Gromov and Piatetski-Shapiro's Nonarithmetic Lattices in $\mathrm{SO}(1,n)$

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Brief Historical Summary

It was known quite classically that there are nonarithmetic lattices in $\mathrm{SO}(1,2)$ (or, in other words, in $\mathrm{SL}(2,\mathbb{R})$). This was extended to $\mathrm{SO}(1,n)$, for $n \leq 5$, by Makarov and Vinberg in the 1960's. Two decades later, nonarithmetic lattices were constructed by Gromov and Piatetski-Shapiro for every n.

Nonarithmetic lattices in $\mathrm{SU}(1,n)$ were constructed by Mostow for $n = 2$, and by Deligne and Mostow for $n = 3$.

These results on $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$ are presented briefly in Appendix C (pp. 353–368) of Margulis's book.

G = conn, simple, noncpct, real, linear Lie group

Γ = lattice in G
- Γ is a discrete subgroup of G, and
- G/Γ has finite volume

Locally symm space $\Gamma\backslash G/K$ has finite volume (if Γ is torsion free).

Eg. $\mathrm{SL}(n,\mathbb{Z})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$.

Defn. Lie subalgebra $\mathfrak{g}$ of $\mathfrak{sl}(\ell,\mathbb{R})$ is *defined over* $\mathbb{Q}$ if it is solution space of linear eqs with $\mathbb{Q}$-coeffs.
- G is *defined over* $\mathbb{Q}$ if $\mathfrak{g}$ is.

Thm (Borel, Harish-Chandra). *G defined over $\mathbb{Q}$ $\Rightarrow$ $G_{\mathbb{Z}} = G \cap \mathrm{SL}(\ell,\mathbb{Z})$ is a lattice in G.*

Defn. $G_{\mathbb{Z}}$ is an *arithmetic* lattice in G.

Different embeddings of G in (various) $\mathrm{SL}(\ell,\mathbb{Z})$ can yield different lattices $G_{\mathbb{Z}}$.
All of these are *arithmetic*.

Also:
- Γ_0 arithmetic in $G \times$ cpct
  $\Rightarrow$ proj of Γ_0 to G is *arithmetic*.
- Γ_1 finite index in Γ_2:
  Γ_1 arithmetic $\Leftrightarrow$ Γ_2 arithmetic.

Defn. Γ_1 is *commensurable* to Γ_2:
$\Gamma_1 \cap \Gamma_2$ is finite index in both Γ_1 and Γ_2.

Lem.
- *Γ_1, Γ_2* **arithmetic** *lattices in G,*
- *$\Gamma_1 \cap \Gamma_2$ Zariski dense in G*

$\Rightarrow$ *Γ_1 is commensurable to Γ_2.*

Thm (Margulis Arithmeticity Theorem).
$G \neq \mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$ $\Rightarrow$ Γ is arithmetic.

Thm (Gromov, Piatetski-Shapiro).
$\exists$ nonarithmetic lattices in $\mathrm{SO}(1,n)$ (for $n \geq 2$).

Rem. $\exists$ nonarith latts in $\mathrm{SU}(1,n)$ for $n \leq 3$.
Not known whether they exist when $n \geq 4$.

Eg. If $a_2,\ldots,a_{n+1} \in \mathbb{Z}^+$, then
- $G = \mathrm{SO}(x_1^2 - a_2x_2^2 - a_3x_3^2 - \cdots - a_{n+1}x_{n+1}^2)^\circ \cong \mathrm{SO}(1,n)^\circ$;
- $\Gamma = G_{\mathbb{Z}}$ is an arithmetic lattice in G; and
- G/Γ is compact iff $\nexists$ nontriv $\mathbb{Z}$-solns of $x_1^2 = a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2$.

Rem. Γ noncocpct, arith latt in $\mathrm{SO}(n,1)$
$\Rightarrow$ Γ is as described in the example
(up to commensurability and conjs)
if $n \neq 7$ (triality causes a problem).

The cocpct latts can also be constructed explicitly.

To make **nonarith** latts, take geometric view:
consider the locally symmetric space $\Gamma \backslash \mathfrak{H}^n$.

Defn. Riemannian manifold M is *hyperbolic*:

- M is locally isometric to $\mathfrak{H}^n$, and
- M is connected, complete and orientable.

Prop. *M hyperbolic of finite volume*
$\Leftrightarrow M \cong \Gamma \backslash \mathfrak{H}^n$, $\exists$ *tors-free latt* $\Gamma \subset \mathrm{PO}(1,n)$.

