Groups of polynomial growth
(after Gromov, Kleiner, and Shalom-Tao)

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Abstract. M. Gromov proved in 1981 that every group of polynomial growth has a nilpotent subgroup of finite index. This is a fundamental result in Geometric Group Theory, but Gromov's proof is too difficult to include in many standard courses on the subject. A tremendous simplification of the proof of the key lemma was achieved by B. Kleiner in 2010 (using methods of T. H. Colding and W. P. Minicozzi II). A further improvement by Y. Shalom and T. Tao has made the theorem truly accessible. We present the simplified proof.

Geometric Group Theory

Γ = group (e.g., $\mathbb{Z}^2$)
S = symmetric finite generating set
(e.g., $\{(\pm 1, 0), (0, \pm 1)\}$)

$\forall x, y \in \Gamma$:
$y = x \cdot \boxed{?} = x s_1 s_2 \cdots s_k, \quad \exists s_i \in S, \ \exists k \in \mathbb{N}$
Metric: $d(x, y)$ = least such k.

So Γ has both algebraic structure (group) and geometric structure (metric space).

Basic question: How are they related?

Gromov's Thm

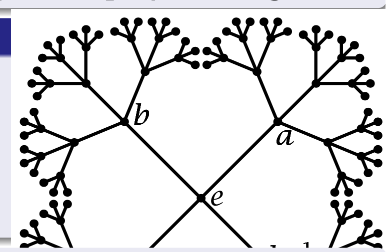
Definition
- ball $B(r) = \{\, x \in \Gamma \mid d(x, e) \le r \,\}$
- Γ has **polynomial growth** if $\#B(r) \le r^d$ $(\exists d \in \mathbb{N})$
(does not depend on choice of generating set)

Example
$\Gamma = \mathbb{Z}^2$: $\#B(r) \asymp r^2$. So $\mathbb{Z}^2$ has polynomial growth.
- *Easy:* abelian grps have polynomial growth.
- *Exer:* Nilpotent grps have polynomial growth.
- *Exer:* Spse H is *finite-index* subgroup of Γ.
Γ has poly growth $\iff$ H has poly growth.
- *Cor:* Virtually nilpotent grps have poly growth.

Virtually ______ $\Leftrightarrow$ finite index subgrp is ______.

Cor. Virtually nilp groups have polynomial growth.

Example
Γ = free group F_2.
$S = \{a^{\pm 1}, b^{\pm 1}\}$.
$\#B(r)$ is exp'l $\gg r^d$.
So F_2 does not have polynomial growth.



Exer: Groups of polynomial growth are amenable.

Theorem (Gromov)
Groups of polynomial growth are virtually nilpotent.

Assume: Γ has polynomial growth.
Prove: Γ is ~~nilpotent*~~ **solvable***. (polycyclic)
Exer. polycyclic with poly growth $\Rightarrow$ nilpotent*.

Key Lem. $\exists$ f.d. rep $\rho \colon \Gamma \to \mathrm{GL}(V)$, s.t. $\#\rho(\Gamma) = \infty$.

Gromov's proof assumes **hard** theorems.
Kleiner, Shalom-Tao: elementary proof.

Idea: Let $V = \{f \colon \Gamma \to \mathbb{R}\}$. $(f^g)(x) = f(gx)$
This representation is not finite-dimensional.
Kleiner: $\exists$ finite-dim'l invariant subspace.

Let $V_0 = \{$ Lipschitz, harmonic $f \colon \Gamma \to \mathbb{R}$ $\}$.
- $|f(x) - f(y)| \le C\, d(x, y), \quad \exists C \in \mathbb{R}$
- $f(x) = \frac{1}{\#S} \sum_{s \in S} f(xs)$

Theorem (Kleiner, Colding-Minicozzi)
$\dim V_0 < \infty$ *if Γ has polynomial growth.*

Prop. $\exists f \in V_0$ not constant. (do not need poly growth)
So f^Γ is infinite. (nonconstant harmonic func has no max)

~~Thm.~~ **Cor.** $\dim\{$ Lipschitz, harmonic $f\,\} < \infty$.

Defn. For $X \subseteq \Gamma$, $\|f\|_{2,X}^2 = \sum_{x \in X} f(x)^2$.

Note: f Lipschitz (& Γ has polynomial growth)
$\Rightarrow \|f\|_{2,B(r)}^2 \le$ polynomial r^D.

