VANISHING OF FIRST COHOMOLOGY FOR ARITHMETIC GROUPS OF HIGHER REAL RANK (AFTER G. A. MARGULIS)

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Abstract. Let Γ be an arithmetic subgroup of $G = \mathrm{SL}(n,\mathbb{R})$, with $n > 2$. (More generally, Γ could be any irreducible arithmetic subgroup of any semisimple Lie group of higher real rank.) We will describe a proof (due to Margulis) that the first cohomology of Γ vanishes (with coefficients in any finite-dimensional real representation). The result is a consequence of the much stronger statement, known as the Margulis Superrigidity Theorem, that, roughly speaking, every finite-dimensional representation of Γ extends to a finite-dimensional representation of G.

Lecture 1. Introduction

Let Γ be an arithmetic subgroup of a semisimple Lie group G, and let V be a Γ-module. (All modules in these lectures are finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. We could easily consider vector spaces over any field of characteristic zero, but we will not have anything to say about coefficient groups that are not vector spaces.) We would like to understand $H^*(\Gamma; V)$.

Example. Suppose W is a submodule of V. If W is a direct summand (i.e., $V \cong W \oplus (V/W)$), then it is obvious that

$$H^*(\Gamma; V) \cong H^*(\Gamma; W) \oplus H^*(\Gamma; V/W),$$

but, in general, there is no easy way to calculate $H^*(\Gamma; V)$ from the cohomology with coefficients in W and V/W.

Therefore, in situations where every Γ-module is semisimple (a direct sum of simple modules), it suffices to calculate the cohomology with coefficients in the simple modules. In fact, this is often the case:

Main Theorem [4, Thm. 7.6.16, p. 248]**.** *If Γ is any arithmetic subgroup of $\mathrm{SL}(n,\mathbb{R})$, with $n \geq 3$, then every Γ-module is semisimple.*

Remark [7]**.** The result is true much more generally. More precisely, essentially the only exceptions are arithmetic subgroups of $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$: if there is a Γ-module that is not semisimple, then some finite-index subgroup of Γ is isomorphic to $\Gamma_1 \times \Gamma_2$, where Γ_1 is an arithmetic subgroup of either $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, for some n.

Unfortunately, $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ are isogenous to $\mathrm{SO}(1,2)$ and $\mathrm{SO}(1,3)$, respectively. (Recall that two Lie groups are said to be “isogenous” if they are isomorphic, modulo ignoring some finite groups.) Therefore, even the generalized theorem does not apply to arithmetic subgroups of these two important groups.

Exercise. Show that some arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$ has a module that is **not** semisimple.
[*Hint:* Let Γ be a free group or a surface group. In either case, the abelianization is infinite, so there is a nontrivial homomorphism $\rho\colon \Gamma \to \mathrm{GL}(2,\mathbb{C})$ whose image is in the abelian group $\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$. Then $\rho(\Gamma)$ acts nontrivially on $\mathbb{C}^2$, but trivially on both $(0,*)$ and $\mathbb{C}^2/(0,*)$.]

The main theorem can also be stated as a cohomology calculation:

Corollary [4, Cor. 7.6.17, p. 249]**.** *If* Γ *is any arithmetic subgroup of* $\mathrm{SL}(n,\mathbb{R})$*, with* $n \geq 3$*, then* $H^1(\Gamma; V) = 0$ *for every* Γ*-module* V*.*

Proof [4, Lem. 5.22, p. 249]. Suppose

- $\rho\colon \Gamma \to \mathrm{GL}(k,\mathbb{R})$ is a homomorphism, and
- $\alpha\colon \Gamma \to \mathbb{R}^k$ is a 1-cocycle representing some cohomology class in $H^1(\Gamma;\rho)$.

Define $\hat{\rho}\colon \Gamma \to \mathrm{GL}(k+1,\mathbb{R})$ by

$$\hat{\rho}(g) = \left[\begin{array}{c|c} \rho(g) & \alpha(g) \\ \hline 0 \cdots 0 & 1 \end{array}\right].$$

Since α is a 1-cocyle, which means $\alpha(gh) = \alpha(g) + \rho(g)\,\alpha(h)$, it is easy to verify that $\hat{\rho}$ is a homomorphism.

