3$.

Another approach is via Cayley **di**graphs. (**directed**)

Defn. G = finite [redacted] group
S = generating set of G
Cayley digraph $\overrightarrow{\mathrm{Cay}}(G;S)$:
vertices = elements of G
edge $v \to vs$ for $v \in G$ and $s \in S$.

Exer. If G is abelian, then $\overrightarrow{\mathrm{Cay}}(G;S)$ has a Hamilton *path*.

Exer. $\overrightarrow{\mathrm{Cay}}\big(\mathbb{Z}_{12};\{3,4\}\big)$ does *not* have a Hamilton cycle.

Problem. *Which circulant digraphs have a Ham cycle?*

Answer (Rankin, 1948) when $\#S = 2$ (and G is abelian).

Conj (Curran-Witte). *If*

- *G is cyclic (or abelian), and*
- *$\#S \geq 3$, and*
- *S is minimal (no proper subset of S generates G),*

then $\overrightarrow{\mathrm{Cay}}(G;S)$ has a Hamilton cycle.

Exer. Conj $\Rightarrow$ $\mathrm{Cay}(D_{2n}, S)$ has a Ham cycle, for every S.

Conj (Curran-Witte). *G abelian, $\#S \geq 3$, S minimal $\Rightarrow \overrightarrow{\mathrm{Cay}}(G;S)$ has a Hamilton cycle.*

Thm (Curran-Witte).
$\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ *has an obvious gen set* S_{nat}.
$r \geq 3 \Rightarrow \overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}; S_{\mathrm{nat}})$ *has a Ham cyc.*

Cor. *If $\overrightarrow{C}_1, \ldots, \overrightarrow{C}_r$ are directed cycles, and $r \geq 3$, then $\overrightarrow{C}_1 \square \cdots \square \overrightarrow{C}_r$ has a Hamilton cycle.*

Thm (Locke-Witte).
$\overrightarrow{\mathrm{Cay}}\big(\mathbb{Z}_{12k};\{6k, 6k+2, 6k+3\}\big)$ *has no Hamilton cycle.*

Ques. $\overrightarrow{\mathrm{Cay}}(G;S)$ *has Ham cyc if $\#S \geq 4$ (& G cyclic)?*

Conj (Curran-Witte). *G abelian, $\#S \geq 3$, S minimal $\Rightarrow \overrightarrow{\mathrm{Cay}}(G;S)$ has a Hamilton cycle.*

Conj. *If $\overrightarrow{C}_1, \ldots, \overrightarrow{C}_r$ are directed cycles, and $r \geq 3$, and* $\gcd(n_1, \ldots, n_r) = 1$, *then $\overrightarrow{C}_1 \square \cdots \square \overrightarrow{C}_r$ is Ham conn.*
(I.e., $\forall$ vertices v, w, $\exists$ Ham path from v to w.)

Exer. In $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}; S_{\mathrm{nat}})$,
if there is a Hamilton path from v to w, then
$w_1 + \cdots + w_r \equiv v_1 + \cdots + v_r - 1 \pmod{\gcd(n_1, \ldots, n_r)}$.

Conj (Austin-Gavlas-Witte). *Converse if $r \geq 3$.*

Thm (Austin-Gavlas-Witte). *True when $n_1 = \cdots = n_r$.*

There are many results on *specific* generating sets.

Thm (Alspach). $\mathrm{Cay}\big(G;\{x,y\}\big)$ *has a Hamilton cycle if $\langle x \rangle$ is a normal subgroup of G.*

Thm (Rankin). $\overrightarrow{\mathrm{Cay}}\big(G;\{x,y\}\big)$ *has a Hamilton cycle if xy^{-1} has order 2 (i.e., $(xy^{-1})^2 = e$).*

Cor. *G simple $\Rightarrow \exists\, x, y \in G$, $\overrightarrow{\mathrm{Cay}}\big(G;\{x,y\}\big)$ has H cyc.*

Cor (Pak). *$\forall G$, $\exists S$, $\overrightarrow{\mathrm{Cay}}\big(G;S\big)$ has a Hamilton cycle, and $\#S \leq \log_2 \#G$.*

Conj. *$\forall G$, $\exists S$, $\overrightarrow{\mathrm{Cay}}\big(G;S\big)$ has a Hamilton cycle, and S is a minimal generating set (or minimum?).*

Survey articles.

B. Alspach,
The search for long paths and cycles in vertex-transitive graphs and digraphs.
Combinatorial mathematics, VIII (Geelong, 1980), pp. 14–22,
Lecture Notes in Math., 884, Springer, Berlin-New York, 1981.
MR 83b:05080

D. Witte and J.A. Gallian,
A survey: Hamiltonian cycles in Cayley graphs,
Discrete Math. 51 (1984), no. 3, 293–304.
MR 86a:05084

S.J. Curran and J.A. Gallian,
Hamiltonian cycles and paths in Cayley graphs and digraphs–a survey,
Discrete Math. 156 (1996), no. 1–3, 1–18.
MR 97f:05083