Cartan-decomposition subgroups

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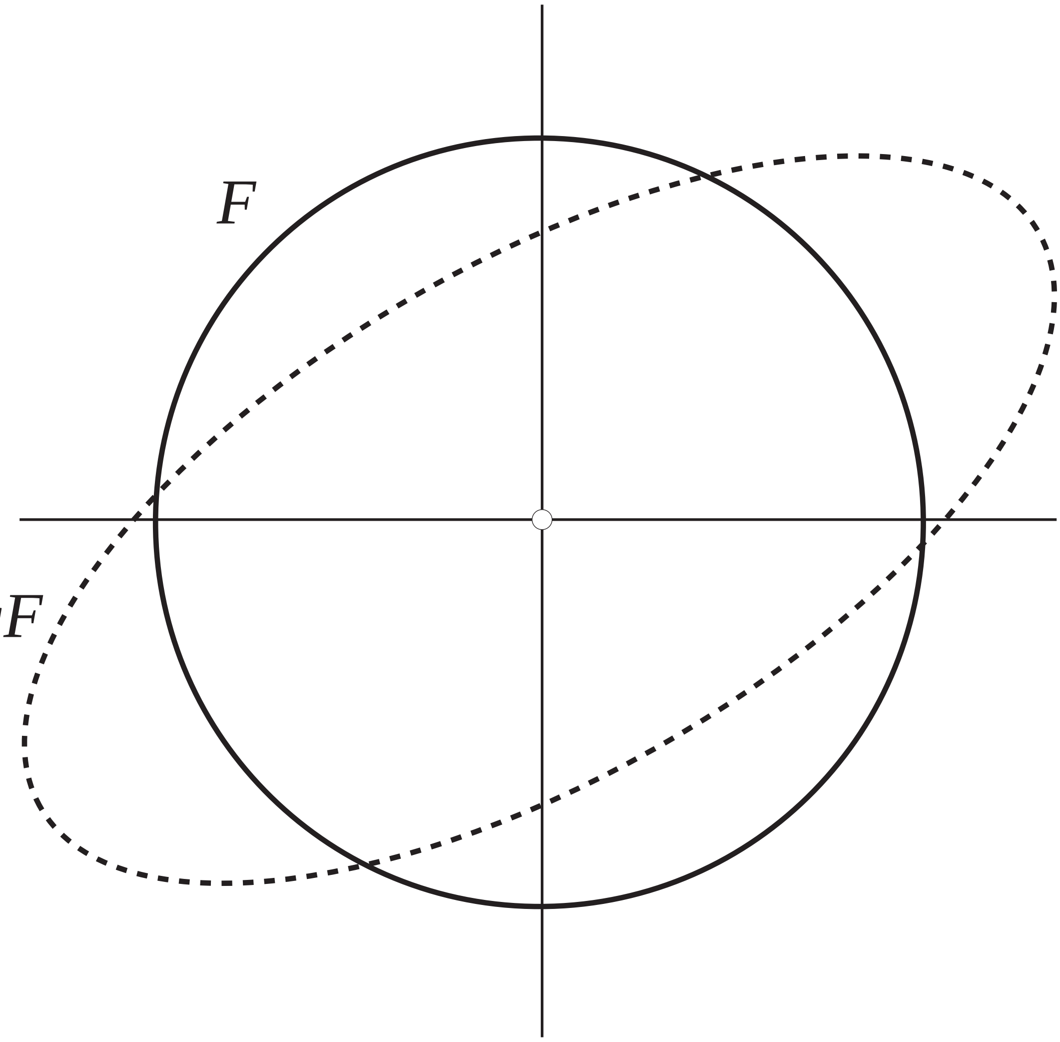
Eg. $G = \mathrm{SL}(2,\mathbb{R})$ is transitive on $X = \mathbb{R}^2 - \{0\}$.
(So X is a *homogeneous space*.)