The main theorem tells us that every Γ-module is semisimple, so there is a (1-dimensional) $\hat{\rho}(\Gamma)$-invariant complement W to the invariant subspace $(*,\ldots,*,0)$. Chose $w \in \mathbb{R}^k$, such that $(w,1) \in W$. Then, letting

$$T = \left[\begin{array}{c|c} I & w \\ \hline 0 \cdots 0 & 1 \end{array}\right],$$

we have $T(0,\ldots,0,*) = W$, so

$$\big(T^{-1}\hat{\rho}(g)T\big)(0,\ldots,0,1) \in (0,\ldots,0,*),$$

which implies $\alpha(g) = w - \rho(g)w$, for all $g \in \Gamma$. Therefore $[\alpha] = 0$ in $H^1(\Gamma;\rho)$. □

Remark [3, §1.4], [5, Chap. 13]**.** For modules with an invariant inner product, the desired conclusion usually follows from *Kazhdan's property* (T).

- **Definition.** Γ has *Kazhdan's property* (T) if $H^1(\Gamma;\pi) = 0$, for every unitary representation $\pi\colon \Gamma \to \mathcal{U}(\mathbb{H})$, where $\mathbb{H}$ is any Hilbert space (possibly ∞-dimensional).
- **Theorem** (Kazhdan et el.). Recall that Γ is an arithmetic subgroup of G. If $G = \mathrm{SL}(n,\mathbb{R})$ with $n \geq 3$, or, more generally, if no simple factor of G is isogenous to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, then Γ has Kazhdan's property (T).
- **Corollary.** Suppose $\rho\colon \Gamma \to \mathrm{SU}(k)$. If $G = \mathrm{SL}(n,\mathbb{R})$ with $n \geq 3$, then $H^1(\Gamma;\rho) = 0$.

The approach of Margulis exploits the fact that G-modules are semisimple:

Lemma. *If* G *is any semisimple Lie group, then every* G*-module is semisimple.*

Proof (Weyl's unitary trick). Assume, for simplicity, that $G = \mathrm{SL}(n,\mathbb{R})$, and that V is a vector space over $\mathbb{C}$. Then V is an $\mathrm{SL}(n,\mathbb{C})$-module. By restriction, it is also an $\mathrm{SU}(n)$-module. Since a subspace of V is $\mathrm{SL}(n,\mathbb{R})$-invariant if and only if it is $\mathrm{SU}(n)$-invariant, it suffices to show that V is semisimple as an $\mathrm{SU}(n)$-module.

Since $\mathrm{SU}(n)$ is compact, there is an $\mathrm{SU}(n)$-invariant Hermitian form $\langle \mid \rangle$ on V. Then, for any $\mathrm{SU}(n)$-invariant subspace W of V, we have the $\mathrm{SU}(n)$-invariant direct sum decomposition $V = W + W^{\perp}$. So V is semisimple as an $\mathrm{SU}(n)$-module. □

Thus, it would suffice to show that every Γ-module is (the restriction of) a G-module, and that is (essentially) what Margulis proved:

Theorem (Margulis Superrigidity Theorem [4, §7.6], [5, Chap. 16], [8, §5.1])**.**

- *Recall that* Γ *is an arithmetic subgroup of* $G = \mathrm{SL}(n,\mathbb{R})$.
- *Assume* $n \geq 3$.
- *For simplicity, also assume* G/Γ *is not compact.*
- *Let* $\rho\colon \Gamma \to \mathrm{GL}(k,\mathbb{R})$ *be a homomorphism.*

Then there is a finite-index subgroup Γ' *of* Γ*, such that* $\rho|_{\Gamma'}$ *extends to a continuous homomorphism* $\hat{\rho}\colon G \to \mathrm{GL}(k,\mathbb{R})$.

Remark. Here is a more general statement of the superrigidity theorem.

- Let Γ be an irreducible, arithmetic subgroup of a connected, semisimple Lie group G with finite center.
- Let $\rho\colon \Gamma \to \mathrm{GL}(k,\mathbb{R})$ be a homomorphism.
- Let H be the identity component of the Zariski closure of $\rho(\Gamma)$. This means that H is the (unique) smallest connected, closed subgroup of $\mathrm{GL}(k,\mathbb{R})$ that contains a finite-index subgroup of $\rho(\Gamma)$.
- To avoid minor complications, assume that G is simply connected, and has no compact factors.

