ZERO-ENTROPY AFFINE MAPS ON HOMOGENEOUS SPACES

By Dave Witte

1. Introduction. If Γ is a closed subgroup of a connected Lie group G, then each element g of G acts by translation on the homogeneous space $\Gamma\backslash G$, namely $T_g\colon \Gamma s \mapsto \Gamma sg$. Generalizing theorems of W. Parry [16] and M. Ratner [18], this paper shows any measure-theoretic isomorphism of ergodic unipotent translations on finite-volume homogeneous spaces of connected Lie groups is "algebraic." It also presents a fairly successful attack on the isomorphism question for more general translations of zero entropy, but the results are not definitive.

Definition. Let Γ and Λ be closed subgroups of Lie groups G and H, and suppose $\sigma\colon G \to H$ is a group homomorphism with $\Gamma^\sigma \subset \Lambda$. For any $h \in H$, the map $T_{\sigma,h}\colon \Gamma\backslash G \to \Lambda\backslash H\colon \Gamma s \mapsto \Lambda s^\sigma h$ is said to be an *affine* map.

Remark. Any translation T_g on $\Gamma\backslash G$ is an affine map from $\Gamma\backslash G$ to itself (let σ be the identity map, and set $h = g$ in the definition).

Definition. A homogeneous space $\Gamma\backslash G$ is *faithful* if Γ contains no nontrivial normal subgroup of G. Since one can mod out any normal subgroup of G contained in Γ, it generally causes no real loss of generality to consider only faithful homogeneous spaces.

Definition 1.1. Recall that a matrix A is *unipotent* if it has no eigenvalue other than 1 (i.e., if $A - \mathrm{Id}$ is nilpotent, where Id is the identity matrix). An affine map $T_{g,\sigma}\colon \Gamma\backslash G \to \Gamma\backslash G$ of a homogeneous space onto itself is *unipotent* if the composition $D\sigma \circ \mathrm{Ad}g$ is a unipotent linear transformation on the Lie algebra of G. (Where $D\sigma$ is the derivative of σ at $e \in G$.)

Theorem 2.1′. *Suppose T_1 and T_2 are volume-preserving invertible ergodic unipotent affine maps on faithful finite-volume homogeneous*

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spaces $\Gamma\backslash G$ and $\Lambda\backslash H$ of connected Lie groups G and H. If $\psi: \Gamma\backslash G \to \Lambda\backslash H$ is a measure-preserving Borel map which conjugates T_1 to T_2 (i.e., if $T_1\psi = \psi T_2$), then ψ is an affine map (a.e.).

It is natural to try to extend Theorem 2.1$'$ to the class of zero-entropy translations. (An affine map $T_{g,\sigma}$ has zero entropy iff $|\lambda| = 1$ for all eigenvalues λ of $D\sigma \circ \mathrm{Ad} g$ (see Lemma 6.1), so any unipotent affine map has zero entropy.) But isomorphisms of general zero-entropy translations need not be affine maps. For example, as a part of his definitive study of translations on homogeneous spaces of solvable groups in the 1960's (see [1]), L. Auslander showed that a zero-entropy ergodic translation on a finite-volume homogeneous space of any solvable group is isomorphic to a translation on a homogeneous space of some nilpotent group. If the solvable group is not nilpotent, this isomorphism cannot be realized by an affine map, because the groups involved are not isomorphic. Analogously, we show (roughly) that any zero-entropy ergodic translation is finitely covered by a translation on a homogeneous space of a group whose radical is nilpotent. (The *radical* of a group is its largest connected solvable normal subgroup.) Essentially this means we can restrict attention to groups whose radical is nilpotent, but there is an unresolved bit of ambiguity resulting from the passage to a finite cover.

LEMMA 6.5$'$. *Let T be an ergodic volume-preserving zero-entropy invertible affine map on a finite-volume homogeneous space $\Gamma\backslash G$ of a connected Lie group G. Then there is an ergodic volume-preserving invertible zero-entropy affine map T' on a finite-volume homogeneous space $\Gamma'\backslash G'$ of a connected Lie group G' whose radical is nilpotent, such that, for some nonzero $n \in \mathbf{Z}$, $(T', \Gamma'\backslash G')$ finitely covers $(T^n, \Gamma\backslash G)$. I.e., there is a finite-to-one measure-preserving Borel map $\psi': \Gamma'\backslash G' \to \Gamma\backslash G$ which conjugates T' to T^n.*

Because of the finite cover introduced in Lemma 6.5$'$, it is convenient to study a generalized notion of conjugacy.

Definition. An ergodic $T_1 \times T_2$-invariant measure ψ on $S_1 \times S_2$ is an *ergodic joining* of two ergodic measure preserving dynamical systems (T_1, S_1, μ_1) and (T_2, S_2, μ_2) if ψ projects to the measure μ_1 on S_1 and to μ_2 on S_2.

Definition. Let ψ be an ergodic joining of (T_1, S_1, μ_1) with (T_2, S_2, μ_2). It is well known (see [22]) that there is an essentially unique family $\{\psi_s : s \in S_1\}$ of measures on S_2 such that, for any measurable $A \subset S_1 \times S_2$,

$\psi(A) = \int_{S_1} \psi_s(A \cap (\{s\} \times S_2)) d\mu_1(s)$. We say ψ has *finite fibers over* S_1 if the support of a.e. ψ_s is a finite set.

Remark. If $\psi: (T_1, S_1, \mu_1) \to (T_2, S_2, \mu_2)$ is a measure preserving conjugacy, then the graph of ψ supports an ergodic joining of (T_1, S_1, μ_1) with (T_2, S_2, μ_2) having finite fibers over S_1.

Definition. Suppose $\psi: \Gamma\backslash G \to \Lambda\backslash H$ is affine, and let $K \subset H$ be a compact subgroup of H. Suppose there is a continuous group homomorphism $G \to K: x \mapsto \dot{x}$ with $\dot{\Gamma} = e$ (this happens whenever there is an ergodic translation on $\Gamma\backslash G$ which is not weak-mixing). If ψ is a conjugacy of T_g with T_h, and if K centralizes h, then the map $\dot{\psi}: \Gamma\backslash G \to \Lambda\backslash H: \Gamma s \mapsto \Gamma s\psi \cdot \dot{s}$ is a conjugacy of T_g with $T_{\dot{h}h}$, but it is usually not an affine map. We say $\dot{\psi}$ is a *twisted affine map*. One can similarly define *twisted affine joinings* (see Theorem 6.7).

THEOREM 6.7′. *Suppose T_1 and T_2 are ergodic zero-entropy affine maps on finite-volume homogeneous spaces $\Gamma\backslash G$ and $\Lambda\backslash H$ of connected Lie groups G and H, and let ψ be an ergodic joining of $(T_1, \Gamma\backslash G)$ with $(T_2, \Lambda\backslash H)$ with finite fibers over $\Gamma\backslash G$. Assume* rad G *and* rad H *are nilpotent. Then ψ is a twisted affine joining.*

In combination with Lemma 6.5′, Theorem 6.7′ determines whether two ergodic zero-entropy translations have a joining with finite fibers. This would settle the isomorphism question if one could cross the gap between maps and joinings, but the author has no serious ideas on how to do so.

We have seen in Theorem 2.1′ that for unipotent translations there is no need to introduce finite covers or twists. The same is true for weak-mixing translations.

COROLLARY 6.8′. *Suppose T_1 and T_2 are ergodic volume-preserving zero-entropy invertible affine maps on finite-volume homogeneous spaces $\Gamma\backslash G$ and $\Lambda\backslash H$ of connected Lie groups G and H. Assume $(T_1, \Gamma\backslash G)$ is weak-mixing. Then any measure-preserving conjugacy $\psi: (T_1, \Gamma\backslash G) \to (T_2, \Lambda\backslash H)$ is an affine map (a.e.).*

This paper does not consider translations of nonzero entropy because the situation there is quite different. For example, D. Ornstein and B. Weiss [15] proved all translations of nonzero entropy on finite-volume homogeneous spaces of $SL_2(\mathbf{R})$ are isomorphic to Bernoulli shifts. (See [6] for an extension of this result.) The isomorphism question for translations of nonzero entropy remains open. In particular, which translations are Bernoulli?

