Some arithmetic groups that cannot act on the circle

Dave Witte

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078

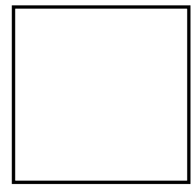
`dwitte@math.okstate.edu`

Notes of this talk are available at
`http://www.math.okstate.edu/~dwitte`

Symmetries.

A *symmetry* (or *automorphism*) of an object is a structure-preserving permutation of the points in the object.

Eg. Square: rotational and reflective symmetry.



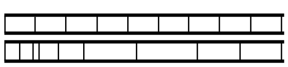
Henceforth, we **ignore** mirror symmetry:
only allow symmetries that *preserve orientation.*

A square has 4 symmetries:

- quarter rotation (order 4)
- half rotation (order 2)
- 3/4 rotation (order 4)
- full rotation (order 1) "trivial"

Symmetries of a line segment.

A line segment has no symmetries
unless we allow distortion (stretching).



A line segment has infinitely many symmetries
(if we allow distortion).

$\mathrm{Symm}(L) = \{$symmetries of $L\}$
$=$ group of (or-pres) homeomorphisms of L.

Prop. *A line segment has no symmetry of finite order (except the trivial symmetry).*

I.e., $\mathrm{Symm}(L) \not\supset$ finite group.

Prop. *A line segment has no (nontrivial) symmetry of finite order.*

Proof. Let f be a symmetry of $[-1,1]$.
Some point is moved by f. It might as well be 0.

Suppose $f(0) < 0$.
Then $f\big(f(0)\big) < f(0)$.
($x < y \Rightarrow f(x) < f(y)$ since f pres orient'n)
So $f^2(0) < 0$.

Then $f\big(f^2(0)\big) < f(0)$. I.e., $f^3(0) < f(0)$.
Hence $f^3(0) < 0$.

...

No power of f fixes 0.

No power of f is trivial, so f has infinite order. □

Defn. $\mathrm{SL}(2,\mathbb{Z}) = \left\{ A \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \;\middle|\; \det A = 1 \right\}$.

Cor. $\mathrm{SL}(2,\mathbb{Z})$ *does not act on a line segment.*

I.e., $\mathrm{Symm}(L) \not\supset$ subgroup isomorphic to $\mathrm{SL}(2,\mathbb{Z})$.

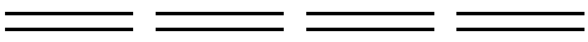
I.e., $\nexists$ homomorphism $\phi\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Symm}(L)$ such that $a \neq b \Rightarrow \phi(a) \neq \phi(b)$.

Proof. $\exists$ element a of finite order in $\mathrm{SL}(2,\mathbb{Z})$.

Eg. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a would have to act via a symm of finite order. □

Rem. $\mathrm{SL}(2,\mathbb{Z})$ acts on a finite union of line segs.



$\mathrm{SL}(2,\mathbb{Z})$ *has a free subgroup of finite index.*

Since $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(3,\mathbb{Z})$ it follows that $\mathrm{SL}(3,\mathbb{Z})$ cannot act on a line segment. $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thm. $\mathrm{SL}(3,\mathbb{Z})$ *cannot act on a finite union of line segments.*

Recall proof that $\nexists$ symmetry of finite order.

"Suppose $f(0)$ is to the left of 0."

For any symmetry f of L, we say that

- f is **positive** if $f(0) > 0$
- f is **negative** if $f(0) < 0$

(If $f(0) = 0$, but f is not trivial, it is harder to decide whether f should be positive or negative.)

Prop. *Every nontrivial symmetry of a line segment is either positive or negative (and not both).*

Rem. f, g pos $\Rightarrow$ fg pos & f^{-1} neg.

Defn. $f < g$ if $f^{-1}g$ is positive.

Rem. $\forall f, g, h \in \mathrm{Symm}(L)$, $f < g \Rightarrow hf < hg$.