If G is not isogenous to either $\mathrm{SO}(1,m)$ or $\mathrm{SU}(1,m)$, for any m, then:

(1) H is semisimple, and
(2) if we assume, for simplicity, that H has no compact factors, then there is a finite-index subgroup Γ' of Γ, such that $\rho|_{\Gamma'}$ extends to a continuous homomorphism $\hat{\rho}\colon G \to H$.

Remarks.

(A) All we need is Conclusion (1) of the preceding remark: this implies that the Γ-module $\mathbb{R}^k$ is semisimple.
(B) In textbooks (e.g., [8, Thm. 5.1.2, p. 86]), proofs of the Margulis Superrigidity Theorem often make the additional assumption that the Zariski closure of $\rho(\Gamma)$ is a noncompact, simple group. This special case suffices for many purposes (such as the proof of the important Margulis Arithmeticity Theorem, which states that every lattice in G is an arithmetic

subgroup), but it is not enough for the proof of our main theorem. Indeed, Remark A points out that, for this purpose, the whole problem is to show that the Zariski closure is semisimple.

(C) For the special case where $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, Bass-Lazard-Serre [1] proved that Γ has the *Congruence Subgroup Property*, and it was observed by Bass-Milnor-Serre [2, §16] that this implies the conclusion of the Margulis Superrigidity Theorem.

Recall

- Γ is an arithmetic subgroup of $G = \mathrm{SL}(n,\mathbb{R})$, with $n \geq 3$.
- Every Γ-module is assumed to be a finite-dimensional vector space over either $\mathbb{R}$ or $\mathbb{C}$.

We wish to prove any of the three equivalent versions of the following result:

Main Theorem (Margulis [4, Thm. 7.6.16 and Cor. 7.6.17, pp. 248 and 249])**.**

(1) *Every* Γ*-module is semisimple* (*i.e., a direct sum of simple modules*)*.*

(2) $H^1(\Gamma; V) = 0$ *for every* Γ*-module* V*.*

(3) *Assume*
- $\rho\colon \Gamma \to \mathrm{GL}(k,\mathbb{R})$ *is a homomorphism, and*
- H *is the identity component of the Zariski closure of* $\rho(\Gamma)$ (*so* H *is the smallest connected subgroup* Γ' *of* $\mathrm{GL}(k,\mathbb{R})$ *that contains a finite-index subgroup of* $\rho(\Gamma)$)*.*

Then H *is semisimple.*

This is a consequence of the following stronger result:

Theorem (Margulis Superrigidity Theorem [4, §7.6], [5, Chap. 16], [8, §5.1])**.** $\rho|_{\Gamma'}$ *extends to a continuous homomorphism* $\hat{\rho}\colon G \to H \subseteq \mathrm{GL}(k,\mathbb{R})$*.*

Remark. In order to extend $\rho|_{\Gamma'}$ to a homomorphism $\hat{\rho}\colon G \to H$, the embedding of H in $\mathrm{GL}(k,\mathbb{R})$ is irrelevant: we will re-embed H into some other $\mathrm{GL}(m,\mathbb{R})$ before we construct the extension.

Topologist's viewpoint. Γ acts on $G \times \mathbb{R}^k$ via $\gamma \cdot (x, v) = \big(x\gamma^{-1}, \rho(\gamma)\, v\big)$. Let $\mathcal{V} = (G \times \mathbb{R}^k)/\Gamma$, so $\mathcal{V}$ is a vector bundle over G/Γ whose fiber is $\mathbb{R}^k$. Furthermore, G acts on $\mathcal{V}$: multiplying x by g on the left is a bundle automorphism (since it commutes with multiplication by γ on the right).

The following are equivalent:

(1) ρ extends to $\hat{\rho}\colon G \to \mathrm{GL}(k,\mathbb{R})$.

(2) $\mathcal{V} \cong G/\Gamma \times \mathbb{R}^k$ (G-equivariantly), where $g \cdot (x, v) = \big(gx, \hat{\rho}(g)v\big)$.

(3) $\exists$ G-invariant subspace $V \subset \mathrm{Sect}(\mathcal{V})$, such that $V \xrightarrow{\cong} V|_{[e]}$.