Overview. Section 2 presents the statement and proof of the main theorem (2.1) on joinings of unipotent affine maps. The following sections (Sections 3, 4, 5) supply some details which were omitted from Section 2. Finally, Section 6 shows how to derive the results on zero-entropy translations from the main theorem.

Acknowledgments. Theorem 2.1′ is the main theorem of my Ph.D. thesis (University of Chicago, 1985). This research was greatly aided by discussions with L. Auslander, S. G. Dani, C. C. Moore, M. Ratner, and my thesis advisor, Robert J. Zimmer. I am indebted to G. Bergman for Theorem 3.13, which is a great simplification of my original treatment, to N. Wallach for Lemma 3.14, which I had been unable to prove, and to S. G. Dani for pointing out several deficiencies in my original manuscript, including a serious error in the statement and proof of Theorem 3.16. The work was supported by a Sloan Doctoral Dissertation Fellowship and an NSF Postdoctoral Fellowship.

2. The main theorem. See Section 1 and Section 3A for definitions used here.

THEOREM 2.1. *Suppose T_1 and T_2 are ergodic volume-preserving invertible unipotent affine maps on faithful finite-volume homogeneous spaces $\Gamma\backslash G$ and $\Lambda\backslash H$ of connected Lie groups G and H, and let ψ be any ergodic joining of $(T_1, \Gamma\backslash G)$ with $(T_2, \Lambda\backslash H)$ having finite fibers over $\Gamma\backslash G$. Then ψ is an affine joining. I.e., there is a finite cover G' of G, a lattice Γ' in G', and a measure-preserving affine map $\phi: \Gamma'\backslash G' \to \Lambda\backslash H$ such that, under the natural map $\Gamma'\backslash G' \times \Lambda\backslash H \to \Gamma\backslash G \times \Lambda\backslash H$,* the joining on *$\Gamma'\backslash G' \times \Lambda\backslash H$ associated to ϕ projects to ψ.*

No new ergodic-theoretic ideas are needed in this paper. All the necessary techniques were developed by Ratner [18, 19, 20, 21] and were used in the author's previous work [24]. Where possible, instead of reproducing these arguments we refer the reader to Ratner's original work or the author's reformulation of it. Proofs (including most of the algebra) which might not be routine even for the expert familiar with [18, 19, 20] and [24] are presented in later sections.

Assumption 2.2. We give the proof of the main theorem (2.1) only for translations, because this provides for some simplification of the notation. Remarks 4.16 and 5.1 indicate why affine maps are not much more difficult to treat. Furthermore, we deal only with the case of a map $\psi: \Gamma\backslash G$

$\to \Lambda\backslash H$ instead of the more general joinings with finite fibers. This entails enormous notational (and conceptual) simplification. For an illustration of Ratner's idea which extends these arguments to the general case, see [19, Lemmas 4.2 and 4.4] or [24, Lemmas 7.7 and 7.8]. (See also [20].)

Definition 2.3. Suppose $g \in G$, $h \in H$, and $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is measure preserving. We say ψ is *affine for g* (via h) if, for a.e. $s \in \Gamma\backslash G$, we have $sg\psi = s\psi h$. Note that h is uniquely determined by g if $\Lambda\backslash H$ is faithful. In note of this we often write $h = \tilde{g}$. When X is some subset of G, we often say ψ is *affine for X* to indicate that ψ is affine for every element of X. If $\Lambda\backslash H$ is faithful, it is not hard to see that ψ is an affine map (a.e.) if and only if ψ is affine for G.

It is sometimes convenient to assume G is simply connected. Because this may conflict with the assumption that $\Gamma\backslash G$ is faithful, it is useful to introduce a weaker notion.

Definition 2.4. $\Gamma\backslash G$ is *locally faithful* if Γ contains no *connected* nontrivial normal subgroup of G.

Remark. Previous authors [1, 4] used the term *presentation* to refer to a locally faithful homogeneous space, but this doesn't seem to be a very descriptive word.

The main theorem can be restated in this language.

THEOREM 2.5. *Let u and $\tilde{u}$ be ergodic unipotent translations on locally faithful finite-volume homogeneous spaces $\Gamma\backslash G$ and $\Lambda\backslash H$ of connected Lie groups G and H, and assume $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is a measure-preserving Borel map. If ψ is affine for u via $\tilde{u}$, then ψ is affine for G.*

The proof of the theorem begins with a technical result establishing that $\Gamma\backslash G$ and $\Lambda\backslash H$ are very nice homogeneous spaces.

PROPOSITION 2.6 (see Propositions 4.23 and 4.19). *Because $\Gamma\backslash G$ supports an ergodic unipotent translation, we have:*

1. Γ *is discrete;*
2. rad G *is nilpotent (so G is locally algebraic);*
3. Γ *intersects each of $Z(G)$ and* rad G *in a lattice; and*
4. Γ *projects densely into the maximal compact semisimple factor of G.*

Analogous results hold for H and Λ. ∎

Assumptions. For simplicity, we assume $\Gamma\backslash G$ and $\Lambda\backslash H$ are compact. The general case requires no new ideas not present in Ratner's original work [18]. Furthermore, we avoid technical complications by pretending G and H are real algebraic groups. Since we know they are in fact locally algebraic it is not hard to modify these arguments for the general case.

The following lemma presents one consequence of the "polynomial divergence of orbits."

LEMMA 2.7. *Let I be an interval on the real line. For any $\epsilon, \theta > 0$, there is $\delta > 0$ satisfying: if $s, t \in \Gamma\backslash G$ are such that $\{r \in I \mid d(su^r, tu^r) < \delta\}$ has relative measure at least θ on I, then $d(su^r, tu^r) < \epsilon$ for all $r \in I$.*

Sketch of proof (cf. [18, Lemma 2.1] or Step 2 of the proof of [24, Lemma 3.1]). Since G is algebraic, any one-parameter algebraically unipotent subgroup of G is an algebraic subgroup. So, for any $x, y \in G$, the distance between the two points xu^r and yu^r is a polynomial function of r, whose degree is bounded by some constant D independent of x and y. Because the u-flow on G covers that on $\Gamma\backslash G$, this implies the distance between the two points su^r and tu^r on $\Gamma\backslash G$ is "locally" a polynomial. I.e., there is some $\tau > 0$ such that if I_0 is an interval on the real line with $d(su^r, tu^r) < \tau$ for all $r \in I$, then $d(su^r, tu^r)$ is a polynomial function (of degree $\leq D$) when restricted to $r \in I_0$. There is no loss in shrinking τ or ϵ so that $\tau = \epsilon$. Given ϵ, D, θ, we can choose $\delta > 0$ such that if f is any polynomial of degree $\leq D$ satisfying $|f(r)| < \delta$ for $r \in [0, \theta/D]$, then $|f(r)| < \epsilon/2$ for $r \in [-1, 1]$.

Let I_0 be a subinterval of I with $d(su^r, tu^r) < \epsilon$ for all $r \in I_0$. Supposing (for a contradiction) that $d(su^r, tu^r)$ is not bounded by ϵ on I, we may choose I_0 so that $d(su^r, tu^r) = \epsilon$ at one endpoint of I_0. Furthermore, we may assume the relative measure on I_0 of the set of r with $d(su^r, tu^r) < \delta$ is at least θ. Since $d(su^r, tu^r)$ is a polynomial of degree $\leq D$ on I_0, the set $\{r \in I_0 \mid d(su^r, tu^r) < \delta\}$ has at most D components. One of them (call it I_0') has relative measure at least θ/D on I_0. Since $d(su^r, tu^r) < \delta$ on I_0', the choice of δ then implies $d(su^r, tu^r) < \epsilon/2$ on all of I_0—contradicting the assumption that $d(su^r, tu^r) = \epsilon$ at one endpoint. ■

PROPOSITION 2.8. *ψ maps $C_G(u)^\circ$-orbits into $C_H(\tilde{u})^\circ$-orbits (where $C_G(u)$ is the centralizer of u in G).*

Sketch of proof (cf. [18, Lemma 3.2] or Step 2 of the proof of [24, Lemma 3.1]). Let g be some small element of $C_G(u)$. For any $s \in \Gamma\backslash G$, the orbits of s and sg are parallel and close together. If ψ were uniformly continuous, it would follow immediately that the orbits of $s\psi$ and $sg\psi$ were close together forever—that they were parallel. This would mean $sg\psi = s\psi g^s$ for some small $g^s \in C_H(\tilde{u})$, as desired.