Prop. $\mathrm{Symm}(L)$ *is* **left orderable**.

Corollary. Any group that acts on a line segment must be left orderable.

Thm. $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *is not left orderable.*

(Nor are subgroups of finite index.)

Impossible to do:

- $\Gamma = X \sqcup \{e\} \sqcup X^{-1}$,
- $X \cdot X \subset X$.

Cor. $\mathrm{SL}(3,\mathbb{Z})$ *cannot act on a line segment (or finite union).*

Thm. $\mathrm{SL}(3,\mathbb{Z})$ *cannot act on a line seg (or $\cup$).*

Cor. $\mathrm{SL}(3,\mathbb{Z})$ *cannot act on a circle (or $\cup$).*

Exer. $\mathrm{SL}(3,\mathbb{Z})$ **can** act on a sphere.

Open Question. Can $\mathrm{SL}(3,\mathbb{Z})$ act on a plane? (or a finite union?)

Open Question. Can $\mathrm{SL}(3,\mathbb{Z})$ act on a torus? (or a finite union?)

Open Question. Can $\mathrm{SL}(4,\mathbb{Z})$ act on a sphere? (or a finite union?)

Eg. Let $Q(x,y,z,w) = 2x^2 + y^2 - z^2 - w^2$ (or other "quadratic form").

$\Gamma = \{ A \in \mathrm{SL}(4,\mathbb{Z}) \mid \forall v \in \mathbb{R}^4,\ Q(Av) = Q(v) \}$
$=$ "arithmetic group"

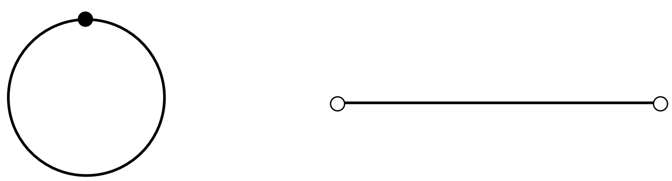
Ques. *Can Γ act on a finite union of line segs? or a plane, torus, or sphere?*

The relation between lines and circles.

(Let us be sloppy about finite unions.)

Γ = arithmetic group (with $\mathbb{R}$-rank ≥ 2).

Thm (Ghys). $\phi\colon \Gamma \hookrightarrow \text{Symm(circle)}$
$\Rightarrow$ Γ *has fixed point.*
(*or a finite orbit*).



Cor. $\phi\colon \Gamma \hookrightarrow \text{Symm(circle)}$
$\Rightarrow \exists\ \phi'\colon \Gamma \hookrightarrow \text{Symm(line seg)}$.

Cor. *No action on line segment* ($or \cup$)
$\Rightarrow$ *no action on circle* ($or \cup$).

Thm. $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ *is* **not** *left orderable.*

Proof. H = Heisenberg grp = $\begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{bmatrix}$.

$a = \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix},$

$c = \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}.$

$ac = ca, \qquad bc = cb.$

$ab = bac$
$\Rightarrow a^p b^q = b^q a^p c^{pq}$.

$e = \mathrm{Id} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}.$

Lem. $\forall$ *left-ordering of* H,
$\exists s \in \{a,b\},\ c \ll |s|, \quad$ *i.e.*, $\forall n \in \mathbb{Z},\ c^n < |s|$.

Proof. Wolog $a, b, c > e$.
(Replace a, b, c with inverse.)
(Interchange a and b: $[b,a] = c^{-1}$.)

Suppose $c^p > a$ and $c^q > b$.

Then $e > a^{-1}$, b^{-1}, $c^{-p}a$, $c^{-q}b$.

So $e > a^{-n}b^{-n}(c^{-p}a)^n(c^{-q}b)^n$
$= a^{-n}b^{-n}a^n b^n c^{-(p+q)n}$
$= (a^{-n}b^{-n})(b^n a^n) c^{n^2} c^{-(p+q)n}$
$= c^{n^2-(p+q)n}$
$= c^{\text{positive}}$
$> e$. $\to\leftarrow$

H = Heisenberg grp = $\begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{bmatrix}$.