Polynomials have **bounded doubling**:
$\varphi(r) \sim r^d \quad \Rightarrow \quad \varphi(2r) \le C\,\varphi(r), \ \forall r$.

Technical issue that we will ignore:
$\varphi(r) \le r^d \quad \Rightarrow$ bdd dbling for most r, not all r.

Theorem $\exists \epsilon, \forall k, \forall r$,
f harmonic $\Rightarrow \|f\|_{2,B(6kr)} \ge \epsilon k \,\|f\|_{2,B(kr)}$
if f has mean 0 on each ball in a certain set $\mathcal{B}$
of balls of radius r with $\#\mathcal{B} = C'(k)$

This contradicts bounded doubling if dim $\gg C'$.

Thm. $\exists$ set $\mathcal{B}$ of C' balls of radius r:
f harmonic with mean 0 on each $B \in \mathcal{B}$
$\Rightarrow \|f\|_{2,B(6kr)} \geq \epsilon k \, \|f\|_{2,B(kr)}$.

Analysis on Γ

Definitions. Let $f\colon \Gamma \to \mathbb{R}$.

- $\partial_s f(x) = f(xs) - f(x)$.
- $|\partial_s f| \leq C$: f is **Lipschitz**. $|f(x) - f(y)| \leq C\, d(x,y)$.
- $\Delta f(x) = \#S \cdot f(x) - \sum_{s \in S} f(xs)$.
- $\Delta f = 0$: f is **harmonic**.

(Assume S is **symmetric** (closed under inverses): $S^{-1} = S$.)

Thm. f harmonic with mean 0 on each $B \in \mathcal{B}$
$\Rightarrow \|f\|_{2,B(6kr)} \geq \epsilon k \, \|f\|_{2,B(kr)}$.

Step 1 (Poincaré Inequality)

Calculus: $|f(x)| \leq \int_0^x |f'(t)|\, dt$ if $f(0) = 0$.
So $\|f\|_{\infty,B(r)} \leq r \, \|f'\|_{\infty,B(r)}$

$\|f\|^2_{2,B(r)} \leq C_1 r^2 \sum_{s \in S} \|\partial_s f\|^2_{2,B(3r)}$ if mean 0 on $B(r)$

Idea of proof.

$d(x, x_0) = n$: $x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \cdots \xrightarrow{s_n} x_n = x$.

$f(x) = f(x_0) + \sum_{i=1}^n \partial_{s_i} f(x_i) \doteq \sum_{i=1}^n \partial_{s_i} f(x_i)$.

$f(x)^2 \doteq \left(\sum_{i=1}^n \partial_{s_i} f(x_i)\right)^2 \leq n \sum_{i=1}^n \left(\partial_{s_i} f(x_i)\right)^2$.

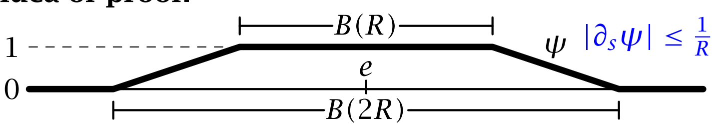
Sum over $x \in B(r)$. (Note that $n \leq 2r$.)

1. Poincaré ≤: $\|f\|^2_{2,B(r)} \leq C_1 r^2 \sum_{s \in S} \|\partial_s f\|^2_{2,B(3r)}$

Step 2 (Reverse Poincaré Inequality)

$\sum_{s \in S} \|\partial_s f\|^2_{2,B(R)} \leq C_2 \, \|f\|^2_{2,B(3R)} / R^2$ if f harmonic

Idea of proof.



Let $f' = \partial_s f$:

$$\|\partial_s f\|^2_{2,B(R)} = \sum_{x \in B(R)} (f')^2 \leq \sum_{x \in \Gamma} (f')^2 \psi^2$$

Integration by parts (or "partial summation")
moves derivative from f to ψ and ψ' is small.

1. Poincaré ≤: $\|f\|^2_{2,B(r)} \leq C_1 r^2 \sum_{s \in S} \|\partial_s f\|^2_{2,B(3r)}$

2. Reverse P I: $\sum_{s \in S} \|\partial_s f\|^2_{2,B(R)} \leq C_2 \, \|f\|^2_{2,B(3R)} / R^2$

Step 3 (finish): $\|f\|_{2,B(6kr)} \geq \epsilon k \;\; \|f\|_{2,B(kr)}$

Vitali Covering Lemma: $\exists \epsilon, \forall k, \exists C', \forall r, \exists \mathcal{B}$

- Cover $B(kr)$ with $\frac{\#B(2kr)}{\#B(r/2)} < C'$ balls of radius r
- No pt in more than $\frac{\#B(4r)}{\#B(r/2)} < C$ tripled balls.

