WHAT IS A COXETER GROUP?
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ABSTRACT. Coxeter groups arise in a wide variety of areas, so every mathematician should know some basic facts about them, including their connection to "Dynkin diagrams." Proofs about these "groups generated by reflections" mainly use group theory, geometry, and combinatorics.

This talk will briefly explain:

- what it means to say that G is a "group generated by reflections" (or, equivalently, that G is a "Coxeter group");
- what it means to say that G is "of type A_n, B_n, or D_n", and
- that all but finitely many of the "irreducible" finite groups generated by reflections are either dihedral groups or belong to these types.

For further reading, see [1, Chap. 1], [2], or [3].

1. What is a group generated by reflections?

Example. In $\mathbb{R}^2$:

- The reflection across the y-axis is the map $(x, y) \mapsto (-x, y)$.
- The reflection across the line $y = x$ is the map $(x, y) \mapsto (y, x)$.

In general, it is easy to see that the reflection σ_L across a line L is the unique isometry of $\mathbb{R}^2$ that fixes each point on the line L, and has order 2 (i.e., $\sigma_L^2 = \mathrm{Id}$, but $\sigma_L \neq \mathrm{Id}$).

Definition. Let H be a hyperplane in $\mathbb{R}^n$. (E.g., a line in $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$.) The corresponding ***reflection*** σ_H is the unique isometry of $\mathbb{R}^n$ that fixes each point on H, and has order 2.

Remark. If H passes through the origin, and e is a unit vector that is orthogonal to H, then

$$\sigma_H(x) = x - 2(x \cdot e)\, e.$$

Example. Let L_1 and L_2 be two lines in $\mathbb{R}^2$, and assume $\angle L_1 L_2 = \pi/m$, for some $m \in \mathbb{Z}^+$. Then (using σ_i as a shorthand for σ_{L_i}) $\sigma_1\sigma_2$ is a rotation through angle $2\pi/m$, so $(\sigma_1\sigma_2)^m = \mathrm{Id}$. Indeed, $\langle \sigma_1, \sigma_2 \rangle$ is the dihedral group D_{2m}, so it has the presentation

$$D_{2m} = \langle\, \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^m = 1 \,\rangle.$$

This is a (fairly trivial) example of a finite group that is generated by reflections. In this theory, it is denoted $I_2(m)$: the subscript 2 means that we are in $\mathbb{R}^2$, and the m tells us that the order of $\sigma_1\sigma_2$ is m. Combined with the fact that reflections always have order 2, this information determines the entire presentation. (There is no deep reason for using the letter I for this particular group; the letters $A, B, \ldots, H$ are used for other Coxeter groups.)

Note that if H_1 and H_2 are any two hyperplanes in $\mathbb{R}_n$, then $\langle \sigma_1, \sigma_2 \rangle \cong I_2(m)$ for some $m \in \mathbb{Z}^+ \cup \{\infty\}$. (We use the convention that the relation $(\sigma_1\sigma_2)^\infty = 1$ does not impose any restriction on σ_1 and σ_2.) Therefore:

Lemma. *Suppose $\sigma_1, \ldots, \sigma_\ell$ are (distinct) reflections in $\mathbb{R}^n$. Then, for each $i, j \in \{1, \ldots, \ell\}$, there exists $m_{i,j} \in \mathbb{Z}^+ \cup \{\infty\}$, such that*

(1) $(\sigma_i\sigma_j)^{m_{i,j}} = \mathrm{Id}$,
(2) $m_{i,i} = 1$,
(3) $m_{i,j} \geq 2$ *if* $i \neq j$,
(4) $m_{j,i} = m_{i,j}$, *and*
(5) $m_{i,j} = 2$ *iff* σ_i *commutes with* σ_j.