$a(x,y,z) = (x+y, y, z),$
$b(x,y,z) = (x, y+z, z),$
$c(x,y,z) = (x+z, y, z).$

Permuting the coordinates (e.g., $x \leftrightarrow y$)
yields another Heisenberg group:
$a'(x,y,z) = (x, y+x, z),$
$b'(x,y,z) = (x+z, y, z),$
$c'(x,y,z) = (x, y+z, z).$

$H' = \begin{bmatrix} 1 & & \mathbb{Z} \\ \mathbb{Z} & 1 & \mathbb{Z} \\ & & 1 \end{bmatrix}.$

This yields 6 Heisenberg groups in $\mathrm{SL}(3,\mathbb{Z})$.

Suppose $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ is left ordered.

$$\mathrm{SL}(3,\mathbb{Z}): \qquad \begin{bmatrix} * & 1 & 2 \\ 4 & * & 3 \\ 5 & 6 & * \end{bmatrix}$$

There are 6 Heisenberg groups in Γ:

1, 2, 3, 2, 3, 4, 3, 4, 5
4, 5, 6, 5, 6, 1, 6, 1, 2.

1, 2, 3 = Heisenberg group $\Rightarrow 2 \ll 1$ or $2 \ll 3$.
Wolog $2 \ll 3$.

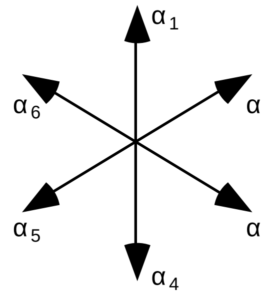
2, 3, 4 = Heisenberg group $\Rightarrow 3 \ll 2$ or $3 \ll 4$.
Must have $3 \ll 4$.

3, 4, 5 = Heisenberg group $\Rightarrow 4 \ll 3$ or $4 \ll 5$.
Must have $4 \ll 5$. etc.

$2 \ll 3 \ll 4 \ll 5 \ll 6 \ll 1 \ll 2$
$\Rightarrow 2 \ll 2$. $\rightarrow\leftarrow$

More conceptual version of the proof:

$\mathrm{SL}(3,\mathbb{Z})$:



$\alpha_6 + \alpha_2 = \alpha_1 \quad \Rightarrow \alpha_6, \alpha_1, \alpha_2$ Heisenberg
$\Rightarrow \alpha_1 < \alpha_6$ or $\alpha_1 < \alpha_2$.
Wolog $\alpha_1 < \alpha_2$.

$\alpha_1, \alpha_2, \alpha_3$ Heisenberg
$\Rightarrow \alpha_2 < \alpha_1$ or $\alpha_2 < \alpha_3$.
Must have $\alpha_2 < \alpha_3$.

Continuing: $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6 < \alpha_1$
$\Rightarrow \alpha_1 < \alpha_1 \quad \rightarrow\leftarrow$

References

É. Ghys: Actions de réseaux sur le cercle. *Invent. Math.* 137 (1999) 199–231.

É. Ghys: Groups acting on the circle. *L'Enseignement Mathématique* 47 (2001) 1-79.

A. Navas: Actions de groupes de Kazhdan sur le cercle (preprint). `http://www.umpa.ens-lyon.fr/~anavas`

V. Platonov and A. Rapinchuk: *Algebraic Groups and Number Theory.* Academic Press, New York, 1994

D. Witte: *Introduction to Arithmetic Groups* (in preparation). `http://www.math.okstate.edu/~dwitte`

D. Witte: Arithmetic groups of higher **Q**-rank cannot act on 1-manifolds, *Proc. Amer. Math. Soc.* 122 (1994) 333–340.

D. Witte and R. J. Zimmer: Actions of semisimple Lie groups on circle bundles, *Geometriae Dedicata* 87 (2001) 91–121.