$\|f\|^2_{2,B(kr)} \leq \sum_{B \in \mathcal{B}} \|f\|^2_{2,B}$ (balls cover $B(kr)$)

$\leq C_1 r^2 \sum_{s \in S} \sum_{B \in \mathcal{B}} \|\partial_s f\|^2_{2,3B}$ (Poincaré)

$\leq C C_1 r^2 \sum_{s \in S} \|\partial_s f\|^2_{2,B(2kr)}$ (no pt in more than C balls)

$\leq C C_1 r^2 \cdot C_2 \|f\|^2_{2,B(3(2kr))} / (2kr)^2$ (Reverse P I)

Review of Kleiner's proof

Assume Γ has polynomial growth.

Poincaré ≤. $\|f\|^2_{2,B(r)} \leq C' r^2 \sum_{s \in S} \|\partial_s f\|^2_{2,B(3r)}$

Reverse P I. $\sum_{s \in S} \|\partial_s f\|^2_{2,B(R)} \leq \dfrac{C'}{R^2} \|f\|^2_{2,B(3R)}$

Thm. $\|f\|_{2,B(6kr)} \geq \epsilon k \, \|f\|_{2,B(kr)}$.

Cor. $\dim\{\text{Lipschitz, harmonic } f\} < \infty$.

Prop. $\exists$ Lipschitz, harmonic, nonconstant f.

Key Lem. $\exists$ f.d. rep $\rho\colon \Gamma \to GL(V)$, s.t. $\#\rho(\Gamma) = \infty$.

Gromov's Theorem. Γ is virtually nilpotent.

Step 1 (Poincaré Inequality)

$\|f\|^2_{2,B(r)} \leq C_1 r^2 \sum_{s \in S} \|\partial_s f\|^2_{2,B(3r)}$ if mean 0 on $B(r)$

$$\|f\|^2_{2,B(r)} = \sum_{x \in B(r)} |f(x)|^2$$
$$= \sum_{x \in B(r)} \left| f(x) - \frac{1}{\#B(r)} \sum_{y \in B(r)} f(y) \right|^2$$
$$= \frac{1}{\#B(r)^2} \sum_{x \in B(r)} \left| \sum_{y \in B(r)} (f(x) - f(y)) \right|^2$$
$$\leq \frac{1}{\#B(r)} \sum_{x \in B(r)} \sum_{y \in B(r)} |f(x) - f(y)|^2$$
$$\leq \frac{1}{\#B(r)} \sum_{g \in B(2r)} \sum_{y \in B(r)} |f(yg) - f(y)|^2$$

Step 1 (Poincaré Inequality)

$$\|f\|^2_{2,B(r)} \le C_1 r^2 \sum_{s\in S} \|\partial_s f\|^2_{2,B(3r)} \quad \text{if mean 0 on } B(r)$$

$$\|f\|^2_{2,B(r)} \le \frac{1}{\#B(r)} \sum_{g\in B(2r)} \sum_{y\in B(r)} |f(yg) - f(y)|^2.$$

$$e = g_0 \overset{s_1}{—} g_1 \overset{s_2}{—} g_2 \overset{s_3}{—} \cdots \overset{s_n}{—} g_n = g$$

$$\sum_{y\in B(r)} |f(yg) - f(y)|^2 \ = \ \le$$

$$2r \sum_{i=1}^{n} \sum_{y\in B(r)} |\partial_{s_i} f(yg_i)|^2$$

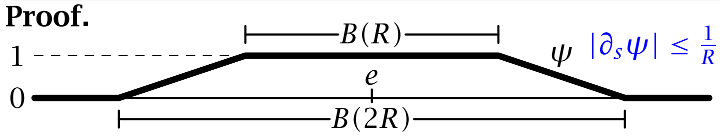
$$\le 2r \cdot 2r \sum_{s\in S} \|\partial_{s_i} f\|^2_{2,B(3r)}$$

$$\|f\|^2_{2,B(r)} \le \frac{1}{\#B(r)} \cdot \#B(2r) \cdot (2r)^2 \sum_{s\in S} \|\partial_s f\|^2_{2,B(3r)}$$

Step 2 (Reverse Poincaré Inequality)

$$\sum_{s\in S} \|\partial_s f\|^2_{2,B(R)} \le \frac{C_2}{R^2} \|f\|^2_{2,B(3R)} \quad \text{if } f \text{ harmonic.}$$

Proof.