This observation provides the axioms for an abstract definition:

Definition. Let M be an $\ell \times \ell$ symmetric matrix with entries in $\mathbb{Z}^+ \cup \{\infty\}$, such that $m_{i,i} = 1$, but $m_{i,j} \geq 2$ when $i \neq j$. The corresponding ***Coxeter group*** (or ***group generated by reflections***) is the group with the following presentation (i.e., generators and relations):

$$\langle\, \sigma_1, \ldots, \sigma_\ell \mid (\sigma_i\sigma_j)^{m_{i,j}} = 1 \text{ for } 1 \leq i, j \leq \ell \,\rangle.$$

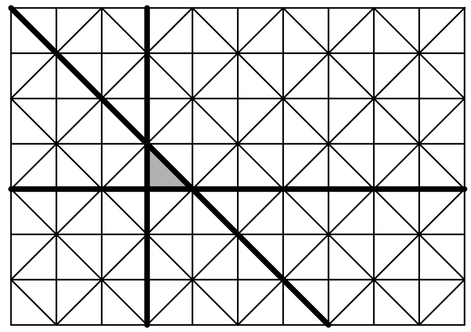
Warning. If $\sigma_1, \ldots, \sigma_\ell$ are reflections in $\mathbb{R}^n$, the group $\langle \sigma_1, \ldots, \sigma_\ell \rangle$ that they generate is usually *not* a "group generated by reflections." The problem is that there will usually be additional relations between $\sigma_1, \ldots, \sigma_\ell$ that are not consequences of the relations in the definition of a Coxeter group.

Conversely, since "group generated by reflections" is an abstract definition, a typical Coxeter group will **<u>not</u>** be isomorphic to any group that is generated by reflections of Euclidean space.

Example (an infinite Coxeter group)**.** For an isosceles right triangle $\triangle a_1a_2a_3$ in $\mathbb{R}^2$, let L_1, L_2, and L_3 be the lines obtained by extending the sides of the triangle. (For definiteness, require that $a_i \notin L_i$, and that the right angle is at a_2.) Then $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is a Coxeter group, with

$$(m_{i,j}) = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}.$$

This group acts as symmetries of a tiling of $\mathbb{R}^2$ by copies of $\triangle a_1 a_2 a_3$.

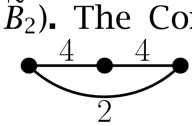


(In fact, there is a unique element of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ that moves $\triangle a_1 a_2 a_3$ to any other tile. This observation is the key to proving that $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is indeed a Coxeter group; i.e., that the generators σ_1, σ_2, and σ_3 do not satisfy any relations other than the ones forced by the defining relations of the Coxeter group.)

Definition. The "Coxeter matrix" $(m_{i,j})$ can be replaced with a (labelled) graph that presents the same information:

- the vertices are $\{1, 2, \ldots, \ell\}$ (or the reflections $\sigma_1, \ldots, \sigma_\ell$), and
- the number $m_{i.j}$ is written on the edge $i \overset{m_{i,j}}{—} j$ (when $i \neq j$).

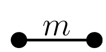
When $m_{i,j}$ is small, we usually draw a multiple edge, instead of a labelled edge. Since $m_{i,j} \geq 2$, but the number of multiple edges can be as small as 0 (meaning there is no edge from i to j), the number of edges drawn from i to j is taken to be $m_{i,j} - 2$. The result is called a ***Coxeter diagram***.

Example (Coxeter diagram $\widetilde{B}_2$)**.** The Coxeter diagram of the above-mentioned infinite Coxeter group is  or  .

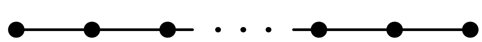
Exercise. If a Coxeter diagram is not connected, then the associated Coxeter group is the direct product of the groups corresponding to the connected components.

2. Finite Coxeter groups

Theorem ([1, pp. 5–6], [2, pp. 214 and 242–246])**.** *Every finite Coxeter group is a direct product of groups on the following list* (*where the subscript in the notation, usually n, specifies the number of vertices in the Coxeter diagram*):

$I_2(m)$: *dihedral group D_{2m} of order $2m$* 

A_n: *symmetric group S_{n+1}, with $\sigma_i = (i\ i+1)$.*
Note that $(\sigma_i \sigma_{i+1})^3 = 1$, and σ_i commutes with σ_j if $|i - j| \neq 1$.



