Automorphisms of direct products of some circulant graphs

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Abstract. The direct product of two graphs X and Y is denoted $X \times Y$. This is a natural construction, so any isomorphism from X to X' can be combined with any isomorphism from Y to Y' to obtain an isomorphism from $X \times Y$ to $X' \times Y'$. Therefore, the automorphism group $\operatorname{Aut}(X \times Y)$ contains a copy of $(\operatorname{Aut} X) \times (\operatorname{Aut} Y)$. It is not known when this inclusion is an equality, even for the special case where $Y = K_2$ is a connected graph with only 2 vertices.

Recent work of B. Fernandez and A. Hujdurović solves this problem when X is a "circulant" graph with an odd number of vertices (and $Y = K_2$). We will present a short, elementary proof of this theorem.

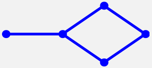
Graph products

Given two graphs X and Y, construct a new graph $X * Y$.
Most important: Cartesian $\square$, strong $\boxtimes$, direct $\times$. (wreath $\wr$)

- commutative: $X * Y \cong Y * X$ (not wreath)
- associative: $(X * Y) * Z \cong X * (Y * Z)$

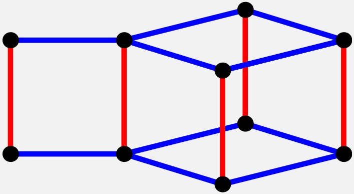
Definition (Cartesian product $X \square Y$)

Horizontal copies of

$X =$ 

Vertical copies of

$Y = K_2 =$ 

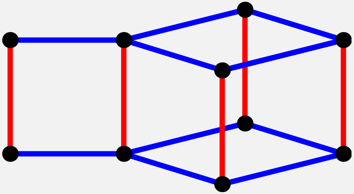


Cartesian product $X \square Y$:

horizontal copies of X,

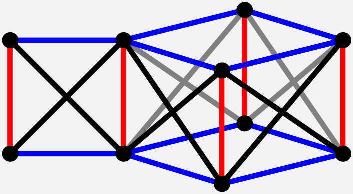
vertical copies of $Y = K_2$

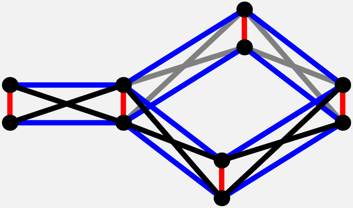
Has many  rectangles, and each rectangle has two diagonals 



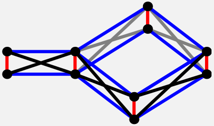
Definition (strong product $X \boxtimes Y$)

$X \square Y$ + diagonals of all  rectangles.



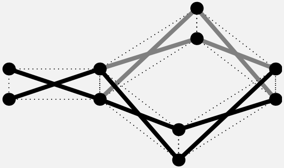


$X \boxtimes Y = X \square Y$ + diagonals of all rectangles



Definition (direct product $X \times Y$)

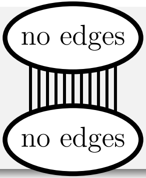
only has the diagonals



$$(x_1, y_1) \overset{X\times Y}{\text{——}} (x_2, y_2) \quad \Leftrightarrow \quad x_1 \overset{X}{\text{——}} x_2 \ \textbf{and} \ y_1 \overset{Y}{\text{——}} y_2$$

Note. $X \times K_2$ is **bipartite**.

Canonical bipartite double cover of X.



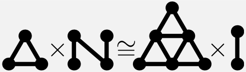
Exercise

Choose a graph product ($\square, \boxtimes, \times$) and call it $*$.
Show that every (finite) graph X has a prime decomposition for $*$:

- $X \cong X_1 * X_2 * \cdots * X_n$.
- No X_i can be written as $Y_1 * Y_2$ (with Y_1, Y_2 smaller than X_i).

Theorem (Sabidussi-Vizing 1960/1963, Dörfler-Imrich 1970)

Assume X connected. *(There is a path of edges from any vertex to any other vertex.)*
Then the prime decomposition is unique for $\square$ and $\boxtimes$.
(up to permutation of the factors and isomorphism)

Fact. Prime decomposition is **not** unique for $\times$: 

Rem. Prime decomp is not unique for $\square$ if graphs not connected:

$$(1 + x + x^2)(1 + x^3) = (1 + x^2 + x^4)(1 + x) \text{ in } \mathbb{Z}^+[x]$$

is a non-unique prime factorization.

Let $x = K_2$ (a graph). (+ is disjoint union and $x^n = x \square x \square \cdots \square x$)

$\square, \boxtimes, \times$ are natural graph-theoretic constructions:

$$X \overset{\alpha}{\cong} X',\ Y \overset{\beta}{\cong} Y' \quad \Rightarrow \quad X * Y \overset{\alpha\times\beta}{\cong} X' * Y'.$$

So $\quad \operatorname{Aut} X \times \operatorname{Aut} Y \subseteq \operatorname{Aut}(X * Y)$.

