Infinitely many nonsolvable groups whose Cayley graphs are hamiltonian

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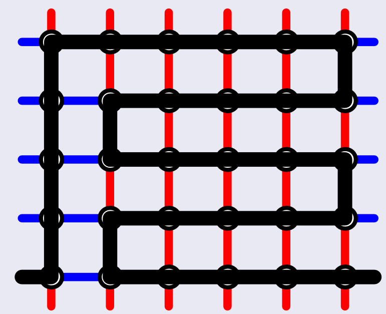
Abstract. It has been conjectured that if G is any finite group, then every connected Cayley graph on G has a hamiltonian cycle. This conjecture has been verified for numerous groups that either are small or are close to being abelian, but we provide the first verification that includes infinitely many non-solvable groups. More precisely, we exhibit infinitely many primes p, such that every connected Cayley graph on the direct product $A_5 \times \mathbb{Z}_p$ has a hamiltonian cycle (where A_5 is the alternating group on 5 letters).

Defn. Cay(G;S) for group G and $S \subseteq G$
verts = elt's of G   edge g — $gs^{\pm 1}$ for $g \in G$, $s \in S$.
(assume **connected**, i.e., $\langle S \rangle = G$)

Conjecture (~1970)
Every connected Cayley graph has a hamiltonian cycle.

Example
Cay($\mathbb{Z}_m \oplus \mathbb{Z}_n$; {(1, 0), (0, 1)}) has hamiltonian cycle.



Exercise
Conjecture **true** for abelian groups.

Conjecture (~1970)
Every connected Cayley graph has a hamiltonian cycle.

Known to be true if:

- G is abelian [folklore] or $|[G,G]| = p$ (prime) [1985]
- $|G| = kp$ $(24 \neq k < 32)$, kp^2 $(k \leq 4)$, kp^3 $(k \leq 2)$, kpq $(k \leq 5)$, pqr. [1990, 2001, 2012–2014]
- etc.

All but finitely many of these groups are **solvable** (made from abelian groups).

Theorem (Morris, 2015)
There are infinitely many groups G, such that every Cayley graph on G has a ham cycle, and G is not solvable.

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Smallest group that is not solvable is A_5 (alternating group on 5 letters) of order 60.
(A_5 is **simple**, so $[A_5, A_5] = A_5$, so not solvable.)

Proposition (Kutnar et al., 2010 arxiv)
Every Cayley graph on A_5 has a hamiltonian cycle.

Theorem (Morris, 2015)
There are infinitely many primes p, such that every Cayley graph on $A_5 \times \mathbb{Z}_p$ has a ham cycle.

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Proof

We construct a hamiltonian cycle in $\mathrm{Cay}(A_5 \times \mathbb{Z}_p; S)$ from a hamiltonian cycle in $\mathrm{Cay}(A_5; \overline{S})$ where $\overline{S}$ is the projection of S to A_5.

Factor Group Lemma (well known exercise)

If $\exists$ ham cyc $e \overset{\overline{s_1}}{\to} x_1 \overset{\overline{s_2}}{\to} x_2 \overset{\overline{s_3}}{\to} \cdots \overset{\overline{s_{60}}}{\to} e$ in $\mathrm{Cay}(A_5; \overline{S})$ such that $s_1 s_2 \cdots s_{60} \neq (e, 0)$, then $\exists$ hamiltonian cycle in $\mathrm{Cay}(A_5 \times \mathbb{Z}_p; S)$.

Proof. We construct a hamiltonian cycle in $\mathrm{Cay}(A_5 \times \mathbb{Z}_p; S)$ from a hamiltonian cycle in $\mathrm{Cay}(A_5; \overline{S})$ where $\overline{S}$ is the projection of S to A_5.

Case 1. *Assume $\overline{S}$ is **redundant**.*

Choose $t \in S$, let $S' = S \setminus \{t\}$, such that $\langle \overline{S'} \rangle = A_5$.

Then $\langle S' \rangle = A_5 \times \{0\}$ (wolog S is irredundant).

We have $t = (\overline{t}, k)$, $k \neq 0$.

Lemma (verified by computer)

*Every Cayley graph on A_5 is **hamiltonian connected**:*
$\forall x \neq y \in A_5$, $\exists$ hamiltonian path from x to y.

So $\exists$ hamiltonian path from e to $\overline{t}^{-1}$ in $\mathrm{Cay}(A_5; \overline{S'})$.

$$e \overset{\overline{s_1}}{\to} x_1 \overset{\overline{s_2}}{\to} x_2 \overset{\overline{s_3}}{\to} \cdots \overset{\overline{s_{59}}}{\to} x_{59} = \overline{t}^{-1} \overset{\overline{t}}{\to} e$$

Then $s_1 s_2 \cdots s_{59}\, t = (e, k) \neq (e, 0)$, so Factor Group Lemma applies.

Case 2. *Assume $\overline{S}$ is **irredundant**.*

Exercise

List the 20 irredundant generating sets of A_5.
(up to isomorphism)

For each irredundant S, find ham cycs in $\mathrm{Cay}(A_5; S)$. Show Factor Grp Lem applies to at least one of them if $p \equiv 1 \pmod{60}$.

Theorem (Morris, 2015)

There are infinitely many primes p, such that every Cayley graph on $A_5 \times \mathbb{Z}_p$ has a ham cycle.

Proof of Case 1 does not make any restriction on p.

Work in progress

Prove Case 2 without any restriction on p.

Reference

D. W. Morris: Infinitely many nonsolvable groups whose Cayley graphs are hamiltonian, *JACODESMATH* 3 (2016) 13–30.
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