Hamiltonian cycles in some easy Cayley graphs

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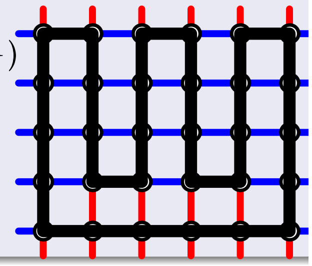
Abstract. It was conjectured 45 years ago that every connected Cayley graph has a hamiltonian cycle, but there is very little evidence for such a broad claim. The talk will explain what Cayley graphs are, describe some of the progress that has been made on this problem, and present a few of the many open questions. Almost all of the talk will be understandable to anyone familiar with the fundamentals of graph theory and group theory.

Example

Cayley graph
$\mathrm{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n; \{\pm(1,0), \pm(0,1)\})$

vertices: elements of $\mathbb{Z}_m \oplus \mathbb{Z}_n$
edges: $x — x \pm (1,0)$
and $x — x \pm (0,1)$

has hamiltonian cycle



Defn. $\mathrm{Cay}(G;S)$ for group G and $S \subseteq G$ with $S = S^{-1}$
vertices = elt's of G edge $x — xs$ for $x \in G$, $s \in S$
(assume **connected**, i.e., $\langle S \rangle = G$)

Exercise

G abelian $\Rightarrow \forall S$, $\mathrm{Cay}(G;S)$ has a hamiltonian cycle.

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Open Problem

Every connected Cayley graph has a hamiltonian cycle?

Conjectures about $\mathrm{Cay}(G;S)$:

- [Parsons?, ~1970] $\exists$ hamiltonian cycle.
- $\exists$ hamiltonian **path**.
- $\exists$ path of length $\epsilon \# G$.
- [Witte, 1982] $\exists$ ham cycle for *some* minimal S.
- [Babai, 1995] $\nexists$ cycle of length $(1-\epsilon)\#G$.

Proposition (Babai, 1979)

$\exists$ *path (& cycle) of length* $> \sqrt{\#G}$.

Conjecture

$\forall G, \forall S$, $\mathrm{Cay}(G;S)$ *has a hamiltonian cycle.*

Many papers pick an interesting group (e.g., Sym_n) that has a natural generating set S
(e.g., $\{(1\,2\,\ldots\,n), (1\,2)\}$ or $\{\ldots, (i\; i+1), \ldots\}$)
and find a hamiltonian cycle in $\mathrm{Cay}(G;S)$.
Applications to design of efficient algorithms.
"Combinatorial Gray Codes"

My viewpoint: Consider *all* S for given G.

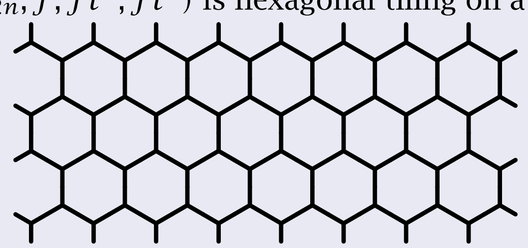
Exercise

G abelian $\Rightarrow \forall S$, $\mathrm{Cay}(G;S)$ has a hamiltonian cycle.

¿ Can we find a ham cycle if G is almost abelian ?

Find a ham cycle in $\mathrm{Cay}(G;S)$ *if G is almost abelian.*

Eg. dihedral $D_{2n} = \langle f, t \mid f^2 = t^n = e,\ ftf = t^{-1} \rangle$.
$\mathrm{Cay}(D_{2n}; f, ft^a, ft^b)$ is hexagonal tiling on a torus.



This has a hamiltonian cycle [Alspach-Zhang, 1989].

Conjecture. $\mathrm{Cay}(D_{2n}; S)$ has ham cyc $\forall S$.

True if n is even [Alspach-Chen-Dean, 2010].