The problem, of course, is that ψ is not known *a priori* to be continuous. On the other hand, Lusin's Theorem asserts ψ is uniformly continuous on a large subset of $\Gamma\backslash G$. Therefore $s\psi$ spends, say, 99% of its life very close to $sg\psi$. Polynomial divergence of orbits (see 2.7) implies then that the two points are *always* close. ■

COROLLARY 2.9. *Suppose U is a closed unipotent subgroup of G containing u. If v is affine for U, then ψ is affine for $N_G(U)^\circ$.*

Sketch of proof (see Section 5A). We wish to show ψ is affine for any $g \in N_G(U)^\circ$. To avoid having to consider commutators $[g, u^r]$, assume for simplicity that $g \in C_G(u)^\circ$. For a.e. $s \in \Gamma\backslash G$, the proposition implies there is some g^s • $C_H(\tilde{u})$ with $sg\psi = s\psi g^s$. We wish to show g^s is essentially independent of s. Since u commutes with g and $\tilde{u}$ commutes with g^s for all $s \in G$, one can easily check that $s\psi g^s = s\psi g^{su}$ for all $s \in G$. Since g^s is actually unique, we conclude that $g^s = g^{su}$. Since u is ergodic, this implies g^s is constant (independent of s) as desired. ■

COROLLARY 2.10. *ψ is affine for the identity component P° of a parabolic subgroup of G.*

Proof. Let U be a maximal connected unipotent subgroup of G (and assume $u \in U$). Since U is nilpotent, if one starts with any subgroup of U, forms its normalizer in U, then forms the normalizer of this normalizer, and so on, one eventually reaches U itself. Hence repeated application of the previous corollary shows ψ is affine for U. Then the preceding corollary asserts ψ is affine for the identity component of $P = N_G(U)$. This is a parabolic subgroup of G. □

PROPOSITION 2.11 (see Section 5B). *We may assume $\Gamma\backslash G$ has no solvmanifold quotient. Furthermore, if X is any connected subgroup of G which does not project to an* Ad*-precompact subgroup of $G/$*rad *G, then X is ergodic on $\Gamma\backslash G$.* □

Notation 2.12. Let S be some subgroup of G which is (locally) isomorphic to $SL_2(\mathbf{R})$, and let U_1, A and V_1 be the subgroups of S corresponding to the group of upper-triangular unipotent matrices, the group of diagonal matrices, and the group of unipotent lower triangular matrices in $SL_2(\mathbf{R})$. Assume P° contains U_1 and A.

PROPOSITION 2.13 (cf. [24, Lemma 5.5]). *The subgroups V_1 as constructed in* 2.12, *together with P°, suffice to generate G. We therefore need only show ψ is affine for each such V_1.* □

To get control on V_1, we use the commutation relations satisfied by A with regard to U_1 and V_1. To carry out the argument, we need exactly the same relations to hold in H as hold in G. Thus we would like to show the homomorphism $P^\circ \to H: g \mapsto \tilde{g}$ extends to an isomorphism of G with H, but we will settle for something slightly weaker. The key step in identifying G with H is showing that ψ is (more-or-less) a one-to-one map.

Remark 2.14. By modding out its kernel, we may assume the homomorphism $\sim : P^\circ \to H$ is one-to-one (see Lemma 5.11).

LEMMA 2.15. *Every fiber of ψ is finite* (*a.e.*).

Sketch of proof (cf. [20, Lemma 3.1] or [24, Lemma 7.4]). This is a consequence of Ratner's "H-property" (see [20, Definition 1] or [24, Theorem 6.1]), which is a manifestation of the fact that unipotent translations are "shearing" transformations. Consider two points $s, t \bullet \Gamma\backslash G$ that are close together and suppose their orbits under U_1 are not parallel (i.e., there is no small $c \in C_G(U_1)$ with $s = tc$). Then the two points wander apart. The shearing property means that s and t move apart much faster in the direction of the $C_G(U_1)$-orbits than in other directions. Therefore there is some c in the unit sphere of $C_G(U_1)$ such that s passes near tc as the points wander apart. Thus, letting u^r ($r \in \mathbf{R}$) be a parametrization of U_1, for some r we have $su^r \doteq tcu^r$.

Now suppose s and t belong to the same fiber of ψ (and ψ is uniformly continuous). Then $su^r\psi \doteq tcu^r\psi = t\psi\tilde{c}\tilde{u}^r = su^r\psi\tilde{c}$. This implies $\tilde{c}$ has a fixed point, which contradicts the fact that we may assume no nonidentity element of $C_H(\tilde{U}_1)$ has a fixed point. We conclude that no two points in the same fiber of ψ are close together. Consequently, the fibers of ψ are finite, as desired.

We use polynomial divergence to eliminate the hypothesis that ψ is uniformly continuous. Namely, polynomial divergence implies the points wander apart slowly, so not only does s pass near tc, but it spends a long time near tc. (Where "a long time" is proportional to the length of time required for s to approach tc.) By the Pointwise Ergodic Theorem, then at some time when s is close to tc, both of these points are in the set of uniform continuity for ψ. So the argument of the previous paragraph applies. ■

PROPOSITION 2.16. *We may assume $G = H$, and $\psi: \Gamma\backslash G \to \Lambda\backslash G$ commutes with U. Furthermore, $\tilde{a} \in a \cdot Z(G)$ for all $a \in A$.*

Sketch of proof (see 5.12). For simplicity of exposition let us assume ψ is one-to-one (i.e., invertible) instead of just finite-to-one. (In the general case, ψ^{-1} is a joining with finite fibers over $\Lambda\backslash H$.) Then by interchanging the roles of G and H in Corollary 2.10 we conclude that ψ^{-1} is affine for the identity component of a parabolic subgroup Q of H. The image $\tilde{P}^\circ$ of P° contains Q°, so it is the identity component of a parabolic subgroup of H. Thus we may assume $\tilde{P}^\circ = Q^\circ$. And, Remark 2.14 shows we may assume the homomorphism ($\sim$) is one-to-one, i.e., it is an isomorphism of P° with Q°. A bit of work (using Proposition 2.11) shows $\widetilde{\operatorname{rad} G} = \operatorname{rad} H$, and then Lie theory (Theorem 3.16) shows there is a (local) isomorphism $\sigma: G \to H$ with $\sigma = \sim$ on U, and $\tilde{p} \in p\sigma \cdot Z(H)$ for all $p \in P^\circ$. If we identify G with H under σ, then the desired conclusions hold. ■

LEMMA 2.17 ("Commutation relations," see Lemma 3.20). *Let $A^+ \subset A$ be the subsemigroup of expanding automorphisms of U_1 (so A^+ contracts V_1, i.e., $a^{-1}va \to e$ as $a \to \infty$ in A^+).*

1. *For $s \in \Gamma\backslash G$ and $v \in V_1$, we have $d(sa, sva) \to 0$ as $a \to \infty$ in A^+.*
2. *Given any compact neighborhood U_ϵ of e in U_1. For any sufficiently small $v \in V_1$, there is a map $U_1 \to U_1: u \mapsto \bar{u}$ such that $vua \doteq \bar{u}a$ and $vu\tilde{a} \doteq \bar{u}\tilde{a}$ for all $u \in U_\epsilon$ and all $a \in A^+$. Furthermore, the derivative of the map $u \mapsto \bar{u}$ is close to 1 for all $u \in U_\epsilon$.*
3. *There is a compact neighborhood U_ϵ of e in U_1 such that, if $s, t \in \Gamma\backslash G$ satisfy $sua \doteq tua$ for all $u \in U_\epsilon$ and all a in some unbounded subset of A^+, then there is some small $c \in C_G(A)$ with $s = tc$.* □