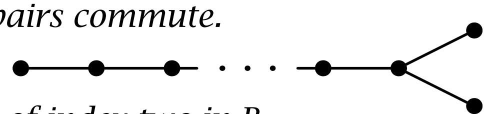
This is the group of symmetries of a regular simplex in $\mathbb{R}^n$ (the higher-dimensional generalization of a regular triangle or regular tetrahedron).

B_n: *The group of* <u>*signed*</u> *permutations*
(permutations σ *of* $\{\pm 1, \ldots, \pm n\}$*, such that* $\sigma(-i) = -\sigma(i)$ *for all* i*),*
with $\sigma_i = (i\ i+1)(-i\ \ -(i+1))$ *for* $i < n$*, and* $\sigma_n = (n\ -n)$*.*
Note that $(\sigma_i \sigma_{i+1})^3 = 1$ *if* $i \leq n-2$*, but* $(\sigma_{n-1}\sigma_n)^4 = 1$*,*
and σ_i *commutes with* σ_j *if* $|i-j| \neq 1$*.*

This is the group of symmetries of the n*-dimensional cube.*

C_n: *Same as* B_n*, so not needed in this list.*

D_n: *Signed permutations of* $\{\pm 1, \ldots, \pm n\}$ *that change an even number of signs*
(i.e., $\{\sigma(1), \ldots, \sigma(n)\} \cap \{-1, \ldots, -n\}$ *has even cardinality),*
with σ_i *as in* B_n *for* $i < n$*, and* $\sigma_n = (n-1\ \ -n)(-(n-1)\ \ n)$*.*
Note that $(\sigma_i \sigma_{i+1})^3 = 1$ *if* $i \leq n-2$*,* $(\sigma_{n-2}\sigma_n)^3 = 1$*,*
and all other pairs commute.



This is a subgroup of index two in B_n*.*

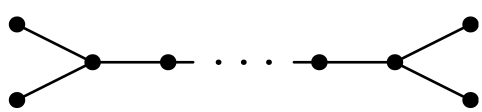
etc: *Seven exceptional examples, called* E_6*,* E_7*,* E_8*,* F_4*,* G_2*,* H_3*, and* H_4*.*

Idea of proof. Suppose X is the Coxeter diagram of a finite group.

Key Fact: If X' is a subdiagram of X (obtained by deleting vertices and/or reducing labels on the edges of X), then X' also represents a finite group.

Every cycle is the Coxeter diagram of an infinite group (see the corollary below). Therefore, the Key Fact implies that X has no cycles, so it is a tree.

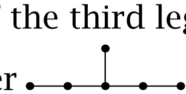
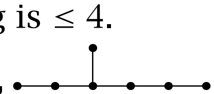
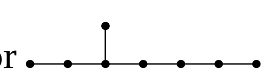
For any $\ell \geq 4$, it can be shown that the following Coxeter diagram represents an infinite group:

$\widetilde{D}_\ell$:  (has $\ell + 1$ vertices).

Assume, for simplicity, that X has no multiple edges. Since $\widetilde{D}_4$ is infinite, every vertex must have degree ≤ 3. Furthermore, there can be at most one vertex of degree 3, since $\widetilde{D}_\ell$ is infinite for $\ell > 4$.

We may assume $X \neq A_n$, so there does exist a (unique) vertex of degree 3. We may also assume $X \neq D_n$, so no more than one of the three legs of X has length one. A few more comparisons (which we omit) show:

- There does exist a (unique) leg of length one.
- One of the other legs has length less than three, so it is of length two.
- The length of the third leg is ≤ 4.

Therefore, X is either , , or . These are the Coxeter diagrams E_6, E_7, and E_8, respectively. □

Remark. Here are diagrams of the remaining Coxeter groups that are finite:

F_4 , G_2  $= I_2(6)$ (not needed in this list),

H_3 , H_4 .

The following result can be used to show that Coxeter groups are infinite (which is required to carry out the above proof).

Proposition. *Suppose $M = (m_{i,j})$ is the Coxeter matrix of G, and let*

$$M^\dagger = \Big(-\cos(\pi/m_{i,j})\Big)_{i,j}.$$

If G is finite, then $M^\dagger$ is positive definite: all of its eigenvalues are strictly positive.