Let $F =$ unit circle (compact).



$$\forall g \in G, \;\; gF \cap F \neq \emptyset.$$

There is a compact subset of X that cannot be moved disjoint from itself.

$$\forall g \in G, \;\; gF \cap F \neq \emptyset.$$

Group-theoretic restatement.

Stabilizer of point $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $H = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

so $X \cong G/H$.

Let $C \subset G$ (compact) with $Cv = F$.

$\emptyset \neq gF \cap F = gCv \cap Cv = gCH \cap CH$

$\Rightarrow gc_1 h_1 = c_2 h_2$

$\Rightarrow g \in CHC^{-1}$

Defn. H is a *Cartan-decomposition subgrp* (CDS):

- $\exists$ compact $C \subset G$, such that $G = CHC$
- H is closed and connected.

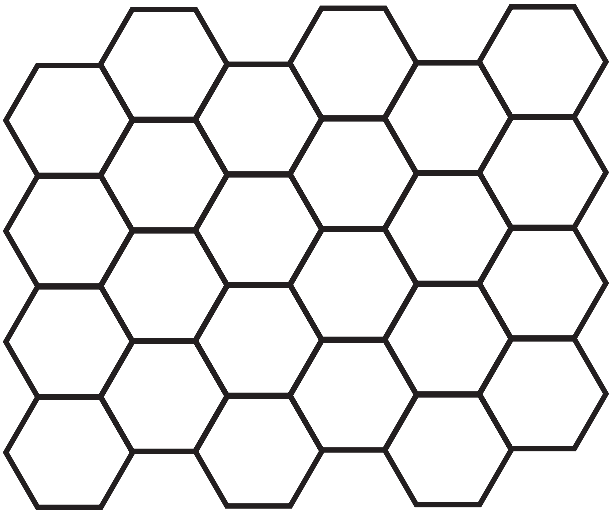
Rem. C is only a sub**set**, not a sub**group**.

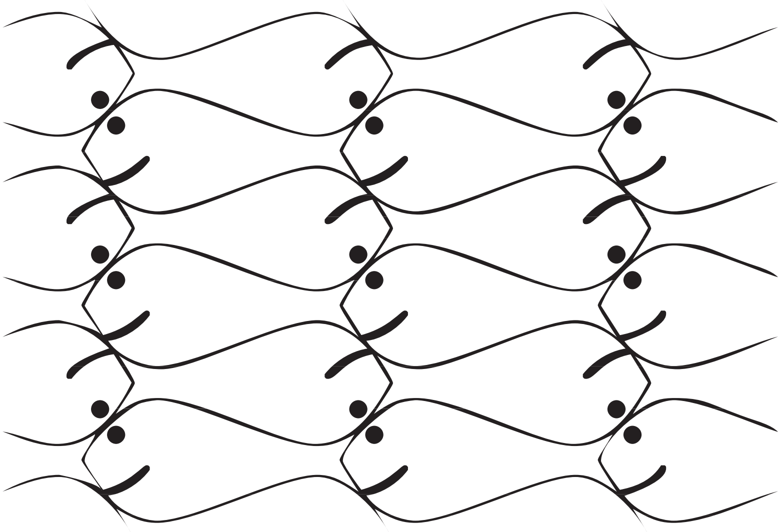
Can C always be chosen to be a subgroup?

(I think not.)

Motivation. Tessellation:

symmetric tiling of a homogeneous space X.





$\forall$ tiles T_1, T_2, $\exists$ isometry ϕ,

$\phi(T_1) = T_2$ and $\phi(\text{tile}) = \text{tile}$

Let Γ = symmetry group of the tessellation.

Any tile is a fundamental domain for $\Gamma \backslash X$.

So $\Gamma \backslash X$ is compact

and Γ acts properly discontinuously on X.

Defn. Γ acts properly discontinuously on X:

$\forall$ cpct $F \subset X$,

$$\{\, \gamma \in \Gamma \mid \gamma F \cap F \neq \emptyset \,\} \text{ is finite.}$$

(In particular, all orbits are discrete.)