Proof. (1⇒2) Define $\varphi\colon \mathcal{V} \xrightarrow{\cong} G/\Gamma \times \mathbb{R}^k$ by $\varphi\big((x,v)/\Gamma\big) = \big(x\Gamma, \hat{\rho}(x)\, v\big)$.

(2⇒3) {constant sections} is G-invariant: for $v \in \mathbb{R}^k$, define $\xi_v(x\Gamma) = (x\Gamma, v)$. Then $g\,\xi_v(g^{-1}x) = \xi_{\hat{\rho}(g)v}(x)$.

(3⇒1) G acts on $V \cong \mathcal{V}_{[e]} \cong \mathbb{R}^k$. This extends ρ. □

A bit more work shows that it suffices to find a (nonzero) G-invariant subspace of $\mathrm{Sect}(\mathcal{V})$ that is finite-dimensional (and satisfies a genericity assumption if the representation ρ is reducible) [4, Prop. 7.4.6, p. 222]. The desired space of sections will be obtained by bootstrapping from the following result, whose proof will be postponed to the end of the lecture.

Key Fact. *Let $a \in A = \{$diagonal matrices in $G\}$. Then there is a (nonzero) a-invariant section of $\mathcal{V}$.*

Remark. Actually, we will replace $\mathcal{V}$ with a bundle that corresponds to a different representation of H.

Lemma. *Assume*

- *A_0 is a noncompact, closed subgroup of A, and*
- *V is an A_0-invariant, finite-dimensional subspace of $\mathrm{Sect}(\mathcal{V})$.*

Then $\langle C_G(A_0) \cdot V \rangle$ is finite-dimensional.

Idea of proof. A_0 has a dense orbit on G/Γ (as we shall see), so if $c \in C_G(A_0)$, then the effect of c on a section is determined by what it does to the section at a single point. □

Proof of superrigidity. Choose $L_1, L_2, \ldots, L_r \cong \mathrm{SL}(2,\mathbb{R})$ in G,

$$\begin{bmatrix} * & * & \\ * & * & \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} * & & * \\ & 1 & \\ * & & * \end{bmatrix}, \quad \begin{bmatrix} 1 & & \\ & * & * \\ & * & * \end{bmatrix}$$

such that

- $G = L_r L_{r-1} \cdots L_1$,
- $A_i = L_i \cap A$ is noncompact, and
- $A_i^{\perp} = C_A(L_i)$ is noncompact.

By the Key Fact, we may let V_0 be the span of an A_0-invariant section (for some $A_0 \subset A$), so V_0 is 1-dimensional and A_0-invariant. Then let

$$V_i = \langle L_i \cdot A \cdot L_{i-1} \cdot A \cdots L_1 \cdot A \cdot V_0 \rangle.$$

Since $G = L_r L_{r-1} \cdots L_1$, we know that V_r is G-invariant. Furthermore, repeated application of the lemma implies that each V_i is finite-dimensional (since $A \subseteq C_G(A_i)$ and $L_i \subseteq C_G(A_i^{\perp})$). □

Technical problem. In the Key Fact, we actually only find a *measurable* invariant section. (Identify two sections that agree a.e.) The proof of superrigidity goes through, but we need more than just a dense orbit: use "*ergodicity*."

Lemma (Special case of the Moore Ergodicity Theorem [5, §4J])**.** *Suppose*

- *Γ is an arithmetic subgroup of $G = \mathrm{SL}(2,\mathbb{R})$,*
- *$a = \mathrm{diag}(t, 1/t) \in \mathrm{SL}(2,\mathbb{R})$ with $t > 1$, and*

- *f is an a-invariant, measurable function on G/Γ.*

Then f is constant (a.e.).

Idea of proof. Let $u = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$, so $a^n u a^{-n} = \begin{bmatrix} 1 & 0 \\ a^{-2n}s & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as $n \to +\infty$. Then

$$f(ux) = f(a^n ux) = f(a^n u a^{-n}\, a^n x) \approx f(a^n x) = f(x),$$

so f is $\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$-invariant (a.e.). Similarly, it is also $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$-invariant (a.e.). Since these subgroups generate G, it follows that f is G-invariant (a.e.), and, hence, constant (a.e.). □

Our construction of an a-invariant section is based on a generalization of the following simple observation.