Product rule $(\varphi\psi)' = \varphi'\,\psi + {}^s\varphi\,\psi'$ $\quad ({}^s\varphi(x) = \varphi(xs))$

$$\Rightarrow (f\psi^2)' = f'\psi^2 + {}^s f(\psi'\psi + {}^s\psi\,\psi')$$
$$= f'\psi^2 + {}^s f\,\psi'(\psi' + 2\psi)$$
$$= f'\psi^2 + \tfrac{C'}{R^2}\,{}^s f + \tfrac{C'}{R}\,{}^s f\psi$$

$$\langle \varphi \mid \psi\rangle_X = \sum_{x\in X} \varphi(x)\,\psi(x)$$

$$\langle f' | (f\psi^2)'\rangle_\Gamma = \|f'\psi\|^2_{2,\Gamma} + \tfrac{C'}{R^2}\langle f' | {}^s f\rangle_{B(2R)} + \tfrac{C'}{R}\langle f'\,{}^s f | \psi\rangle_\Gamma$$

Reverse P I: $\sum_{s\in S} \|\partial_s f\|^2_{2,B(R)} \le \frac{C'}{R^2} \|f\|^2_{2,B(3R)}$

$$\langle f' | (f\psi^2)'\rangle_\Gamma = \|f'\psi\|^2_{2,\Gamma} + \tfrac{C'}{R^2}\langle f' | {}^s f\rangle_{B(2R)} + \tfrac{C'}{R}\langle f'\,{}^s f | \psi\rangle_\Gamma$$

Sum over $s \in S$: $0 = (\ge \text{LHS})^{\checkmark} + (\le \text{RHS}) + (\text{smaller})$.

$$\sum_s \langle \partial_s f \mid \partial_s(f\psi^2)\rangle = \sum_s \langle \partial_{s^{-1}}\partial_s f \mid f\psi^2\rangle$$
$$= 2\langle \Delta f \mid f\psi^2\rangle = 2\langle 0 \mid f\psi^2\rangle = 0. \ \checkmark$$

$$\left|\langle f' | {}^s f\rangle_{B(2R)}\right| = \left|\langle {}^s f - f \mid {}^s f\rangle_{B(2R)}\right|$$
$$\le \langle {}^s f | {}^s f\rangle_{B(2R)} + \left|\langle f | {}^s f\rangle_{B(2R)}\right|$$
$$\le \|f\|^2_{2,B(2R+1)} + \|f\|_{2,B(2R)}\,\|{}^s f\|_{2,B(2R)}$$
$$\doteq 2\|f\|^2_{2,B(2R+1)}. \ \checkmark$$

Schwarz Inequality: $|\langle \varphi \mid \psi\rangle| \le \|\varphi\|_2\,\|\psi\|_2$

Reverse P I: $\sum_{s\in S} \|\partial_s f\|^2_{2,B(R)} \le \frac{C'}{R^2} \|f\|^2_{2,B(3R)}$

$$\langle f' | (f\psi^2)'\rangle_\Gamma = \|f'\psi\|^2_{2,\Gamma} + \tfrac{C'}{R^2}\langle f' | {}^s f\rangle_{B(2R)} + \tfrac{C'}{R}\langle f'\,{}^s f | \psi\rangle_\Gamma$$

Sum over $s \in S$: $0^{\checkmark} = (\ge \text{LHS})^{\checkmark} + (\le \text{RHS})^{\checkmark} + (\text{smaller})$

$$\frac{C'}{R}\left\langle f'\,{}^s f \,\middle|\, \psi\right\rangle_\Gamma = \left\langle f'\psi \,\middle|\, \frac{C'}{R}\,{}^s f\right\rangle_{B(2R)}$$

$$\le \|f'\psi\|_{2,\Gamma} \cdot \frac{C'}{R}\|{}^s f\|_{2,B(2R)} \quad \text{(Schwarz Inequality)}$$

$$\le \tfrac{1}{2}\|f'\,\psi\|^2_{2,\Gamma} + \frac{C''}{R^2}\|{}^s f\|^2_{2,B(2R)}$$

$$\le \tfrac{1}{2}\|f'\,\psi\|^2_{2,\Gamma} + \frac{C''}{R^2}\|f\|^2_{2,B(2R+1)}. \ \checkmark$$

Expository

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