PROPOSITION 2.18. *ψ is affine for V_1.*

Sketch of proof (cf. [18, Lemma 3.4] or [24, Proposition 8.5]). Given $v \in V_1$. If ψ were uniformly continuous, for all $u \in U_\epsilon$ and all $a \in A^+$ we would have:

$$\Gamma s v\psi u\tilde{a} = \Gamma svua\psi \qquad \text{(since } \psi \text{ commutes with } U)$$

$$\doteq \Gamma s\bar{u}a\psi \qquad \text{(by 2 and because } \psi \text{ is uniformly continuous)}$$

$$= \Gamma s\psi\bar{u}\tilde{a} \qquad \text{(since } \psi \text{ commutes with } U\text{)}$$

$$\doteq \Gamma s\psi vu\tilde{a} \qquad \text{(by 2).}$$

By part 3 of the lemma, we conclude that $sv\psi = s\psi \cdot vc^s$ for some $c^s \in C_H(\tilde{A})$. Since $d(sa, sva) \to 0$ as $a \to \infty$ in A^+, we have

$$sa\psi \doteq sva\psi = s\psi vc^s\tilde{a} = s\psi v\tilde{a}c^s \doteq s\psi\tilde{a}c^s = sa\psi c^s.$$

Therefore $c^s = e$ as desired.

Unfortunately ψ is not known to be uniformly continuous, but from Lusin's theorem and the Pointwise Ergodic Theorem, we conclude that $\Gamma sv\psi u\tilde{a} \doteq \Gamma s\psi vu\tilde{a}$ for most $u \in U_\epsilon$, for each fixed $a \in A^+$. By Lemma 2.7, it follows that the approximation indeed holds for *all* $u \in U_\epsilon$. ■

This completes the proof of the main theorem.

3. Lie Theory.

3A. Definitions. Save explicit mention to the contrary, all Lie groups and Lie algebras are real, separable, and finite-dimensional.

Notation 3.1. For a closed subgroup X of a Lie group G, we use $C_G(X)$, $N_G(X)$, and X°, respectively, to denote the centralizer, normalizer, and identity component (in the Hausdorff topology) of X. We use $Z(G)$ to denote the center of G, i.e., $Z(G) = C_G(G)$. We use a corresponding script letter $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, . . . to denote the Lie algebra of a Lie group A, B, C, ... As is customary, we identify the Lie algebra of a subgroup of G with the corresponding subalgebra of $\mathcal{G}$.

Definition 3.2. Two Lie groups G and H are *locally isomorphic* if they have isomorphic Lie algebras or, equivalently, if the universal cover of G° is isomorphic to the universal cover of H° (cf. [23, Section 2.8, pp. 72–74]).

Definition 3.3. An element u of a Lie group G is *unipotent* if $\mathrm{Ad}_G u$ is a unipotent linear transformation on $\mathcal{G}$. A subgroup U of G is *unipotent* if each element of U is unipotent.

Caution 3.4. The theory of algebraic groups provides a notion of unipotence for elements of a real algebraic group. To avoid hopeless confusion with the preceding definition, we refer in this context to elements (or subgroups) as being *algebraically* unipotent. (In a real algebraic group, every algebraically unipotent element is unipotent, but the converse fails if $Z(G)$ is not algebraically unipotent.)

Remark 3.5. Any unipotent subgroup of a connected Lie group G is nilpotent (cf. Engel's Theorem [11, Section V.2, pp. 63–67]). As a partial converse, any nilpotent normal subgroup of G is unipotent.

Definition 3.6. For any Lie group G, we let rad G (the *radical* of G) be the largest connected solvable normal subgroup of G, and nil G (the *nilradical* of G) be the largest connected nilpotent normal subgroup of G. Obviously nil $G \subseteq$ rad G.

Definition 3.7. A Lie algebra $\mathcal{L}$ is *perfect* if it coincides with its derived algebra, i.e., if $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$. This is equivalent to the assertion that $\mathcal{L}$ has no nonzero abelian (or solvable) homomorphic images.

Definition 3.8. A *Borel* subalgebra of a complex Lie algebra $\mathcal{H}$ is a maximal solvable subalgebra, and any subalgebra containing a Borel subalgebra is said to be *parabolic*. A subalgebra $\mathcal{P}$ of a real Lie algebra $\mathcal{G}$ is *parabolic* if its complexification $\mathcal{P} \otimes \mathbf{C}$ is parabolic in $\mathcal{G} \otimes \mathbf{C}$. For a connected Lie group G, the normalizer $N_G(\mathcal{P})$ of any parabolic subalgebra of $\mathcal{G}$ is said to be a *parabolic subgroup* of G.

Definition 3.9. Suppose g is an element of a Lie group G. Then

$$\{x \in G \mid g^{-n}xg^{n} \to e \text{ as } n \to +\infty\}$$

is a subgroup of G, called the *horospherical subgroup* associated to g.

Remark 3.10. Any horospherical subgroup is unipotent. As a partial converse, any connected unipotent subgroup of a semisimple Lie group is contained in a horospherical subgroup.

Definition 3.11. A *real algebraic group* is a Lie group which is a subgroup of finite index in the real points of an (affine) algebraic group defined over $\mathbf{R}$. A Lie group is *locally algebraic* if it is locally isomorphic to some real algebraic group.

LEMMA 3.12. *Any Lie group G whose radical is nilpotent is locally isomorphic to an essentially unique connected real algebraic group whose radical and center are algebraically unipotent.*

Proof. The proof of [11, Theorem XVIII.1.1, p. 250] shows G is locally isomorphic to a connected real algebraic group whose radical and center are algebraically unipotent. It follows from (the proof of) [11, Theorem XVIII.2.2, p. 252] that a local isomorphism between any two such groups comes from an isomorphism of real algebraic groups. □

3B. Isomorphisms of parabolic subalgebras. An isomorphism of parabolic subalgebras may or may not extend to an isomorphism of the ambient Lie algebras. (Theorem 3.13(iii) shows that an extension, if it exists, is unique.) The author has previously shown there is always an extension if the ambient Lie algebras are semisimple (cf. 3.15). We now give a criterion (Corollary 3.17) for the existence of an extension, under the weaker assumption that the ambient Lie algebras have trivial center and that their radicals are nilpotent.

THEOREM 3.13 (Bergman [2]). *If $\mathcal{P}$ is a parabolic subalgebra of a real or complex Lie algebra $\mathcal{G}$, then:*

(i) *For any finite-dimensional $\mathcal{G}$-module V, we have $H^0(\mathcal{P}, V) = H^0(\mathcal{G}, V)$. In particular, $Z(\mathcal{P}) = Z(\mathcal{G})$;*

(ii) *Suppose V and W are finite-dimensional representations of $\mathcal{G}$. If $\sigma\colon V \to W$ is a $\mathcal{P}$-module homomorphism, then σ is $\mathcal{G}$-equivariant; and*

(iii) *The inclusion of $\mathcal{P}$ in $\mathcal{G}$ is a category-theoretic epimorphism. In other words, if $\sigma, \tau\colon \mathcal{G} \to \mathcal{H}$ are Lie algebra homomorphisms with the same restriction to $\mathcal{P}$, i.e., if $\sigma|_{\mathcal{P}} = \tau|_{\mathcal{P}}$, then $\sigma = \tau$.*

Proof. (i) We may assume $\mathcal{G}$ is a complex Lie algebra and $\mathcal{P}$ is a Borel subalgebra. Letting $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$, note that $\mathcal{P} \cap \mathcal{L}$ is a Borel subalgebra of $\mathcal{L}$. Since $H^0(\mathcal{P}, V) = H^0(\mathcal{P} \cap \mathcal{L}, H^0(\text{rad}\, \mathcal{G}, V))$ and $H^0(\mathcal{G}, V) = H^0(\mathcal{L}, H^0(\text{rad}\, \mathcal{G}, V))$, we may assume $\mathcal{G} = \mathcal{L}$ is semisimple. In this case, Weyl's Theorem asserts V is completely reducible, so we may assume V is irreducible. Let $\mathfrak{I}$ be a Cartan subalgebra of $\mathcal{G}$ contained in $\mathcal{P}$. Then the Borel subalgebra $\mathcal{P}$ determines an ordering of the weights of $\mathcal{G}$ (w.r.t. $\mathfrak{I}$). Letting λ be the maximal weight of V, we have $H^0(\text{nil}\, \mathcal{P}, V) = V_\lambda$. If $H^0(\mathcal{P}, V) \neq 0$, we conclude that $\lambda = 0$, and hence V is the trivial $\mathcal{G}$-module.