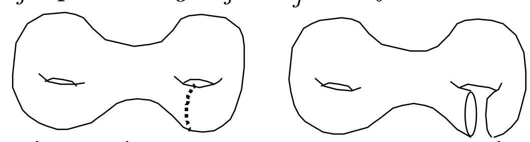
Defn. $\mathrm{PO}(1,n) = \mathrm{O}(1,n)/\{\pm 1\} = \mathrm{Isom}(\mathfrak{H}^n)$
$\sim \mathrm{SO}(1,n)$.

We combine ("interbreed") two hyperbolic mflds
to create a new hyperbolic mfld ("hybrid").
The hybrid is often nonarithmetic.

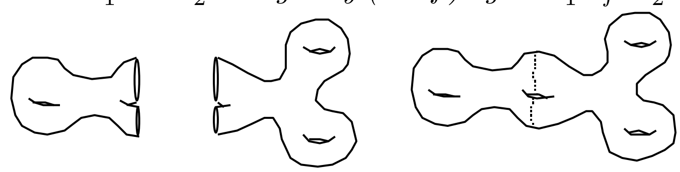
Thm. *Suppose*

- *M_1, M_2 = connected, orientable n-mflds,*
- *C_j = closed $(n-1)$-submanifold of M_j,*
- *$f \colon C_1 \to C_2$ any homeomorphism.*

Cut M_j open along C_j: M_j' = mfld with bdry.



Glue M_1' to M_2' along bdry (via f): get $M_1' \cup_f M_2'$.



Then $M_1' \cup_f M_2'$ is an n-manifold (without bdry).

- *Compact $\Leftrightarrow$ M_1 and M_2 are compact.*
- *Conn $\Leftrightarrow$ either $M_1 \setminus C_1$ or $M_2 \setminus C_2$ is conn.*

May not be Riemannian (in natural way), even if f is an isometry.

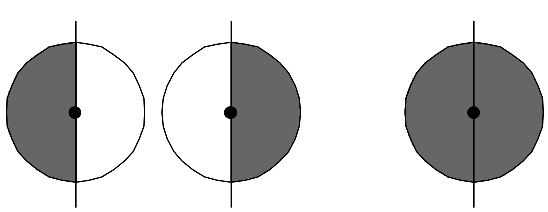
Eg. Let $M_1' = M_2' = D^2$ closed unit disk in $\mathbb{R}^2$.
Glue M_1' to M_2' along boundary to get 2-sphere.
M_1' and M_2' are flat, but S^2 has no flat metric.

Require C_j to be *totally geodesic hypersurface*:
(closed) image of $\mathfrak{H}^{n-1}$ in $\Gamma \backslash \mathfrak{H}^n$.

Prop.

- *M_1, M_2 hyperbolic (of finite volume),*
- *C_1, C_2 totally geodesic hypersurfaces,*
- *$f \colon C_1 \to C_2$ isometry*

$\Rightarrow$ *$M_1' \cup_f M_2'$ is hyperbolic (of finite volume).*



Thm.

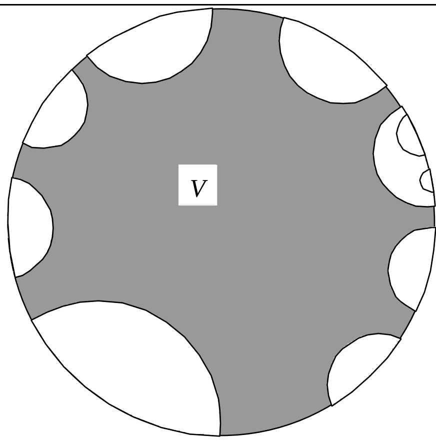
- *$M_1' \cup_f M_2'$ arithmetic,*
- *C_1, C_2 finite volume (as $(n-1)$-manifolds),*
- *$M_1 \setminus C_1, M_2 \setminus C_2$ connected*

$\Rightarrow$ *M_1 is commensurable to M_2.*
I.e., finite cover of one $\cong$ finite cover of other.

Proof. We show $M_1' \cup_f M_2'$ commensurable to M_1.

- Let $M = M_1' \cup_f M_2'$.
- Write $M = \Gamma \backslash \mathfrak{H}^n$.
- Let $\phi \colon \mathfrak{H}^n \to M$ be the covering map.
- Let $B = \phi^{-1}(\partial M_1') = \bigcup$ disjoint $\mathfrak{H}^{n-1}$'s.
- Let V = closure of some component of $\mathfrak{H}^n \setminus B$ that contains a point of $\phi^{-1}(M_1')$.
- Let $\Gamma' = \{ \gamma \in \Gamma \mid \gamma V = V \}$
$= \{ \gamma \in \Gamma \mid \gamma V \cap V \text{ nonempty interior} \}$,

so $M_1' = \phi(V) \cong \Gamma' \backslash V$.