Example. Suppose $T \in \mathrm{GL}(k, \mathbb{C})$. After a change of basis, we can assume T is in Jordan Canonical Form. It is not difficult to see that if we let W_+, W_0, and W_- be the subspaces spanned by the basis vectors in blocks corresponding to eigenvalues of absolute value > 1, $= 1$, and < 1, respectively, then $\mathbb{R}^k = W_+ \oplus W_0 \oplus W_-$, and

$$(*) \qquad \lim_{m\to\pm\infty} \|T^m v\|^{1/m} \text{ is } \begin{cases} > 1 & \text{for } v \in W_+^\times, \\ = 1 & \text{for } v \in W_0^\times, \\ < 1 & \text{for } v \in W_-^\times. \end{cases}$$

The following result (which is stated in a very weak form) shows that this observation can be generalized by replacing the single vector space V with a family of vector spaces:

Multiplicative Ergodic Theorem (Oseledec [6], [4, App. A, pp. 346–351])**.** *Assume:*

- *$\mathcal{V} \to X$ is a vector bundle (with compact base X), so each fiber $\mathcal{V}_x$ is a real vector space,*
- *T is an automorphism of $\mathcal{V}$, so the restriction of T to any fiber $\mathcal{V}_x$ is a linear isomorphism onto some other fiber $\mathcal{V}_{\overline{T}x}$,*
- *we have a norm $\| \;\|$ on each fiber $\mathcal{V}_x$ (that varies continuously with x), and*
- *μ is a finite $\overline{T}$-invariant measure on the base X.*

Then, for a.e. $x \in X$, the fiber $\mathcal{V}_x$ is a direct sum $\omega_+(x) \oplus \omega_0(x) \oplus \omega_-(x)$, such that $(*)$ holds with $W_\epsilon^\times$ replaced by $\omega_\epsilon(x)^\times$.

Remark. When G/Γ is not compact, the proof utilizes a more general version of the Multiplicative Ergodic Theorem that replaces compactness with the assumption that the functions $\log \|T_x\|$ and $\log \|T_x^{-1}\|$ are in L^1, where T_x is the linear map from $\mathcal{V}_x$ to $\mathcal{V}_{\overline{T}x}$

Proof of the Key Fact. Let $T = a \in A$. Then, for each x, let $\pi_x \in \operatorname{End}(\mathcal{V}_x)$ be the projection to $\omega_0(x)$ with kernel $\omega_+(x) \oplus \omega_-(x)$, so π is a section of the bundle $\operatorname{End}\mathcal{V} = (G \times \operatorname{Mat}_{k\times k}(\mathbb{R}))/\Gamma$, where Γ acts on $\operatorname{Mat}_{k\times k}(\mathbb{R})$ by $\gamma \cdot M = \rho(\gamma)\, M \rho(\gamma)^{-1}$.

It is clear that each ω_ϵ is $C_G(a)$-invariant, in the sense that $c^{-1} \cdot \omega_\epsilon(cx)) = \omega_\epsilon(x)$. Therefore, the section π is also $C_G(a)$-invariant. In particular, it is a-invariant. □

A technical issue. We need to know that H acts faithfully on the space of sections. (E.g., π_x should not be the identity matrix for all x.)

Suppose, for example, that ρ is of the form $\rho(\gamma) = \left[\begin{array}{c|c} \gamma & \alpha(\gamma) \\ \hline 0 & \gamma \end{array}\right] \in \mathrm{GL}(6,\mathbb{R})$, so α is a 1-cocycle. We wish to show $\rho(\Gamma)$ (or H) is conjugate to a subgroup of $\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]$, so the main concern is to be sure that $U = \left[\begin{array}{c|c} I & * \\ \hline 0 & I \end{array}\right]$ is not in the kernel of the action on the space of sections.

Let $a = \operatorname{diag}(t, 1, 1/t)$ with $t > 1$. Let $W = \mathbb{R}^3 \subset \mathbb{R}^6$, so W is a submodule. Since the homomorphism $\gamma \mapsto \gamma$ obviously extends to a homomorphism defined on G, the subbundle of $\mathcal{V}$ with fiber W is G-equivariantly a product. So it is clear that $\omega_0^W = (0, *, 0) \subsetneq \mathbb{R}^3$. Similarly for the quotient bundle with fiber $\mathbb{R}^6/W$.

This implies that $\omega_0(x)$ is not U-invariant, so π_x is not centralized by U.

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