(ii) The vector space $\text{Hom}(V, W)$ of linear transformations $V \to W$ is a $\mathcal{G}$-module in the usual way. Since σ is $\mathcal{P}$-equivariant, we have $\sigma \in H^0(\mathcal{P}, \text{Hom}(V, W))$. Hence (i) implies $\sigma \in H^0(\mathcal{G}, \text{Hom}(V, W))$ as desired.

(iii) Let V be a faithful finite-dimensional representation of $\mathcal{H}$. Our two homomorphisms of $\mathcal{G}$ into $\mathcal{H}$ give us two ways to view V as a $\mathcal{G}$-module. By assumption, the identity map on V is an isomorphism of these as $\mathcal{P}$-modules. Hence (ii) implies the identity map is $\mathcal{G}$-equivariant. Since V is a faithful $\mathcal{H}$-module, it follows that $\sigma = \tau$. □

LEMMA 3.14 (Wallach). *If $\mathcal{B}$ is a parabolic subalgebra of a real or complex semisimple Lie algebra $\mathcal{L}$, and if V is a finite-dimensional $\mathcal{L}$-module with no trivial submodules, then $H^1(\mathcal{B}, V) = 0$.*

Proof. Weyl's Theorem asserts that any $\mathcal{L}$-module is completely reducible, so we may assume V is irreducible. Now apply the Hochschild-Serre spectral sequence [10, Exercise VIII.9.3, p. 305] to the Langlands decomposition $\mathcal{B} = \mathcal{M} + \mathcal{N}$ of $\mathcal{B}$ (where $\mathcal{N} = \text{nil}\,\mathcal{B}$) to determine $H^1(\mathcal{B}, V)$. There are two relevant groups in the E_2 term: $E_2^{0,1} = H^0(\mathcal{M}, H^1(\mathcal{N}, V))$ and $E_2^{1,0} = H^1(\mathcal{M}, H^0(\mathcal{N}, V))$. It suffices to show both of these groups vanish, for then the spectral sequence immediately yields $H^1(\mathcal{B}, V) = 0$.

The $\mathcal{M}$-module structure of $H^1(\mathcal{N}, V)$ is known [13, Theorem 5.14]. In particular, the highest weight α of any irreducible $\mathcal{M}$-submodule of $H^1(\mathcal{N}, V)$ is of the form $\alpha = (\lambda + \delta)w_\beta - \delta$, where δ is one-half the sum of the positive roots, λ is the highest weight of V, and w_β is the reflection corresponding to a simple root β. It follows from this that 0 is not the highest weight of any $\mathcal{M}$-submodule. In other words, $H^0(\mathcal{M}, H^1(\mathcal{N}, V)) = 0$, as desired.

The reductive algebra $\mathcal{M}$ is a direct sum $\mathcal{M} = \mathcal{S} \oplus \mathcal{A}$ of a semisimple and an abelian Lie algebra. The Hochschild-Serre spectral sequence can be applied to this decomposition to show $H^1(\mathcal{M}, H^0(\mathcal{N}, V)) = 0$. There are two relevant groups in the E_2 term:

$$E_2^{0,1} = H^0(\mathcal{A}, H^1(\mathcal{S}, H^0(\mathcal{N}, V))) \quad \text{and} \quad E_2^{1,0} = H^1(\mathcal{A}, H^0(\mathcal{S}, H^0(\mathcal{N}, V))).$$

Since Whitehead's Lemma asserts $H^1(\mathcal{S}, \cdot) = 0$, only the latter of these two groups matters. Note that $H^0(\mathcal{S}, H^0(\mathcal{N}, V)) = H^0(\mathcal{S} + \mathcal{N}, V)$.

Lemma 3.13(i) implies $H^0(\mathcal{B}, V) = 0$, so $H^0(\mathcal{S} + \mathcal{N}, V)$ has no trivial $\mathcal{A}$-submodules. Since $\mathcal{A} \subseteq \mathcal{M}$ is reductive in $\mathcal{L}$, this implies $H^0(\mathcal{S} + \mathcal{N}, V)$ is a direct sum of nontrivial irreducible $\mathcal{A}$-modules. Since $\mathcal{A}$ is abelian, it is then easy to see $H^1(\mathcal{A}, H^0(\mathcal{S} + \mathcal{N}, V)) = 0$. □

LEMMA 3.15 [24, Proposition 5.7 and Lemma 5.8]. *Suppose $\mathcal{G}$ and $\mathcal{H}$ are semisimple real or complex Lie algebras. Let $\mathcal{P}$ and $\mathcal{Q}$ be parabolic subalgebras of $\mathcal{G}$ and $\mathcal{H}$ respectively, and let σ: $\mathcal{P} \to \mathcal{Q}$ be a Lie algebra* *isomorphism. Then* σ *extends to an isomorphism of Lie algebras* $\mathcal{G} \cong \mathcal{H}$. □

THEOREM 3.16. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be parabolic subalgebras of real or complex Lie algebras* $\mathcal{G}$ *and* $\mathcal{H}$ *respectively, and assume* rad $\mathcal{G}$ *is nilpotent. If* $\sigma: \mathcal{P} \to \mathcal{Q}$ *is a Lie algebra isomorphism such that* $(\text{rad } \mathcal{G})\sigma = \text{rad } \mathcal{H}$, *then there is an isomorphism* $\pi: \mathcal{G} \to \mathcal{H}$, *such that* π *and* σ *agree on* rad $\mathcal{G}$ $+ [\mathcal{P}, \mathcal{P}]$, *and* $p\pi \in p\sigma + Z(\mathcal{H})$ *for each* $p \in \mathcal{P}$.

Proof. Let $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$, and set $\mathcal{B} = \mathcal{L} \cap \mathcal{P}$, a parabolic subalgebra of $\mathcal{L}$.

Step 1. If rad $\mathcal{H}$ is abelian, then there is a Levi subalgebra $\mathcal{M}$ of $\mathcal{H}$ with $\mathcal{B}\sigma \subset \mathcal{M} + C_{\text{rad}\,\mathcal{H}}(\mathcal{M})$.

Proof. Let $\mathcal{M}$ be a Levi subalgebra of $\mathcal{H}$, and let $\mathcal{B}' = \mathcal{M} \cap \mathcal{Q}$. There is no loss in assuming $C_{\text{rad}\,\mathcal{H}}(\mathcal{M}) = 0$. (Because rad $\mathcal{H}$ is abelian, $C_{\text{rad}\,\mathcal{H}}(\mathcal{M}) = Z(\mathcal{H})$ is an ideal, so it may be modded out. Because all $\mathcal{M}$-modules are completely reducible, then $\mathcal{M}$ has no centralizer in the quotient.) I.e., the $\mathcal{M}$-module rad $\mathcal{H}$ has no trivial submodules. Hence Lemma 3.14 asserts $H^1(\mathcal{B}', W) = 0$ for every $\mathcal{M}$-submodule W of rad $\mathcal{H}$, so the usual argument that Levi subalgebras are conjugate shows any two subalgebras complementary to rad $\mathcal{H}$ in $\mathcal{Q}$ are conjugate (cf. proof of [23, Theorem 3.14.2, p. 227]). In particular, $\mathcal{B}\sigma$ is conjugate to $\mathcal{B}'$. Replacing $\mathcal{M}$ by a conjugate if necessary, then we may assume $\mathcal{B}\sigma = \mathcal{B}'$ is contained in $\mathcal{M}$. Thus the desired conclusion holds.