Find a ham cycle in $\mathrm{Cay}(G;S)$ *if G is almost abelian.*

Question: What is the next best thing to abelian?
Group theorist's answer: **nilpotent**.
$G = P_1 \times \cdots \times P_r$, $|P_i| =$ prime power $p_i^{k_i}$

$\exists$ ham cycle when $r = 1$ [Witte, 1986]. ($|G| = p^k$)

Open Problem

Find ham cycle when $r = 2$ *(and* $|P_2| = \mathbb{Z}_2$*).*

If $\#S \leq 4$, $\exists$ hamiltonian path [Morris, 2012]. ($\forall r$)

Find a ham cycle in Cay$(G;S)$ *if G is almost abelian.*

Recall. commutator subgroup $G' = \langle\, ghg^{-1}h^{-1} \mid g,h \in G \,\rangle$.

G abelian $\Leftrightarrow G' = \{e\} \Leftrightarrow |G'| = 1$.

Find a hamiltonian cycle if $|G'|$ *is small.*

Theorem (Marušič, Durnberger, Keating-Witte, 1985)

Cay$(G;S)$ *has a ham cycle if* $|G'| = p$ *(prime)*.

Open problem. Find ham cycle if $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Work in progress. $\exists$ ham cyc if $|G'| = p_1 p_2$. $(p_1 \neq p_2)$

- ✓ G nilpotent [Ghaderpour-Morris]
- ✓ $|G|$ odd [Morris]
- ✓? $p_1 = 2$ (in progress)

Cay$(G;S)$ *has a hamiltonian cycle if* $|G'| = p$

Idea of proof. Hamiltonian cycle in Cay$(G/G';\overline{S})$:

$\overline{x_0} \xrightarrow{\overline{s_1}} \overline{x_1} \xrightarrow{\overline{s_2}} \overline{x_2} \xrightarrow{\overline{s_3}} \overline{x_3} \xrightarrow{\overline{s_4}} \cdots \xrightarrow{\overline{s_n}} \overline{x_n} \quad (= \overline{x_0} = \overline{e})$.

Then $\overline{x_i} = \overline{x_{i-1} s_i}$, so $\overline{x_n} = \overline{s_1 s_2 \cdots s_n}$.

Let $\pi = s_1 s_2 \cdots s_n \in G'$. ("voltage")

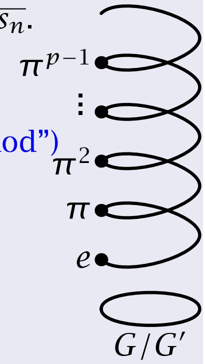
There are *many* ham cycs in G/G'.

Find one with $\pi \neq e$ ("Marušič's Method")

so $\langle \pi \rangle = G'$.

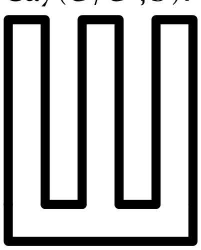
Ham cycle in G/G' lifts to a path in G from e to π.

Repeated lifts extend this to a hamiltonian cycle in G.

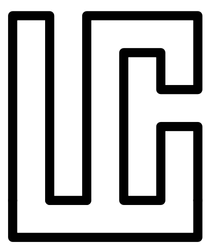


"Marušič's Method"

Ham cycle in Cay$(G/G';\overline{S})$:



Other ham cycles:



Voltage not all same (usually), so some voltage $\neq e$.

Cayley graphs of small order

Theorem (2011–2013)

Cay$(G;S)$ *has ham cycle if* $|G| = kp$ *with* $24 \neq k < 32$.

Also kp^2 $(k \le 4)$, kp^3 $(k \le 2)$, kpq $(k \le 5)$, pqr.

[Marušič, Kutnar, Šparl, Morris², Curran, Ghaderpour, Li, Jungreis-Friedman (∼1990)]

Proof for pqr. [Li Deng Xin, 2001]

Group theory fact: $|G|$ square free $\Rightarrow G'$ cyclic.

Hence $G' \neq G$, so $|G'| = pq$.