Proof. Define a bilinear form $\langle x|y\rangle = x^T M^\dagger y$, and let $\{e_1,\ldots,e_\ell\}$ be the standard basis of $\mathbb{R}^\ell$. For each i, define a linear transformation r_i on $\mathbb{R}^\ell$ by

$$r_i(x) = x - 2\langle x \mid e_i\rangle\, e_i.$$

This is the formula for a reflection through the hyperplane orthogonal to e_i (with respect to the given bilinear form), so it is not difficult to see that:

- $\langle\,|\,\rangle$ is invariant under $\langle r_1,\ldots,r_\ell\rangle = \overline{G}$.
- Since $\langle e_i \mid e_j\rangle = -\cos(\pi/m_{i,j})$, the transformations r_i and r_j interact just like Euclidean reflections through hyperplanes at an angle of $\pi/m_{i,j}$. Therefore $(r_i r_j)^{m_{i,j}} = \mathrm{Id}$. This means that $r_1,\ldots,r_\ell$ satisfy the relations defining G, so there is a homomorphism $G \to \overline{G}$, such that $\overline{\sigma_i} = r_i$.

Now suppose G is finite. Then we can define a G-invariant bilinear form by averaging the usual dot product:

$$\langle x \mid y\rangle' = \frac{1}{|G|}\sum_{g\in G}(gx)\cdot(gy).$$

This is positive definite (i.e., $\langle x \mid x\rangle' > 0$ for all $x \neq 0$). However, if we assume that G is irreducible (i.e., the Coxeter diagram is connected), then it can be shown that the G-invariant bilinear form is unique (up to a scalar multiple). Therefore, $\langle x|y\rangle = \langle x|y\rangle'$ is positive definite, so the matrix $M^\dagger$ is positive definite. □

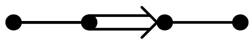
Remark. The converse is true: if $M^\dagger$ is positive definite, then G is finite.

Corollary. *The Coxeter group corresponding to a cycle is infinite.*

Proof. Since $-\cos(\pi/3) = -1/2$, the nonzero entries in each row of $M^\dagger$ are 1, $-1/2$, and another $-1/2$, so the sum of every row is 0. Therefore, $(1,1,\ldots,1)$ is an eigenvector of $M^\dagger$ with eigenvalue 0. Hence, $M^\dagger$ is not positive definite. □

Alternate proof. The group acts transitively on the tiles of a tessellation of Euclidean space by regular simplices. □

Exercise. Show that the Coxeter group of type $\widetilde{D}_\ell$ is infinite.
Hint: $M^\dagger$ is not invertible.

Remark. Coxeter groups of types A to G arise in the theory of Lie groups, where they are called ***Weyl groups*** (but types H_3, H_4, and $I_2(m)$ do not appear). An orientation is assigned to each edge with a label greater than 2 (or, in other words, to each multiple edge) in the Coxeter diagram, and the resulting directed graph is called a ***Dynkin diagram***. For example, the Dynkin diagram of type F_4 is .

However, the Coxeter diagram of type B_n yields two different Dynkin diagrams (if $n > 2$), because the two different orientations of the double edge yield non-isomorphic directed graphs. One orientation yields the Dynkin diagram B_n, and assigning the opposite orientation yields a Dynkin diagram that is called C_n.

Also, a slightly different convention is used in Lie theory, so the Dynkin diagram G_2 has only a triple edge, not quadruple.

References

[1] K. Brown: *Buildings*. Springer, New York, 1989. MR0969123, ISBN 0-387-96876-8.
http://www.math.cornell.edu/~kbrown/buildings/brown1988.pdf

[2] P. de la Harpe: An invitation to Coxeter groups, in *Group Theory from a Geometrical Viewpoint (Trieste, 1990)*. World Sci. Publ., River Edge, NJ, 1991, pp. 193–253. MR1170367

[3] J. E. Humphreys: *Reflection Groups and Coxeter Groups*. Cambridge Univ. Press, Cambridge, 1990. MR1066460, ISBN: 0-521-37510-X