Hamiltonian cycles in some easy Cayley graphs

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Abstract. Some of Chris Godsil's early work (including his Ph.D. thesis) completed the proof of a conjecture of Mark Watkins about automorphism groups of Cayley graphs. We will explain the conjecture, and then we will discuss a different conjecture that is still open: every connected Cayley graph has a hamiltonian cycle. Recent work provides a positive answer when the number of vertices is a small multiple of a prime number, and in some other similar cases.

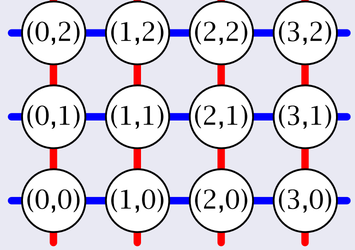
http://people.uleth.ca/~dave.morris/talks.shtml

Definition

Cayley graph **Cay(G; S)** (for $S \subseteq G$ with $S = S^{-1}$):

- vertices = elements of group G ($S = -S$)
- edge $x - xs$ for $x \in G$ and $s \in S$. ($x - x + s$)

Example. $\mathrm{Cay}(\mathbb{Z}_4 \oplus \mathbb{Z}_3; \{\, \pm(1,0),\ \pm(0,1)\})$



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Observation

Edge $x - y \Rightarrow y = xs \Rightarrow \forall g \in G,\ gy = gxs$
  $\Rightarrow$ *edge* $gx - gy$,   *i.e.*, $\varphi_g(x) - \varphi_g(y)$.

Mult by g on the left is an aut: $G \subseteq \mathrm{Aut}(\mathrm{Cay}(G;S))$.

Example

$\mathrm{Aut}(\mathrm{Cay}(G;S))$ can be much larger than G: ($|G| = n$)
  $\mathrm{Aut}(\mathrm{Cay}(G;G)) = \mathrm{Aut}(K_n) = S_n$ has order $n!$.
Structure of $\mathrm{Cay}(G;G)$ has nothing to do with G.

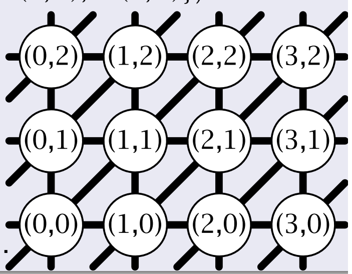
Conjecture (Mark Watkins, 1971)

$\forall G, \exists S, \mathrm{Aut}(\mathrm{Cay}(G;S)) = G$ *unless* $(*)$.
  "Graphical Regular Representation" (GRR)

Example

$\mathrm{Cay}(\mathbb{Z}_4 \oplus \mathbb{Z}_3; \{\pm(1,0),\ \pm(0,1),\ \pm(1,1)\})$

Add generator $(1,1)$.
Still has rotation symmetry (180°):
$\varphi(x) = -x$ in $\mathrm{Aut}(G)$
  and $\varphi(S) = S$
$\Rightarrow \varphi \in \mathrm{Aut}(\mathrm{Cay}(G;S))$.



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$(*)$: $\forall S = S^{-1}, \exists \varphi \in \mathrm{Aut}(G), \varphi(S) = S$.   ($\varphi \neq 1$)

Example

G abelian $\Rightarrow$ can let $\varphi(x) = x^{-1} \Rightarrow (*) \Rightarrow$ no GRR.
  (unless $|x| = 2$, $\forall x \in G$)

Theorem (Babai 1978, Hetzel 1976)

$(*) \Leftrightarrow$ *explicit list.*
(abelian or abel subgrp of index 2 or finitely many)

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Partial [Imrich, Nowitz, Watkins 1970's]. Complete:

- Hetzel (G solvable, 1976),
- Godsil (G not solvable, 1978).

Conjecture (Babai-Godsil 1982)   *unless* $(*)$

Random Cayley graph on G is a GRR with high prob.

- Babai-Godsil: true if G nilpotent of odd order.
- General case is still open [Morris-Spiga-Verret (on arxiv)].

Open Problem (~1970)

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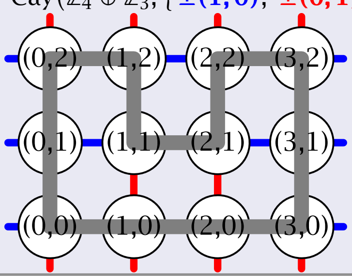

Exercise. Find ham cycle in $\mathrm{Cay}(G;S)$ if G is abelian.

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Remark. Open for nilpotent groups (but not p-groups).
  (**Cubic** Cayley graphs on **nilpotent** groups have a ham **path**.)

Recall. *commutator subgroup* $[G,G] = \langle\, ghg^{-1}h^{-1} \mid g,h \in G \,\rangle$.
  G abelian $\Leftrightarrow [G,G] = \{e\} \Leftrightarrow |[G,G]| = 1$.

¿ Can we find a ham cycle if $|[G,G]|$ *is small ?*

Theorem (Marušič, Durnberger, Keating-Witte 1985)

$\mathrm{Cay}(G;S)$ *has a ham cycle if* $|[G,G]| = p$ (*prime*).

Open problem. Find ham cycle if $[G,G] = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Work in progress: $|[G,G]| = p_1 p_2$.   ($p_1 \neq p_2$)

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