Note: $V = \bigcap$ half-spaces

$\Rightarrow V$ is hyperbolically convex

$\Rightarrow V$ is simply connected

$\Rightarrow V =$ univ cover of M_1',

$\Gamma' =$ fundamental group of M_1'.

Define $\Gamma_*, \phi_*, B_*, V_*, \Gamma_*'$ as above, but with

$M_* = M_1' \cup_{\mathrm{Id}} M_1'$ in place of M.

Uniqueness of univ cover of M_1'

$\Rightarrow$ wolog $V = V_*$ and $\Gamma' = \Gamma_*'$

$\Rightarrow \Gamma' \subset \Gamma \cap \Gamma_*$.

Let $\overline{\Gamma'}$ = Zariski closure of Γ'. Assume $\overline{\Gamma'} \not\supset G$. (This will lead to a contradiction.)

Wolog $\mathfrak{H}^{n-1}$ = component of ∂V.

$\partial M_1'$ finite volume

$\Rightarrow \Gamma' \cap \mathrm{SO}(1, n-1)$ = lattice in $\mathrm{PO}(1, n-1)$

$\Rightarrow \Gamma' \cap \mathrm{SO}(1, n-1)$ Zar dense in $\mathrm{PO}(1, n-1)$

$\Rightarrow \overline{\Gamma'}^\circ = \mathrm{PO}(1, n-1)^\circ$

$\Rightarrow \Gamma' \cap \mathrm{PO}(1, n-1)$ finite index in Γ'

$\Rightarrow \forall$ component H of ∂V,

$\{ \gamma \in \Gamma' \mid \gamma H = H \}$ finite index in Γ'.

Choose two components H_1 and H_2.

Assume $\Gamma' H_1 = H_1$ and $\Gamma' H_2 = H_2$.

For simplicity, assume $\mathrm{dist}(H_1, H_2) > 0$.

(E.g., if $\partial M_1'$ is compact.)

Negative curvature

$\Rightarrow \exists! p \in H_1$, $\mathrm{dist}(p, H_2) = \mathrm{dist}(H_1, H_2)$.

Uniqueness of p

$\Rightarrow \Gamma'$ fixes p

$\Rightarrow \Gamma'$ is finite (bcs properly discontinuous)

$\Rightarrow \partial M_1' \supset \Gamma' \backslash H_1$ has infinite volume. $\rightarrow\leftarrow$

Thm. *$M_1' \cup_f M_2'$ arithmetic*
$\Rightarrow M_1$ is commensurable to M_2.

Cor. *$\exists$ nonarithmetic lattice Γ in $\mathrm{SO}(1, n)$.*

Proof.

- $B_j(x) = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2 - j x_{n+1}^2$
- $\Gamma_1 \approx \mathrm{SO}(B_1; \mathbb{Z})$
- $\Gamma_2 \approx h^{-1}\, \mathrm{SO}(B_2, \mathbb{Z})\, h$
- $h = \mathrm{diag}(1, 1, \ldots, 1, \sqrt{2}) \in \mathrm{GL}(n+1, \mathbb{R})$
- $M_j = \Gamma_j \backslash \mathfrak{H}^n$
- C_j = image of $\mathfrak{H}^{n-1}$ in M_j
- $\hat{\Gamma}_j = \Gamma_j \cap \mathrm{SO}(1, n-1)$

$C_j \cong \hat{\Gamma}_j \backslash \mathfrak{H}^{n-1}$ and $\hat{\Gamma}_1 = \hat{\Gamma}_2$,

$\Rightarrow C_1 \cong C_2$.

$\Rightarrow \exists\ M_1' \cup_f M_2'$.

Suffices to show M_1 not commensurable to M_2.

I.e., Γ_1 not commensurable to conj of Γ_2.

Easy if n is odd.

Lem.

- *n odd,*
- *$\exists g \in \mathrm{O}(1,n)$, s.t. $g\Gamma_1 g^{-1}$ commens to Γ_j*

$\Rightarrow$ j is a perfect square.

Proof. Unique Γ_1-invariant quadratic form is B_1, up to a scalar multiple.

Hence, existence of g implies $B_1 = B_j$, up to

- scalar from $\mathbb{Q}^\times$ and
- change of basis from $\mathrm{GL}(n+1, \mathbb{Q})$.

Discriminant of λB_j is $\lambda^{n+1} j$ (a square times j).
Change of basis multiplies by $(\det)^2$.

Rem. M_1 is commensurable to M_2 if n is even.
But Γ is not arithmetic:

$n-1$ odd $\Rightarrow$ $\Gamma \cap \mathrm{SO}(1, n-1)$ not arithmetic.

Rem. We constructed a noncocompact lattice, but the same method also yields a cocompact lattice.

References

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