Therefore Cay$(G;S)$ has a ham cyc, by previous thm unless $|G| = 2pq$ (separate case). □

The Directed Case

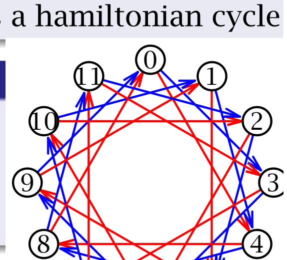
Exer. G abelian $\Rightarrow$ Cay$(G;S)$ has a hamiltonian cycle

Example

Cayley ***di****graph* $\overrightarrow{\text{Cay}}(\mathbb{Z}_{12}; 3, 4)$

vertices: elements of $\mathbb{Z}_{12}$

arcs: $x \rightarrow x+3$ & $x \rightarrow x+4$



Open problem

Which Cayley digraphs have *hamiltonian cycles*?

Example

$\overrightarrow{\text{Cay}}(\mathbb{Z}_{12}; 3, 4)$ does *not* have a hamiltonian cycle.

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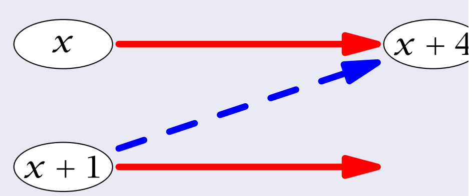
$\overrightarrow{\text{Cay}}(\mathbb{Z}_{12}; 3, 4)$ does *not* have a hamiltonian cycle.

Proof.

Suppose H is a hamiltonian cycle.

Say vertex x travels by 4.

Then $x+1$ cannot travel by 3 (collision at $x+4$).



So $x+1$ must travel by 4.

By induction, every vertex travels by 4. →←

So no vertex travels by 4. All travel by 3. →← □

Eg. $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12}; 3, 4)$ does *not* have a hamiltonian cycle.

Exercise (Rankin, 1948)

$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; a, b)$ has a ham cyc $\Leftrightarrow$ $\exists\, s, t \in \mathbb{Z}^{\geq 0}$, s.t.
$s + t = \gcd(a - b, n) = \gcd(sa + tb, n)$.

Theorem (Locke-Witte, 1999)

$\nexists$ hamiltonian cycle in:

- $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{12k}; 6k, 6k+2, 6k+3)$
- $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{2k}; a, a+1, a+k)$ *if $a + k$ is even*
  (*and* $\gcd(a, 2k) \neq 1$ *and* $\gcd(a+1, 2k) \neq 1$).

Conjecture

These are the only 3-gen'd examples. (*up to* $\cong$)

Exer. $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; a, b)$ has a ham cyc $\Leftrightarrow \ldots$

Conjecture

$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; a, b, c)$ *has a ham cyc* $\Leftrightarrow \ldots$

Conjecture

$\#S > 3 \Rightarrow \overrightarrow{\mathrm{Cay}}(\mathbb{Z}_n; S)$ *has a ham cyc.*

Exercise

Conjecture true
$\Rightarrow$ $\mathrm{Cay}(D_{2n}; S)$ has a hamiltonian cycle (if $\#S > 4$)
$\Rightarrow$ $\overrightarrow{\mathrm{Cay}}(D_{2n}; S)$ has a hamiltonian cycle (if $\#S > 4$)

Surveys

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D. Witte and J. A. Gallian: A survey: Hamiltonian cycles in Cayley graphs, *Discrete Math.* 51 (1984) 293-304. MR 0762322

S. J. Curran and J. A. Gallian: Hamiltonian cycles and paths in Cayley graphs and digraphs — a survey, *Discrete Math.* 156 (1996) 1–18. MR 1405010

C. Savage: A survey of combinatorial Gray codes, *SIAM Rev.* 39 (1997) 605–629. MR 1491049

I. Pak and R. Radoičić: Hamiltonian paths in Cayley graphs, *Discrete Math.* 309 (2009) 5501–5508. MR 2548568

Recent results

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D. W. Morris: 2-generated Cayley digraphs on nilpotent groups have Hamiltonian paths, *Contrib. Discrete Math.* 7 (2012) 41–47. MR 2956335, http://hdl.handle.net/10515/sy5ks6jk4

D. W. Morris: Odd-order Cayley graphs with commutator subgroup of order pq are hamiltonian, *Ars Math. Contemp.* 8 (2015), no. 1, 1–28. MR 3281117, http://amc.imfm.si/index.php/amc/article/view/330