Conversely: if

- $\Gamma \subset \mathrm{Isom}(X)$,
- $\Gamma \backslash X$ is compact and
- Γ acts properly discontinuously on X,

then translates of any fund domain yield a tess.

$G = \mathrm{SL}(n, \mathbb{R})$

$\quad = $ (Zariski) connected, almost simple Lie grp

$H =$ closed, connected subgroup of G

Question. *Does G/H have a tessellation? I.e., is there a discrete subgroup Γ of G, such that*

- *Γ acts properly discontinuously on G/H;*

and

- *$\Gamma \backslash G/H$ is compact?*

Easy examples.

If G/H is compact: let $\Gamma = e$.

If H is compact: let Γ be a lattice in G.

Defn. Γ is a (cocompact) lattice in G:

- Γ is discrete
- $\Gamma \backslash G$ is compact.

A. Borel proved there is a lattice in every simple G.

Assumption. Neither H nor G/H is compact.

Therefore Γ must be infinite
and Γ cannot be a lattice in G.

Prop. *H is a Cartan-decomposition subgroup*
$\Rightarrow G/H$ does not have a tessellation.

Proof. $\exists$ cpct $F \subset G/H$, s.t. $\forall g \in G,\ gF \cap F \neq \emptyset$
$\Rightarrow \Gamma = \{\, \gamma \in \Gamma \mid \gamma F \cap F \neq \emptyset \,\}$ is finite. $\rightarrow\leftarrow$

$G = \mathrm{SL}(n, \mathbb{R})$

$K = \mathrm{SO}(n)$ rotations (compact)

$$A = \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix} \quad \text{diagonal}$$

$$N = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \quad \text{upper triangular}$$

Cartan decomposition. $G = KAK$

so A is a Cartan-decomposition subgroup

Fact. $G = KNK$ [Kostant]

so N is a Cartan-decomposition subgroup

Prop. *Every (connected, noncompact) subgrp H of* $\mathrm{SL}(2,\mathbb{R})$ *is a Cartan-decomposition subgroup.*

Cor. *No (interesting) homogeneous space of* $\mathrm{SL}(2,\mathbb{R})$ *has a tessellation.*

Proof of proposition. H contains either A or N (or a conjugate).

Better proof.

$$\mu(e) = e, \qquad \lim_{h\to\infty} \mu(h) = \infty \qquad \Rightarrow \mu(H) = A^+$$

I.e., $A^+ \subset KHK$.

So $G = KA^+K \subset KHK$.

Rem. $\mu(H) = A^+ \Leftrightarrow KHK = G$
$\Rightarrow H$ is a CDS.

Same proof. $\mathbb{R}$-rank$G = 1 \Rightarrow H$ is a CDS.

Given $g \in G$.

$G = KAK \Rightarrow \exists a \in A$, s.t. $g \in KaK$.

But a is not unique

Let $A^+ = \left\{ \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \middle| \begin{array}{c} a_1 \geq a_2 > 0 \\ a_1 a_2 = 1 \end{array} \right\}$

= "positive Weyl chamber."

Then $\exists! a \in A^+$, s.t. $g \in KaK$.

Defn (Cartan projection). $\mu \colon G \to A^+$

by $g \in K\,\mu(g)\,K$.

μ is continuous and proper.

H is a CDS

$$\Leftrightarrow \exists \text{ cpct } C \subset G, \;\; G \subset CHC$$
$$\Leftrightarrow \exists \text{ cpct } C \subset G, \;\; A^+ \subset C\,\mu(H)\,C$$

Can take C to be in A!

Thm (Benoist, Kobayashi). *H is a CDS iff $\mu(H)$ comes within bdd distance of every pt of A^+*

i.e., $\exists$ cpct $C \subset A$, s.t. $\mu(H)C \supset A^+$.

Not every subgroup of $\mathrm{SL}(3,\mathbb{R})$ is a CDS.

Eg. $\dim H = 1 \Rightarrow H$ is not a CDS.