Step 2. In any case, there is a Levi subalgebra $\mathcal{M}$ of $\mathcal{H}$ with $\mathcal{B}\sigma \subset \mathcal{M} + C_{\text{rad}\,\mathcal{H}}(\mathcal{M})$.

Proof. By Step 1 (applied to $\mathcal{G}/[\text{rad } \mathcal{G}, \text{rad } \mathcal{G}]$ and $\mathcal{H}/[\text{rad } \mathcal{H}, \text{rad } \mathcal{H}]$), there is a Levi subalgebra $\mathcal{M}$ of $\mathcal{H}$ with $\mathcal{B}\sigma \subset \mathcal{M} + C_{\text{rad}\,\mathcal{H}}(\mathcal{M}) + [\text{rad } \mathcal{H}, \text{rad } \mathcal{H}] = \mathcal{H}_0$. We may assume $C_{\text{rad}\,\mathcal{H}}(\mathcal{M}) < \text{rad } \mathcal{H}$ (or else the desired conclusion is obvious). Since rad $\mathcal{H}$ is nilpotent, this implies $C_{\text{rad}\,\mathcal{H}}(\mathcal{M}) + [\text{rad } \mathcal{H}, \text{rad } \mathcal{H}] < \text{rad } \mathcal{H}$ (cf. [9, Corollary 10.33, p. 155]). Hence $\mathcal{H}_0$ is a proper subalgebra of $\mathcal{H}$. By induction on the dimension of $\mathcal{H}$, we conclude there is a Levi subalgebra $\mathcal{M}'$ of $\mathcal{H}_0$ with $\mathcal{B}\sigma \subset \mathcal{M}' + C_{\text{rad}\,\mathcal{H}_0}(\mathcal{M}')$. Note that $\mathcal{M}'$ is a Levi subalgebra of $\mathcal{H}$, and $\text{rad } \mathcal{H}_0 = \text{rad } \mathcal{H} \cap \mathcal{H}_0$ is contained in rad $\mathcal{H}$.

Step 3. There is a Levi subalgebra $\mathcal{M}$ of $\mathcal{H}$ with $\mathcal{B}\sigma \subset \mathcal{M} + Z(\mathcal{H})$.

Proof. Let $\mathcal{M}$ be as given by Step 2 and set $\mathcal{B}' = \mathcal{M} \cap \mathcal{Q}$. Then $\mathcal{B}\sigma$ is the graph of a homomorphism $\tau: \mathcal{B}' \to C_{\text{rad}\,\mathcal{H}}(\mathcal{M})$. We need only show

$\mathcal{B}'\tau \subset Z(\mathcal{H})$. Because we already know $\mathcal{B}\tau \subset C_{\mathrm{rad}\,\mathcal{H}}(\mathfrak{M})$, it is enough to show $\mathcal{B}'\tau \subset Z(\mathrm{rad}\;\mathcal{H})$. Now rad $\mathcal{H}$ is nilpotent, so $\mathcal{B}'/\ker\tau$ is nilpotent. Because $\mathcal{B}'$ is a parabolic subalgebra of a semisimple Lie algebra, its structure is such that this implies $[\mathcal{B}', \mathcal{B}'] \subset \ker\tau$. So, letting $\mathfrak{I}$ be a torus in $\mathcal{B}$ complementary to $[\mathcal{B}, \mathcal{B}]$, we need only show $\mathfrak{I}\sigma \subset \mathcal{B}' + Z(\mathrm{rad}\;\mathcal{H})$. Any element t of $\mathfrak{I}$ is ad-semisimple, so $\mathrm{ad}(t\sigma)$ must be ad-semisimple on rad $\mathcal{H}$. Write $t\sigma = t' + t'\tau$ (with $t' \in \mathcal{B}'$). Then $\mathrm{ad}(t\sigma) = \mathrm{ad}\,t' + \mathrm{ad}(t'\tau)$ is the Jordan decomposition of $\mathrm{ad}(t\sigma)$, because t' is ad-semisimple, $t'\tau \in \mathrm{rad}\;\mathcal{H}$ is ad-nilpotent, and $[t', t'\tau] = 0$. Since $\mathrm{ad}(t\sigma)$ is semisimple on rad $\mathcal{H}$, this implies $\mathrm{ad}(t'\tau)$ is 0 on rad $\mathcal{H}$. I.e., $t'\tau \in Z(\mathrm{rad}\;\mathcal{H})$ as desired.

Step 4. Conclusion.

Proof. Let $\mathfrak{M}$ be as given by Step 3, and let $\mathcal{B}' = \mathfrak{M} \cap \mathcal{Q}$. Then $\mathcal{B}\sigma \subset \mathcal{B}' + Z(\mathcal{H})$, indeed there is an isomorphism $\sigma' : \mathcal{B} \to \mathcal{B}'$ with $b\sigma \in b\sigma' + Z(\mathcal{H})$ for all $b \in \mathcal{B}$. (Note for future reference that σ and σ' agree on $[\mathcal{B}, \mathcal{B}]$.) Now Lemma 3.15 implies σ' extends to an isomorphism σ_1: $\mathcal{L} \to \mathfrak{M}$. Identify $\mathcal{L}$ with $\mathfrak{M}$ under σ_1, so the restriction of σ to rad $\mathcal{G}$ is $\mathcal{B}$-equivariant. Then Theorem 3.13(ii) asserts it is $\mathcal{L}$-equivariant. Therefore the linear map

$$\pi: \mathcal{G} \to \mathcal{H}: l + r \mapsto l\sigma_1 + r\sigma \qquad (l \in \mathcal{L}, r \in \mathrm{rad}\;\mathcal{G})$$

is a Lie algebra isomorphism. It obviously agrees with σ on rad $\mathcal{G}$, and, because σ and σ_1 agree on $[\mathcal{B}, \mathcal{B}]$, it also agrees with σ on $[\mathcal{B}, \mathcal{B}]$. Thus π agrees with σ on $\mathrm{rad}\;\mathcal{G} + [\mathcal{B}, \mathcal{B}] = \mathrm{rad}\;\mathcal{G} + [\mathcal{P}, \mathcal{P}]$. ■

COROLLARY 3.17. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are parabolic subalgebras of real or complex Lie algebras $\mathcal{G}$ and $\mathcal{H}$ respectively. If* $Z(\mathcal{H}) = 0$ *and* rad $\mathcal{H}$ *is nilpotent, then any isomorphism σ: $\mathcal{P} \to \mathcal{Q}$ with* $(\mathrm{rad}\;\mathcal{G})\sigma = \mathrm{rad}\;\mathcal{H}$ *extends to a Lie algebra isomorphism π: $\mathcal{G} \to \mathcal{H}$.* ■

Remark 3.18. The assumption that $(\mathrm{rad}\;\mathcal{G})\sigma = \mathrm{rad}\;\mathcal{H}$ cannot be omitted. For example, if $\mathcal{P}$ is a proper parabolic subalgebra of any Lie algebra $\mathcal{H}$, the isomorphism $\mathcal{P} \cong \mathcal{P}$ obviously does not extend to an isomorphism of $\mathcal{P}$ with $\mathcal{H}$.

3C. Miscellaneous technical results

LEMMA 3.19. *Let M be any finite-dimensional real or complex S-module, where S is a connected Lie group locally isomorphic to* $\mathrm{SL}_2(\mathbf{R})$. *Let A be a split Cartan subgroup of S, and choose a maximal unipotent sub-*

group U of S normalized by A. The selection of U corresponds to an ordering of the weights of S (w.r.t. A), and this determines a decomposition $M = M^- \oplus M^0 \oplus M^+$ of M into the direct sum of its negative, zero, and positive weight spaces. Then:

(a) $C_M(U) \subseteq M^0 + M^+$;

(b) *Any U-submodule of M contained in $M^- + M^0$ is contained in $C_M(S) \subseteq M^0$.*

Proof. Weyl's Theorem asserts M is completely reducible, so, by projecting to irreducible summands, we may assume M is irreducible. Any nonzero vector centralized by U is a maximal vector (cf. [12, Section 20.2, p. 108]). Since the highest weight of M is nonnegative, this proves (a).

