Flows that are sums of hamiltonian cycles in abelian Cayley graphs

Joy Morris
University of Lethbridge
morris@cs.uleth.ca

David Petrie Moulton
Center for Communications Research
moulton@idaccr.org

Dave Witte*
Oklahoma State University
dwitte@math.okstate.edu
http://www.math.okstate.edu/~dwitte

Abstract

If X is any connected Cayley graph on any finite abelian group, we determine precisely which flows on X can be written as a sum of hamiltonian cycles. In particular, if the degree of X is at least 5, and X has an even number of vertices, then it is precisely the even flows, that is, the flows f, such that $\sum_{\alpha \in E(X)} f(\alpha)$ is divisible by 2. On the other hand, there are infinitely many examples of degree 4 in which not all even flows can be written as a sum of hamiltonian cycles. Analogous results were already known 10 years ago, from work of Brian Alspach, Stephen Locke, and Dave Witte, for the case where X is cubic, or has an odd number of vertices.

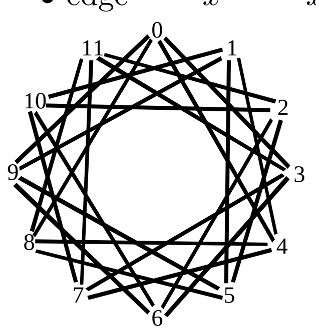
References

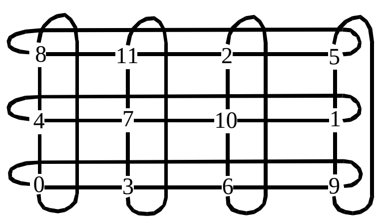
B. Alspach, S. C. Locke, and D. Witte:
The Hamilton spaces of Cayley graphs on abelian groups,
Discrete Math. 82 (1990) 113–126.

S. C. Locke and D. Witte:
Flows in circulant graphs of odd order are sums of Hamilton cycles,
Discrete Math. 78 (1989) 105–114.

Circulant graph. $\mathrm{Circ}\big(12; \{3,4\}\big)$:

- $V(X) = \mathbb{Z}_{12}$
- edge x —— $x \pm s$ for $s \in \{3,4\}$.





Rem. $\mathrm{Circ}\big(12; \{3,4\}\big) \cong C_4 \square C_3$.

Abelian Cayley grf. Replace $\mathbb{Z}_{12}$ with abel grp G.

Eg. $\mathrm{Cay}\big(\mathbb{Z}_4 \oplus \mathbb{Z}_3; \{(1,0),(0,1)\}\big)$
$\cong C_4 \square C_3$.

Always assume Cayley graphs are connected.
I.e., S generates G.

Recall. Any flow in any graph is a sum of cycles.

Thm (Alspach, Locke, Witte, 1990).
Every flow in any abelian Cayley graph X
= sum of hamiltonian cycles
(unless $X \cong C_{\mathrm{odd}} \square K_2$*)*
(mod 2).

Thm (Locke, Witte, 1989).
Every flow in any abelian Cayley graph X
= sum of hamiltonian cycles
if X has odd order
(unless $X \cong K_3 \square K_3$*).*

Locke-Witte also settled the cubic graphs.

Remains: graphs of even order, with degree ≥ 4.

X has even order $\Rightarrow$ every ham cyc is **even** flow,
i.e., $\displaystyle\sum_{e \in E(X)} f(e)$ is even.

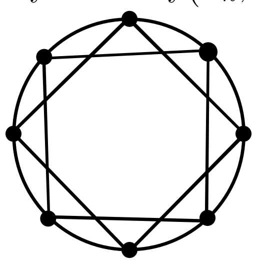
So sum of ham cycs is always an even flow.

Thm (Morris, Moulton, Witte).
Even flow in any abel Cayley grf X of even order
$=$ sum of hamiltonian cycles
if $\deg X \geq 5$.

Thm (M, M, W). *X = abel Cay grf of deg 4.*
Every even flow is sum of ham cycles unless

- *$4 \nmid |V(X)|$ and X is not bipartite; or*
- *X = square of cycle.*

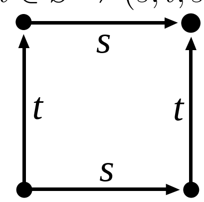
Defn. Square of cycle $= \mathrm{Cay}\big(\mathbb{Z}_n; \{1,2\}\big)$



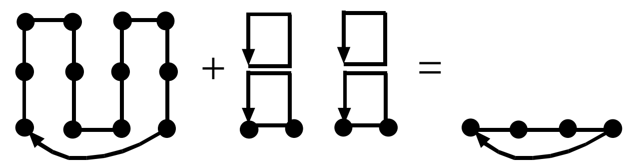
Rem. We are still working on the converse.

- $\mathcal{E}$ = {even flows on X}
- $\mathcal{H}$ = {sums of hamiltonian cycles}

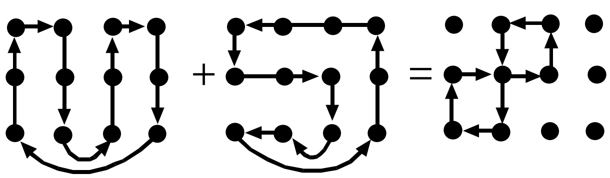
Basic 4-cycle. $s,t \in S \Rightarrow (s,t,s^{-1},t^{-1}) \in \mathcal{F}$



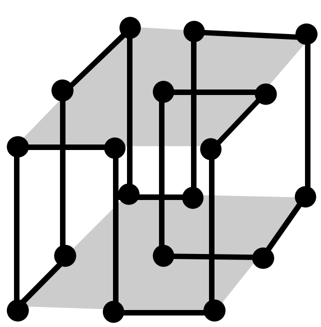
Lem. *If every basic 4-cycle is in $\mathcal{H}$,*
then $\mathcal{E} \subset \mathcal{H}$.



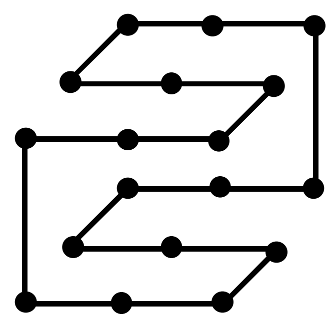
Cor. *If $\mathcal{H} \ni$ sum of odd number of basic 4-cycles,*
then every basic 4-cycle is in $\mathcal{H}$,
so $\mathcal{E} \subset \mathcal{H}$.



Prop. *Some ham cyc in $P_3 \,\square\, P_3 \,\square\, P_2$*
is the sum of an odd number of 4-cycles



Prop. *Some ham cyc in $P_3 \,\square\, P_3 \,\square\, P_2$*
is the sum of an even number of 4-cycles



Cor. *If $P_3 \,\square\, P_3 \,\square\, P_2 \subset X$,*
then $\mathcal{E} \subset \mathcal{H}$.

Cor. *If* $\deg X \geq 5$, *then $\mathcal{E} \subset \mathcal{H}$.*