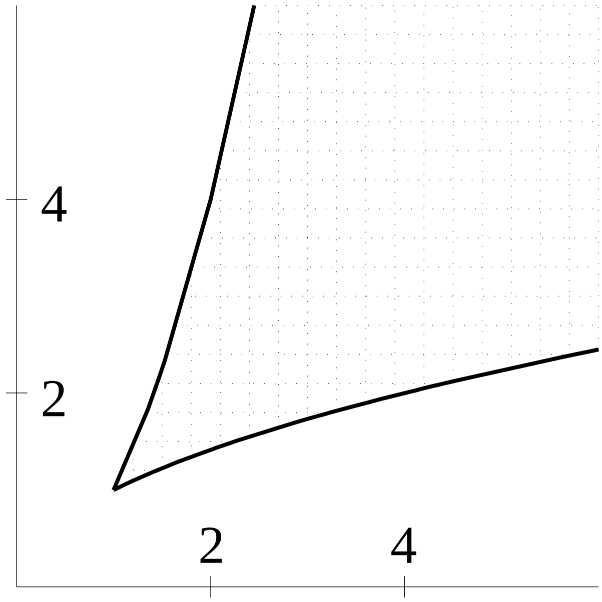
$$A^+ = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} a_1 \geq a_2 \geq a_3 > 0 \\ a_1 a_2 a_3 = 1 \end{matrix} \right\}$$

$$\leftrightarrow \left\{ (s,t) \in (\mathbb{R}^+)^2 \;\middle|\; \begin{pmatrix} s & & \\ & t/s & \\ & & 1/t \end{pmatrix} \in A^+ \right\}$$

$$= \{ (s,t) \in (\mathbb{R}^+)^2 \mid s \geq t/s \geq 1/t \}$$

$$= \{ (s,t) \in (\mathbb{R}^+)^2 \mid \sqrt{s} \leq t \leq s^2 \}$$

$\mathrm{SL}(3,\mathbb{R})$: $A^+ \leftrightarrow \{(s,t) \in (\mathbb{R}^+)^2 \mid \sqrt{s} \le t \le s^2\}$



Thm (Benoist, Kobayashi). *H is a CDS iff $\mu(H)$ comes within bdd distance of every pt of A^+*

Cor. $\dim H = 1 \Rightarrow$ *H is not a CDS.*

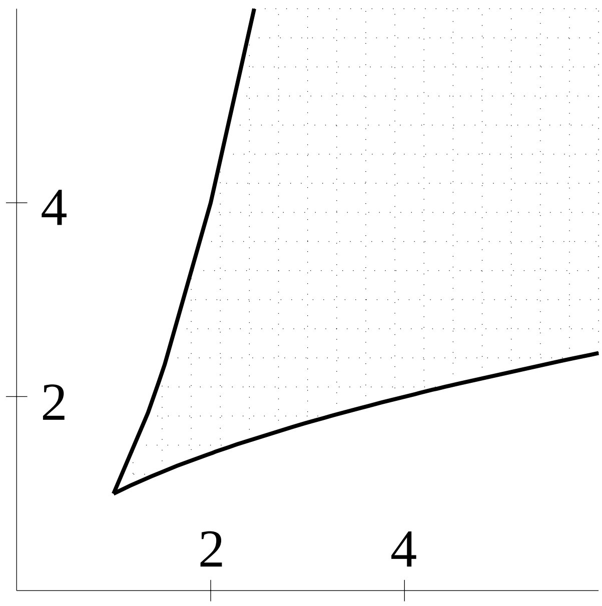
Cor. $H = \begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ *is not a CDS.*

$\exists k \in \begin{pmatrix} 1 & \\ & \mathrm{SO}(2) \end{pmatrix}$, $k^{-1}hk \in \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} = U$.

So $\mu(H) = \mu(U)$.

Prop. $\left\{ \begin{pmatrix} 1 & u & v \\ & 1 & u \\ & & 1 \end{pmatrix} \;\middle|\; u, v \in \mathbb{R} \right\}$ *is a CDS.*

Suffices: $\mu(H)$ within bdd dist of every pt of ∂A^+.



