Survey of hamiltonian cycles in Cayley graphs

Dave Witte Morris

University of Lethbridge, Alberta, Canada

http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Abstract. It was conjectured about 40 years ago that every connected Cayley graph has a hamiltonian cycle. This is easy to prove for Cayley graphs on abelian groups, but we are nowhere near a proof of the general case. The talk will discuss some of the progress that has been made, and some of the many open problems.

Notation

- G = finite group
- S = generating set for G
- $\mathrm{Cay}(G;S)$ = *Cayley graph*
  - vertices = elements of G
  - edge g — gs for $g \in G$ and $s \in S$

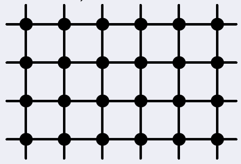
Conjecture (~1970)

$\mathrm{Cay}(G;S)$ has a hamiltonian cycle.

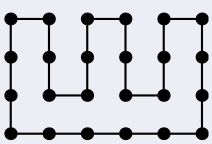
Not much progress.

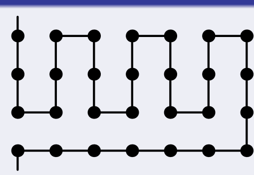
Example

$\mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_n; \{(1,0),(0,1)\})$ has a hamiltonian cycle.



Proof.





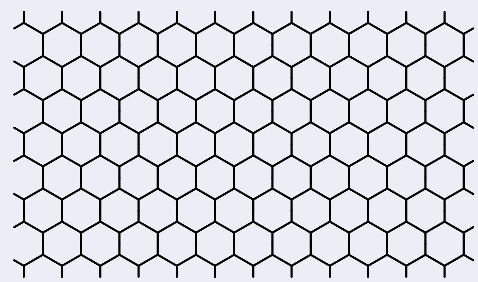
□

Exercise: $\mathrm{Cay}(G;S)$ has a hamiltonian cycle if G is abelian.

Example

- G = dihedral group $D_{2n} = \langle f, t \mid f^2 = t^n = e,\ ftf = t^{-1} \rangle$
- $S = \{f, ft^a, ft^b\}$

$\mathrm{Cay}(D_{2n}; f, ft^a, ft^b)$ is hexagonal tiling on a torus.



This has a hamiltonian cycle [Alspach-Qiang]. (But not easy!)

Conjecture

- $\mathrm{Cay}(G;S)$ has a hamiltonian cycle.
- $\mathrm{Cay}(G;S)$ has a hamiltonian path.
- $\mathrm{Cay}(G;S)$ has a path of length $\epsilon\, \#G$.
- $\mathrm{Cay}(G;S)$ has a hamiltonian cycle for *some* (irredundant) S.
- [Babai] Opposite conjecture: not always a ham path.

Proposition

- [Babai] $\exists$ *path (& cycle) of length* $\approx \sqrt{\#G}$.
- [Pak] $\forall G, \exists S$, $\mathrm{Cay}(G;S)$ *has a ham cyc, and* $\#S \le \log_2 \#G$.
- [Witte] $\forall S, \exists S'$, $\mathrm{Cay}(G;S')$ *has a ham cyc, and* $\#S' \le (\#S)^2$.

Theorem

$\mathrm{Cay}(G;S)$ *has a hamiltonian cycle if:*

- *G is dihedral and* $4 \mid \#G$. [Alspach et al.]
- $\#G = p^n$ *(prime power).* [Witte]
- *commutator subgroup of G is cyclic of prime-power order.* [Keating-Witte]

Problem

Find a hamiltonian cycle if:

- G is dihedral.
- $G = P \times Q$ where $\#P$ and $\#Q$ are prime powers. (G is "nilpotent.")

The directed case

Conjecture. Cay$(G;S)$ has a hamiltonian cycle.

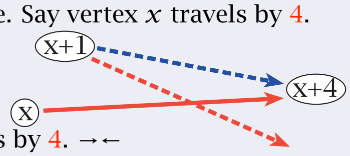
Remark

Cayley digraph $\overrightarrow{\text{Cay}}(G;S)$ might *not* have a ham cyc (or path).

Example. $\nexists$ hamiltonian cycle in $\overrightarrow{\text{Cay}}(\mathbb{Z}_{12};3,4)$.

Proof.

Suppose H is a hamiltonian cycle. Say vertex x travels by 4.
Then $x+1$ cannot travel by 3
(collision at $x+4$).
So $x+1$ must travel by 4.
By induction, every vertex travels by 4. →←
So no vertex travels by 4. They all travel by 3. →← □



Theorem (Rankin, ~1940)

$\overrightarrow{\text{Cay}}(\mathbb{Z}_n;a,b)$ *has a ham cyc* $\iff$
$\exists\, s,t \in \mathbb{Z}^{\geq 0}$, *s.t.* $s+t = \gcd(a-b,n) = \gcd(sa+tb,n)$.

Similar if we replace $\mathbb{Z}_n$ with any abelian group.

Conjecture

G is abelian, $\#S \geq 3$, but $\overrightarrow{\text{Cay}}(G;S)$ has no hamiltonian cycle ⇒

- $\#S = 3$,
- G is cyclic,
- $\#G$ is even,
- S has an explicit description.

Example (Locke-Witte)

$\overrightarrow{\text{Cay}}(\mathbb{Z}_{12k};\,6k,\,6k+2,\,6k+3)$ has no hamiltonian cycle.

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