Exercise

$\operatorname{Aut} X \times \operatorname{Aut} Y = \operatorname{Aut}(X * Y) \quad \Rightarrow \quad X$ relatively prime to Y for $*$.

Theorem (Sabidussi-Vizing 1960/1963)

Converse is true for $\square$. *(if X and Y are connected)*

Also for $\boxtimes$, but need an additional technical condition.

Bad news

Converse is **not** true for $\times$:

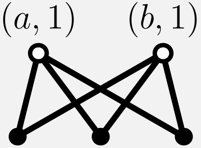
we do not understand $\operatorname{Aut}(X \times Y)$, even if $Y = K_2 =$ •—•.

Defn. X is ***stable*** if $\operatorname{Aut}(X \times K_2) = \operatorname{Aut} X \times \operatorname{Aut} K_2$.

Exercise (an obvious cause of instability)

$\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}\, X \times \mathrm{Aut}\, K_2$ if X has "twin" vertices. even if connected

Hint: Assume neighbours of a = neighbours of b. ("twins")
Then $(a,1)$ and $(b,1)$ are twins in $X \times K_2$.
There is an automorphism that interchanges $(a,1)$ and $(b,1)$, but fixes all other vertices.



Converse is not true. (*Lots* of counterexamples that are connected.)

Theorem (Fernandez-Hujdurović, 2020^+)

Converse is true if X is "circulant" graph with odd number of vertices.

Generalization (Morris, 2020^+)

X can be a "Cayley graph" on an abelian group of odd order.

(**Defn.** *Circulant graph* = Cayley graph on a cyclic group.)

Remark (Hujdurović-Mitrović, 2020^+)

Cannot delete "abelian." (Computer found counterexample with 21 vertices.)

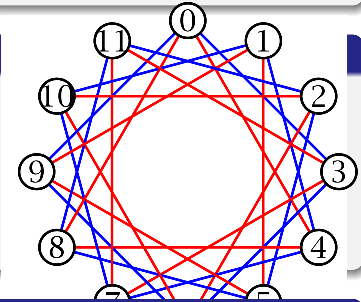
Thm. If X is a Cayley graph on an abelian group of odd order, then
$$\operatorname{Aut}(X \times K_2) = \operatorname{Aut} X \times \operatorname{Aut} K_2.$$ (Assume X is connected and twin-free.)

For any abelian group G, and $S \subseteq G \setminus \{0\}$: $\exists$ *Cayley graph* $\operatorname{Cay}(G; S)$.

Example

$\operatorname{Cay}(\mathbb{Z}_{12}; \{3, 4\})$
($\mathbb{Z}_{12}$ cyclic: this is a *circulant* graph.)

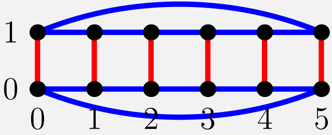
vertices: elements of $\mathbb{Z}_{12}$
edges: v — $v \pm 3$ & v — $v \pm 4$



Example

$\operatorname{Cay}(\mathbb{Z}_6 \times \mathbb{Z}_2; \{(1,0), (0,1)\})$.

vertices: elements of $\mathbb{Z}_6 \times \mathbb{Z}_2$
edges: v — $v \pm (1,0)$ & v — $v \pm (0,1)$



Theorem (Morris 2020^+)

*$X = \mathrm{Cay}(G;S)$ with G abelian of **odd order*** *(connected, twin-free)*
$\Rightarrow \quad \mathrm{Aut}(X \times K_2) = \mathrm{Aut}\,X \times \mathrm{Aut}\,K_2$.

Lemma (will prove later)

$X = \mathrm{Cay}(G;S)$ with G abelian. Assume
$\forall s_1, s_2 \in \pm S: \quad s_1 \neq s_2 \implies 2s_1 \neq 2s_2$.
Then $\mathrm{Aut}\,\mathrm{Cay}(G;S) \subseteq \mathrm{Aut}\,\mathrm{Cay}(G;\mathbf{2}S)$ *where $2S = \{\, 2s \mid s \in S \,\}$.*

Proof of Theorem. $X \times K_2 = \mathrm{Cay}\big(G \times \mathbb{Z}_2;\ S \times \{1\}\big)$. (can take as definition)
$2(s_1,1) = 2(s_2,1) \Rightarrow (2s_1,0) = (2s_2,0) \Rightarrow 2s_1 = 2s_2 \Rightarrow s_1 = s_2$.

$$
\begin{aligned}
&\mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ S \times \{1\}\big)\\
&\quad \subseteq \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ \mathbf{2}(S \times \{1\})\big)\\
&\quad = \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ 2S \times \{0\}\big)\\
&\quad \subseteq \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ \mathbf{2^k}S \times \{0\}\big)\\
&\quad = \mathrm{Aut}\,\mathrm{Cay}\big(G \times \mathbb{Z}_2;\ S \times \{0\}\big) \qquad (\text{choose } 2^k \equiv 1 \pmod{|G|})
\end{aligned}
$$