[Does not work for $\mathrm{SL}(4,\mathbb{R})$ (or $\mathbb{R}$-rank$G \geq 2$).]

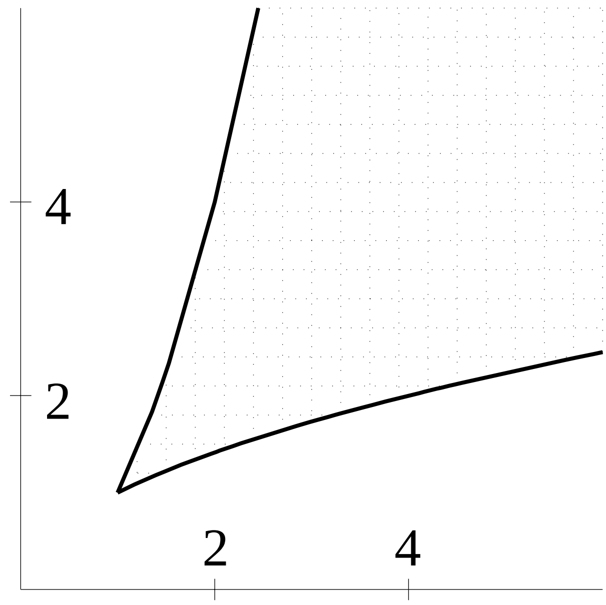
Actually only need one wall.

(*h* near one wall $\Rightarrow h^{-1}$ near other wall.)

This is special for $\mathrm{SL}(n,\mathbb{R})$, not other G.

How to calculate $\mu(h)$.

$$A^+ = \left\{ \begin{pmatrix} s & & \\ & t/s & \\ & & 1/t \end{pmatrix} \;\middle|\; \sqrt{s} \le t \le s^2 \right\}$$



For $a \in A^+$:

$$s_a \approx \|a\|$$
$$t_a \approx \|a^{-1}\|$$

For $g \in G$:

$$s_{\mu(g)} \approx \|\mu(g)\| = \|k_1 g k_2\| = \|g\|$$
$$t_{\mu(g)} \approx \|\mu(g)^{-1}\| = \|g^{-1}\|$$

Thus, $\mu(g) \leftrightarrow \big(\|g\|, \|g^{-1}\|\big)$.

so $\mu \begin{pmatrix} 1 & u & 0 \\ & 1 & u \\ & & 1 \end{pmatrix} \approx \big(|u|, u^2\big)$ is near a wall.

Thm (O–W).
Every CDS of $\mathrm{SL}(3,\mathbb{R})$ *contains a conjugate of:*

A,

$$\left\{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; r, s \in \mathbb{R} \right\},$$

$$\left\{ \begin{pmatrix} e^t & te^t & s \\ 0 & e^t & r \\ 0 & 0 & e^{-2t} \end{pmatrix} \;\middle|\; r, s, t \in \mathbb{R} \right\},$$

$$\left\{ \begin{pmatrix} e^{pt} & r & 0 \\ 0 & e^{qt} & 0 \\ 0 & 0 & e^{-(p+q)t} \end{pmatrix} \;\middle|\; r, t \in \mathbb{R} \right\},$$

$(\max\{p,q\} = 1,\ \min\{p,q\} \geq -1/2)$, or

$$\left\{ \begin{pmatrix} e^t \cos pt & e^t \sin pt & s \\ -e^t \sin pt & e^t \cos pt & r \\ 0 & 0 & e^{-2t} \end{pmatrix} \;\middle|\; r, s, t \in \mathbb{R} \right\}$$

$(p \neq 0)$.

Thm (Benoist, O-W). *If* $G = \mathrm{SL}(3,\mathbb{R})$, *then* G/H *does not have a tessellation.*

[Benoist proved for $H = \mathrm{SL}(2,\mathbb{R})$.
Same method for other subgroups.]