Any U-submodule contains a maximal vector of M. If the submodule is contained in $M^- + M^0$, this implies the highest weight of M is 0. Because an irreducible S-module is determined by its highest weight, we conclude that M is trivial, and (b) follows. □

LEMMA 3.20. *Let Γ be a lattice in a Lie group G. Let U_1 and V_1 be one-parameter unipotent subgroups of G such that $S = \langle U_1, V_1 \rangle$ is locally isomorphic to $SL_2(\mathbf{R})$. Let $A = (N_S(U_1) \cap N_S(V_1))^\circ$, and let $A^+ \subset A$ be the subsemigroup of expanding automorphisms of U_1 (so A^+ contracts V_1, i.e., $a^{-1}va \to e$ as $a \to \infty$ in A^+).*

1. *For $s \in \Gamma\backslash G$ and $v \in V_1$, we have $d(sa, sva) \to 0$ as $a \to \infty$ in A^+.*
2. *Let U_ϵ be any compact neighborhood of e in U_1. For any sufficiently small $v \in V_1$, there is a map $U_1 \to U_1 : u \mapsto \bar{u}$ such that $vua \doteq \bar{u}a$ for all $u \in U_\epsilon$ and all $a \in A^+$. Furthermore, the derivative of the map $u \mapsto \bar{u}$ is close to 1 for all $u \in U_\epsilon$.*
3. *There is a compact neighborhood U_ϵ of e in U_1 such that, if $s, t \in \Gamma\backslash G$ satisfy $sua \doteq tua$ for all $u \in U_\epsilon$ and all a in some unbounded subset of A^+, then there is some small $c \in C_G(A)$ with $s = tc$.*

Proof. (1) is a consequence of the fact that A^+ contracts V_1. For (2), see [24, p. 20]. We now prove (3) (cf. Step 2 of the proof of [24, Lemma 7.8]).

The Lie algebra $\mathcal{G}$ is an S-module, and thus $\mathcal{G}$ splits into a direct sum $\mathcal{G}^- \oplus \mathcal{G}^0 \oplus \mathcal{G}^+$ of negative, zero, and positive weight spaces as in Lemma 3.19. Let $\mathcal{B}^-, \mathcal{B}^0, \mathcal{B}^+$ be compact convex neighborhoods of 0 in $\mathcal{G}^-, \mathcal{G}^0, \mathcal{G}^+$, respectively. For each a in some unbounded subset $[A^+]$ of A^+, there is some $y_a \in G$ with $t = sy_a$ and $(ua)^{-1}y_a ua \bullet \operatorname{Exp}(\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+)$ for all $u \bullet U_\epsilon$.

We may assume the neighborhoods are small enough that Exp is one-to-one on $\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+$. Thus there is a unique $\dot{y}_a \in \mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+$ with $\mathrm{Exp}(\dot{y}_a) = y_a$, and

$$\dot{y}_a.(\mathrm{Ad}u) \in [\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+]\cdot(\mathrm{Ad}a^{-1}) \subseteq \mathcal{G}^- + \mathcal{B}^0 + \mathcal{B}^+.$$

Therefore $\dot{y}_a.(\mathrm{Ad}u)\pi \in \mathcal{B}^0 + \mathcal{B}^+$, where π is projection onto $\mathcal{G}^0 + \mathcal{G}^+$. Since $C_{\mathcal{G}}(u) \subseteq \mathcal{G}^0 + \mathcal{G}^+$ (see Lemma 3.19(a)), the projection into $C_{\mathcal{G}}(u)$ is even smaller. We conclude from [24, Lemma 6.2] that if U_ϵ is large enough, then $u^{-1}y_a u \doteq e$ for $u \in U_\epsilon$.

Letting $u = e$, in particular we have $y_a \doteq e$ for all $a \in [A^+]$. Because $y = xy_a$ and no small element of G has a fixed point near s [6, Lemma 2.1], this implies $y_a = x$ is independent of a. Thus $\dot{y}.(\mathrm{Ad}u) \in [\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+].(\mathrm{Ad}a^{-1})$ for all $u \in U_\epsilon$ and all $a \in [A^+]$. Letting $a \to \infty$, we get $\dot{y}.(\mathrm{Ad}u) \in \mathcal{G}^- + \mathcal{B}^0$ for all $u \in U_\epsilon$. Hence $\dot{y}.(\mathrm{Ad}u) \in \mathcal{G}^- + \mathcal{G}^0$ for all $u \in U_1$ (since U_ϵ is Zariski dense in U_1). By the structure of $\mathrm{SL}_2(\mathbf{R})$-modules (see Lemma 3.19(b)), this implies $\dot{y} \in \mathcal{G}^0$. Therefore $x = \mathrm{Exp}(\dot{y}) \in C_G(A)^\circ$. ■

LEMMA 3.21. *For any Lie algebra* $\mathcal{G}$, *we have* $C_{\mathcal{G}}(\mathrm{rad}\ \mathcal{G}) \subseteq Z(\mathcal{G}) + [\mathcal{G}, \mathcal{G}]$.

Proof. Let $\mathcal{L}$ be a Levi subalgebra of G. Since $\mathcal{L}$ is semisimple, Weyl's Theorem asserts every $\mathcal{L}$-module is completely reducible. Hence we may write $C_{\mathcal{G}}(\mathrm{rad}\ \mathcal{G}) = \mathcal{Z} \oplus V$, where $\mathcal{Z} = Z(\mathcal{G})$ is the centralizer of $\mathcal{L}$ in $C_{\mathcal{G}}(\mathrm{rad}\ \mathcal{G})$, and V is a sum of nontrivial irreducible $\mathcal{L}$-modules, so $V = [\mathcal{L}, V] \subseteq [\mathcal{G}, \mathcal{G}]$. □

LEMMA 3.22. *If* $\mathcal{P}$ *is a parabolic subalgebra of a real or complex Lie algebra* $\mathcal{G}$, *then* $[\mathcal{G}, \mathcal{G}] \cap Z(\mathcal{G}) = [\mathcal{P}, \mathcal{P}] \cap Z(\mathcal{P})$.

Proof. Let $\bar{\mathcal{G}} = \mathcal{G}/[\mathrm{rad}\ \mathcal{G}, \mathrm{rad}\ \mathcal{G}]$. Then $\mathrm{rad}\ \bar{\mathcal{G}}$ is abelian, so $[\bar{\mathcal{G}}, \bar{\mathcal{G}}] = [\bar{\mathcal{G}}, \bar{\mathcal{L}}]$ for any Levi subalagebra $\bar{\mathcal{L}}$ of $\bar{\mathcal{G}}$. Since $[\bar{\mathcal{G}}, \bar{\mathcal{L}}]$ is the sum of the nontrivial irreducible $\bar{\mathcal{L}}$-submodules of $\bar{\mathcal{G}}$, then $\bar{\mathcal{L}}$ has trivial centralizer in $[\bar{\mathcal{G}}, \bar{\mathcal{G}}]$. Therefore $[\bar{\mathcal{G}}, \bar{\mathcal{G}}] \cap Z(\bar{\mathcal{G}}) = 0$, which implies $[\mathcal{G}, \mathcal{G}] \cap Z(\mathcal{G}) \subseteq [\mathrm{rad}\ \mathcal{G}, \mathrm{rad}\ \mathcal{G}]$. ■

LEMMA 3.23. *Let K be a closed subgroup of a connected real Lie group G whose radical is nilpotent. Assume there is a closed normal subgroup N of G contained in K such that K projects to an* Ad*-precompact subgroup of G/N, and that K is normalized by the identity component P° of a parabolic subgroup of G. Then K is a normal subgroup of G.*

Proof. Passing to a quotient of G, we may assume K contains no normal subgroup of G. This implies $K \cap Z(G) = e$, and K is Ad-precompact. Since $\mathrm{Ad}_G(K)$ (resp. $\mathrm{Ad}_G(\mathrm{rad}\, G)$) consists only of semisimple (resp. unipotent) elements, we have $K \cap \mathrm{rad}\, G = e$. Since K and $\mathrm{rad}\, G$ normalize each other, this implies $[K, \mathrm{rad}\, G] = e$.