So restriction to bottom layer is in $\mathrm{Aut}\,X$: $\alpha(x,0) = (\varphi(x),0)$.
Since there are no twins: $\alpha(x,1) = (\varphi(x),1)$. (Exercise) □

Lemma

$X = \mathrm{Cay}(G;S)$ *with G abelian. Assume*

$\forall s_1, s_2 \in \pm S : \quad s_1 \neq s_2 \implies 2s_1 \neq 2s_2.$

Then $\mathrm{Aut}\,\mathrm{Cay}(G;S) \subseteq \mathrm{Aut}\,\mathrm{Cay}(G;2S)$ *where $2S = \{\, 2s \mid s \in S \,\}$.*

Proof. Let $\#_2(x,y) = \#$ paths of length 2 from x to y.

Edge: $x \text{---} x + s$ (with $s \in \pm S$).

Path of length 2: $x \text{---} x + s_1 \text{---} x + s_1 + s_2$ (with $s_1, s_2 \in \pm S$).

$\{$paths of length 2 from x to $y\}$ $\leftrightarrow$ $\{(s_1, s_2) \mid x + s_1 + s_2 = y\}$.

These come in pairs (s_1, s_2) and (s_2, s_1) unless $s_1 = s_2$: $y = x + 2s$.

Note: s is **unique** (if it exists) because $s_1 \neq s_2 \implies 2s_1 \neq 2s_2$

So $\#_2(x,y)$ is odd $\iff$ $x \overset{2S}{\text{---}} y$.

Any automorphism of $\mathrm{Cay}(G;S)$ must preserve $\#_2$

and must therefore preserve the edges in $\mathrm{Cay}(G;2S)$. □

Remark

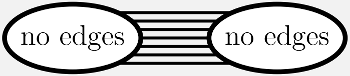
Can replace 2 with any $k \in \mathbb{Z}^+$, but proof is a bit more complicated.

Bad news

We do not understand $\mathrm{Aut}(X \times Y)$, even if $Y = K_2 =$ .

Good news

The problem only arises for graphs that are *bipartite.*



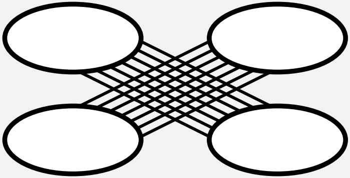
Theorem (Dörfler 1974)

$\mathrm{Aut}(X \times Y) = \mathrm{Aut}\,X \times \mathrm{Aut}\,Y$ *if X and Y are connected, twin-free, and **not** bipartite* *and X is $\times$-coprime to Y.*

Exercise

Assume X and Y are bipartite
(and have more than one vertex).

1. Show $X \times Y$ is not connected.
2. Show $\mathrm{Aut}(X \times Y) \neq \mathrm{Aut}\,X \times \mathrm{Aut}\,Y$ if $\mathrm{Aut}\,X$ and $\mathrm{Aut}\,Y$ are nontrivial.



- **both** X and Y not bipartite: **good**
- **both** X and Y bipartite: **bad**

Open case: X is not bipartite and Y is bipartite.

The simplest nontrivial bipartite graph is K_2.

That is one reason why it is important to study $\operatorname{Aut}(X \times K_2)$.

(Another reason: $X \times K_2$ is the canonical double cover.)

But it is not just **a** special case — it is the **main** case:

Proposition (classical?)

Assume $\operatorname{Aut}(X \times K_2) = \operatorname{Aut} X \times \operatorname{Aut} K_2$. (*and* X *is not bipartite*)

Then $\operatorname{Aut}(X \times Y) = \operatorname{Aut} X \times \operatorname{Aut} Y$

if X *is coprime to* Y *in an appropriate sense.*

Eg., If X and Y are abelian Cayley graphs, then suffices to assume

$$\gcd\big(|V(X)|, |V(Y)|\big) = 1.$$

References

Products of graphs

R. Hammack, W. Imrich, and S. Klavžar:
Handbook of Product Graphs, 2nd ed.
CRC Press, Boca Raton, FL, 2011.
MR 2817074, https://www.routledge.com/9781138199088

Wikipedia: Bipartite double cover.
https://en.wikipedia.org/wiki/Bipartite_double_cover

$\mathrm{Aut}(X \times K_2)$ when X is a circulant graph

S. Wilson,
Unexpected symmetries in unstable graphs.
J. Combin. Theory Ser. B 98 (2008), no. 2, 359–383.
MR 2389604, doi:10.1016/j.jctb.2007.08.001

Y.-L. Qin, B. Xia, and S. Zhou:
Stability of circulant graphs,
J. Combin. Theory Ser. B 136 (2019) 154–169.
MR 3926283, doi:10.1016/j.jctb.2018.10.004

B. Fernandez and A. Hujdurović:
Canonical double covers of circulants
(preprint, 2020). `https://arxiv.org/abs/2006.12826`

D. W. Morris,
Stability of Cayley graphs on abelian groups of odd order
(preprint, 2020). `https://arxiv.org/abs/2010.05285`