In general [Benoist], for $\mathbb{R}$-rank$G = 2$:

$\exists$ representations ρ_1 and ρ_2 of G,
s.t. $\mu(g) \approx \big(\|\rho_1(g)\|, \|\rho_2(g)\|\big)$.

Walls are given by $\|\rho_1(g)\| = \|\rho_2(g)\|^{c_i}$.

Eg. $G = \mathrm{SL}(3,\mathbb{R})$.

$$\rho_1(g) = g, \qquad \rho_2(g) = (g^{-1})^T$$

$$c_1 = 1/2, \qquad c_2 = 2$$

Eg. $G = \mathrm{SO}(2,n)$ or $\mathrm{SU}(2,n)$.

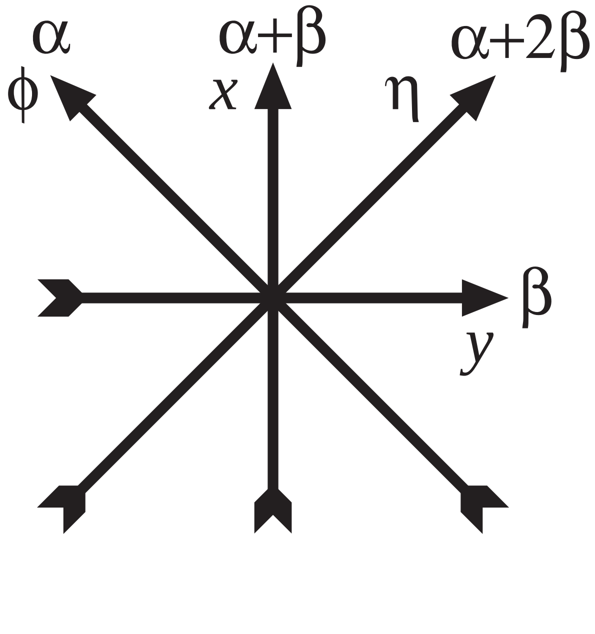
$$\rho_1(g) = g, \qquad \rho_2(g) = g \wedge g$$

$$c_1 = 1, \qquad c_2 = 2$$

$$\mathrm{SO}(2,n) = \mathrm{Isom}\left(v_1 v_{n+2} + v_2 v_{n+1} + \sum\nolimits_{i=3}^{n} v_i^2\right)$$

$$\mathfrak{a} + \mathfrak{n} = \left\{ \begin{pmatrix} \tau_1 & \phi & x & \eta & 0 \\ & \tau_2 & y & 0 & -\eta \\ & & 0 & -y^T & -x^T \\ & & & -\tau_2 & -\phi \\ & & & & -\tau_1 \end{pmatrix} \right\}$$

$$t_1, t_2, \phi, \eta \in \mathbb{R},\ \ x, y \in \mathbb{R}^{n-2}$$



Thm (O–W). *$H \subset N$ is a CDS if*

- $\dim H = 2$, $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, $\exists u \in \mathfrak{h}$ *s.t.* $\phi_u y_u \neq 0$; *or*
- $\dim H \geq 2$,
  - $\exists u \in \mathfrak{h}$ *s.t.* $\dim\langle(\phi_u, x_u), (0, y_u)\rangle = 1$;
  - $\exists v \in \mathfrak{h}$ *s.t. either*
    - $\dim\langle(\phi_v, x_v), (0, y_v)\rangle = 2$ *or*
    - $y_v = 0$ *and* $\|x_v\|^2 = -2\phi_v\eta_v$.

