On interactions of amenability with left-orderings

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Amenability is a fundamental notion in group theory, as evidenced by the fact that it can be defined in more than a dozen different ways. A few of these different definitions will be discussed, together with some commentary on the theorem that left-orderable amenable groups are locally indicable, and perhaps some speculation on other ways that amenability might be useful in the theory of left-orderings.

Motivation

Theorem (Rhemtulla, Chiswell, Kropholler, Linnell, Morris)

- *G left-orderable group*
- *G solvable* (amenable)

$\implies$ *G has a Conradian left-order.* (G is locally indicable)

(In fact, $\exists$ recurrent left-order.)

Proof shows:

- G solvable (amenable)
- $\prec$ any left-order on G

$\implies \exists g_1, g_2, \ldots,$ s.t. $\prec^{g_n} \to$ Conradian (recurrent).

Corollary

- *G left-orderable,*
- *H solvable* (amenable) *subgroup of G* (do not assume normal or convex)

$\implies$ *$\exists$ left-order on G, s.t. restriction to H is Conradian* (in fact, recurrent).

Proof. $\exists$ left-order on G: $\prec$ restriction to H: $\prec\!\!\!\cdot$

$\exists h_1, h_2, \ldots,$ $\prec\!\!\!\cdot^{h_n} \to$ Conradian $\prec\!\!\!\cdot^{\infty}$.

$\mathrm{LO}(G)$ compact $\implies$ (pass to subseq) $\prec^{h_n} \to \prec^{\infty}$.

Restriction of $\prec^{\infty}$ to H is $\prec\!\!\!\cdot^{\infty}$ (Conradian).

Corollary

- *G left-orderable*
- *H nilpotent subgrp of G*

$\implies$ *$\exists$ left-order on G, s.t. restriction to H is bi-inv't.*

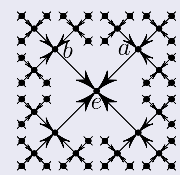
¿ Obvious from classical (algebraic) methods ?

What is *amenable* ?

Example

Every element of the free group F_2 starts with \$1:

$f_0(g) = 1, \quad \forall g \in F_2.$



Everyone passes their dollar to the person next to them who is closer to the identity:

$f_1(g) = \$3$ (except $f_1(e) = \$5$).

Everyone richer, & money only moved bdd distance.

Definition

A *Ponzi scheme* on G is a function $\rho \colon G \to G$, s.t.:

- $\forall g \in G,\ \#\rho^{-1}(g) \geq 2$ (everyone got richer)
- $\exists R,\ \forall g \in G,\ d(g, \rho(g)) \leq R$ (money moves bdd dist)

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Exercise

On $\mathbb{Z}^n$, $\nexists$ Ponzi scheme.

($\exists$ Ponzi scheme $\implies$ exponential growth.)

Solvable grps of exp'l growth do *not* have a Ponzi:

Theorem (Gromov)

$\exists$ Ponzi scheme on G $\iff$ G is not "amenable".

What is amenable really ?

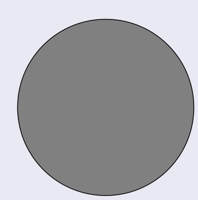
Answer

G is amenable $\iff$ G has *almost-invariant* subsets.

Example

G = abelian group (f.g.) $= \mathbb{Z}^2 = \langle a, b \rangle$.

G acts on itself by left translation.



F = G-inv't subset of G, ($aF = F$, $bF = F$, nonempty)

$\implies F$ is infinite.

$\nexists$ finite, invariant subset.

F = big ball $\implies F$ is 99.99% invariant ("almost inv't"):

$\#(F \cap aF) > (1 - \epsilon)\, \#F$

Definition

F is *almost invariant* (F is a "Følner set"):

$\#(F \cap aF) > (1-\epsilon)\#F \quad \forall a \in S$

Definition

G *amenable* $\iff$ G has almost-inv't finite subsets ($\forall$ finite S, $\forall \epsilon > 0$)

Exercise

Free group F_2 is not amenable.

Idea. $\frac{3}{4}$ of F does not start with a^{-1}.

$\implies \frac{3}{4}$ of aF starts with a.

$\implies \frac{3}{4}$ of baF starts with b.

$aF \approx F \approx baF \implies \approx\frac{3}{4}$ of F starts with a and b. $\to\leftarrow$

Proposition

G amen $\iff$ every bdd func on G has an avg value.
I.e., $\exists$ $A\colon \ell^\infty(G) \to \mathbb{R}$, s.t.

- $A(1) = 1$,
- $A(a\varphi + b\psi) = a\,A(\varphi) + b\,A(\psi)$,
- $A(\,\geq 0\,) \geq 0$,
- $A(\varphi^g) = A(\varphi)$. *(translation invariant)*

Proof.

Choose sequence of almost-inv't sets F_n $(\epsilon = 1/n)$.

Let $A_n(\varphi) = \frac{1}{\#F_n}\sum_{x \in F_n} \varphi(x)$.

Pass to subsequence, so $A_n(\varphi) \to A(\varphi)$.

Can make a consistent choice of $A(\varphi)$ for all φ.

[Axiom of Choice (Zorn's Lemma, Hahn-Banach, Ultrafilter, Tychonoff)] □

Corollary ($\iff$)

- *G amenable,*
- *G acts on compact metric space X* *(by homeos)*

$\implies$ *every continuous function on X has an avg val*

$\implies$ *$\exists$ G-inv't probability measure μ on X.* $(\mu(X) = 1)$

This defn proves LO + amenable $\implies$ $\exists$ Conradian.

Problem

Find a proof that uses a different definition.
(Leads to generalization? other applications?)

Average vals of characteristic funcs of subsets of G:

Corollary (von Neumann's original definition)

G amen $\iff$ $\exists$ finitely additive probability measure.

Other definitions of amenability

Notation

G f.g. $\implies$ $\exists \phi\colon F_n \twoheadrightarrow G$.

Let $B_r = \{\,$words of length $\leq r\,\}$ in F_n.

(Note: $\#B_r \approx (2n-1)^r$.)

Example

$G = F_n \implies \#(B_r \cap \ker\phi) = 1 < (\#B_r)^\epsilon$.

$G = \mathbb{Z}^n \implies \#(B_r \cap \ker\phi) \approx \dfrac{\#B_r}{(2r+1)^n} = (\#B_r)^{1-\epsilon}$.

Theorem (R. I Grigorchuk, J. M. Cohen)

G amenable $\iff$ $\#(B_r \cap \ker\phi) \geq (\#B_r)^{1-\epsilon}$.

I.e., amenable groups have maximal cogrowth.

Bounded cohomology

Define group cohomology as usual, except that all cochains are assumed to be bounded functions.

Theorem (B. E. Johnson)

G amenable

$\iff H^n_{\text{bdd}}(G;V) = 0$, $\forall$ *G-module V* $\left(\text{such that } V \text{ is a Banach space}\right)$.

Proof of $(\Rightarrow)$. If G is finite, and $|G|$ is invertible, one proves $H^n(G;V) = 0$ by averaging:

$\overline{\alpha}(g_1,\ldots,g_n) = \frac{1}{|G|}\sum_{g \in G} \alpha(g, g_1, \ldots, g_n)$.

Since G is amenable, we can do exactly this kind of averaging for any *bounded* cocycle.

(added after the talk)

*I thank the BIRS workshop participants for being such a great audience! The many comments and questions during the talk were very stimulating. Among other things, it was pointed out to me that the corollary near the start about bi-invariant restrictions to a nilpotent subgroup H is valid more generally, for **locally** nilpotent subgroups.*