Thm (O–W). *$H \subset N$ is* **not** *a CDS if*

- $\dim H \leq 1$; *or*
- $\forall u \in \mathfrak{h}$, $\phi_u = 0$ *and* $\dim\langle x_u, y_u\rangle \neq 1$; *or*
- $\forall u \in \mathfrak{h}$, $\phi_u = 0$ *and* $\dim\langle x_u, y_u\rangle = 1$; *or*
- $\exists\ X_0 \subset \mathbb{R}^{n-2}$, $x_0 \in X_0$, $x' \in X_0^{\perp}$, $\eta_0 \in \mathbb{R}$ *s.t.*
  - $\|x_0\|^2 - \|x'\|^2 - 2\eta_0 < 0$,
  - $\forall u \in \mathfrak{h}$, $y_u = 0$, $x_u \in \phi_u x' + X_0$, *and* $\eta_u = \phi_u \eta_0 + x_0 \cdot x$.

For $u \in \mathfrak{n}$, $\exp(u) =$

$$\begin{pmatrix} 1 & \phi & x + \dfrac{\phi y}{2} & \eta - \dfrac{x \cdot y}{2} - \dfrac{\phi \|y\|^2}{6} & -\phi\eta - \dfrac{\|x\|^2}{2} + \dfrac{\phi^2 \|y\|^2}{24} \\ & 1 & y & -\dfrac{\|y\|^2}{2} & -\eta - \dfrac{x \cdot y}{2} + \dfrac{\phi \|y\|^2}{6} \\ & & \mathrm{Id} & -y^{\mathsf{T}} & -x^{\mathsf{T}} + \dfrac{\phi y^{\mathsf{T}}}{2} \\ & & & 1 & -\phi \\ & & & & 1 \end{pmatrix}$$

Thm (Kobayashi). *$H, L \subset AN$ and $L \subset CHC$. If $\dim L > \dim H$, then G/H does not have a tess.*

Thm (O–W). *$H \subset AN$, $\dim H \geq 2$. If $\nexists L \subset AN$, s.t. $L \subset CHC$ and $\dim L > \dim H$, then*

- *$H \sim \mathrm{SO}(1, n) \cap AN$; or*
- *$H \sim L_5 \cap AN$; or*
- *n even and $H \sim H_B$; or*
- *n odd, $\dim H = n - 1$, $\mathrm{SU}(1, \frac{n-1}{2}) \subset CHC$.*

Proof. Inspect list of non-CDS subgroups, compare image of Cartan projection.

Eg. $\forall h \in \mathrm{SU}(1, \lfloor n/2 \rfloor)$, we have $\mu(h) \approx \|h\|^2$.

If $\exists h_n \to \infty$ in H, s.t. $\mu(h_n) \approx \|h_n\|^2$,
then $\exists$ cpct $C \subset G$ with $\mathrm{SU}(1, \lfloor n/2 \rfloor) \subset CHC$.

$$L_5 \cong \mathrm{PSL}(2,\mathbb{R})$$
$$= \text{image of 5-dim'l rep of } \mathrm{SL}(2,\mathbb{R}).$$

$$\mathfrak{h}_B = \left\{ \begin{pmatrix} \tau & 0 & x & \eta & 0 \\ & \tau & B(x) & 0 & -\eta \\ & & \dots & & \end{pmatrix} \;\middle|\; \begin{matrix} x \in \mathbb{R}^{2m-2} \\ t, \eta \in \mathbb{R} \end{matrix} \right\}$$

$$B\colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2} \text{ has no real eigenvalues}$$

- $H \sim \mathrm{SO}(1,n) \cap AN$
  n even: G/H has a tess [Kulkarni]
  ($\Gamma \subset \mathrm{SU}(1,n/2)$)
  n odd: G/H has no tess [Kulkarni]
- $H \sim L_5 \cap AN$
  L_5 is tempered in G [Oh],
  so G/H has no tess [Margulis]
- n even and $H \sim H_B$
  G/H has a tess
  ($\Gamma \subset \mathrm{SO}(1,n)$)
  special case [Kulkarni]: $\mathrm{SU}(1,n/2) \cap AN$
- n odd, $\dim H = n-1$, $\mathrm{SU}(1,\frac{n-1}{2}) \subset CHC$.
  Conj. $G/\mathrm{SU}\big(1,\frac{n-1}{2}\big)$ has no tess. ???

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