Case 1. G is semisimple with trivial center. Note that G is (isomorphic to) a real algebraic group [25, Proposition 3.1.6, p. 35]. Since $Z(G) = e$ and K is a closed Ad-precompact subgroup, K is compact, and hence K is an algebraic subgroup of G (cf. [25, p. 40]). Thus K is a reductive algebraic subgroup of G. This means there is a Cartan involution (*) of G with $K^* = K$ [17, Section 2.6, p. 11]. Then $N_G(K) = N_G(K)^*$. Since $P^\circ \subseteq N_G(K)$, then $N_G(K) \supseteq \langle P^\circ, (P^\circ)^* \rangle = G$.

Case 2. K is connected. We wish to show $[\mathcal{G}, \mathcal{K}] \subseteq \mathcal{K}$. Since $\mathcal{K}$ is reductive in $\mathcal{G}$ and $[\mathcal{K}, \mathrm{rad}\, \mathcal{G}] = 0$, it suffices to show $[\mathcal{G}, \mathcal{K}] \subseteq \mathcal{K} + \mathrm{rad}\, \mathcal{G}$. Thus there is no loss in passing to the maximal semisimple quotient of G with trivial center. Then Case 1 applies.

Case 3. The general case. Case 2 implies K° is normal in G, which implies K is discrete. Hence, showing K is normal is equivalent to showing K is central in G. We already know K centralizes $\mathrm{rad}\, \mathcal{G}$. Since K is reductive, then we need only show K centralizes $\mathcal{G}/\mathrm{rad}\, \mathcal{G}$. Thus we may assume G is semisimple (with trivial center), and Case 1 applies. ■

LEMMA 3.24. *Any faithful lattice* Γ *in a connected Lie group* G *has a torsion-free subgroup of finite index.*

Proof. Γ is finitely generated [17, Remark 6.18, pp. 99–100], so $\mathrm{Ad}_G\Gamma$ is a finitely generated subgroup of the linear group $\mathrm{Ad}G$, and hence $\mathrm{Ad}_G\Gamma$ has a torsion-free subgroup of finite index [17, Theorem 6.11, p. 93]. Since Γ is faithful, we have $\Gamma \cong \mathrm{Ad}_G\Gamma$. ■

4. Finite-volume homogeneous spaces. This section presents a number of technical results which generalize the structure theorems proved by Auslander [1] and others for finite-volume homogeneous spaces of solvable groups. They are more-or-less known, but not in the generality required in this paper. Our development closely follows the presentation of Brezin and Moore [4].

Theorem 4.12 is perhaps the only new result of independent interest. Roughly speaking, this theorem (which generalizes a fundamental theorem of Auslander, see 4.11) shows any finite-volume homogeneous space decomposes into a solvable and a semisimple part.

Added in proof. This result is not new: see [Ta-Sun Wu, Products of subgroups in Lie groups, *Illinois J. Math.* **29** (1985) 687–695].

4A. Dani's generalization of the Borel Density Theorem. The Borel Density Theorem [17, Theorem 5.26(vi), pp. 87–88] asserts that if Γ is a lattice in connected semisimple real algebraic group with no compact factors, then Γ is Zariski dense in G. We adopt the following result of S. G. Dani as an analogue of this theorem which is valid for arbitrary groups G.

THEOREM 4.1 (Dani [8, Corollary 2.6]). *Suppose Γ^* is an algebraic subgroup of a real algebraic group G^*, and let ν be a finite measure on $\Gamma^*\backslash G^*$. Set*

$$G_\nu^* = \{g \in G^* \mid \text{the } g\text{-action on } \Gamma^*\backslash G^* \text{ preserves } \nu\}$$

and

$$N_\nu^* = \{g \in G^* \mid sg = s \text{ for all } s \in \operatorname{supp} \nu\}.$$

Then $G_\nu{}^*$ and N_ν^* are algebraic subgroups of G^*, and N_ν^* is a cocompact normal subgroup of G_ν^*. □

COROLLARY 4.2. *Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a Lie group G, and $\sigma\colon G \to \mathrm{GL}_n(\mathbf{R})$ is a finite-dimensional representation of G. Let G^* and Γ^* be the Zariski closures of $G\sigma$ and $\Gamma\sigma$, respectively, in $\mathrm{GL}_n(\mathbf{R})$. Then $\Gamma^*\backslash G^*$ has finite volume. Hence Γ^* contains a cocompact normal algebraic subgroup of G^*. In particular, Γ^* contains every algebraically unipotent element of G^*.*

Proof. The G-invariant probability measure μ on $\Gamma\backslash G$ pushes to a $G\sigma$-invariant measure $\nu = \mu\sigma_*$ on $\Gamma^*\backslash G^*$. Since $G\sigma$ is Zariski dense, and because G_ν^* is algebraic, we must have $G_\nu^* = G^*$. Hence ν is a finite G^*-invariant measure on $\Gamma^*\backslash G^*$.

The support of the G^*-invariant probability measure ν is obviously all of $\Gamma^*\backslash G^*$; hence $N_\nu^* \subseteq \Gamma^*$. Any compact real algebraic group (e.g., $N_\nu^*\backslash G_\nu^*$) has no algebraically unipotent elements. Since $G_\nu^* = G^*$, this implies N_ν^* contains every algebraically unipotent element of G^*. ■

COROLLARY 4.3. *Suppose* $\Gamma\backslash G$ *is a finite-volume homogeneous space of a Lie group* G. *Then every unipotent element of* G *normalizes (resp. centralizes) every connected Lie subgroup of* G *normalized (resp. centralized) by* Γ. *In particular*, $\operatorname{nil} G \subseteq N_G(\Gamma^\circ)$. □

COROLLARY 4.4. *Let* $\Gamma\backslash G$ *be a finite-volume homogeneous space of a connected Lie group* G, *and assume* Γ *projects densely into the maximal compact semisimple factor of* G. *If* H *is a real algebraic group, and* $\sigma: G \to H$ *is a continuous homomorphism, then the Zariski closure* Γ^* *of* $\Gamma\sigma$ *contains every Levi subgroup of the Zariski closure* G^* *of* $G\sigma$ *in* H. □

COROLLARY 4.5. *Suppose* $\Gamma\backslash G$ *is a finite-volume homogeneous space of a connected Lie group* G, *and assume* Γ *projects densely into the maximal compact semisimple factor of* G. *Then any closed connected subgroup of* G *normalized (resp. centralized) by both* Γ *and* $\operatorname{rad} G$ *is normal (resp. central) in* G. ■

COROLLARY 4.6. *Suppose* $\Gamma\backslash G$ *is a finite-volume homogeneous space of a connected Lie group* G *whose radical is nilpotent, and assume* Γ *projects densely into the maximal compact semisimple factor of* G. *Then any closed connected subgroup of* G *normalized (resp. centralized) by* Γ *is normal (resp. central) in* G. *If* Γ *is discrete, this implies* $\Gamma \cdot Z(G)$ *is closed in* G. □

4B. The Bieberbach-Auslander Theorem revisited.

Definition 4.7. Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a Lie group G, and let N be a closed normal subgroup of G. We say Γ (or $\Gamma\backslash G$) is *compatible with* N if ΓN is closed in G (and hence $(\Gamma \cap N)\backslash N$ is a finite-volume homogeneous space).

Definition 4.8. A finite-volume homogeneous space $\Gamma\backslash G$ of a Lie group G is *strongly rad-compatible* if

(1) Γ is compatible with both $\operatorname{rad} G$ and $\operatorname{nil} G$;
(2) $\Gamma^\circ \subseteq \operatorname{nil} G$; and
(3) $(\Gamma \cap \operatorname{rad} G)\backslash\operatorname{rad} G$ is locally faithful.

Definition 4.9. A finite-volume homogeneous space $\Gamma\backslash G$ is *weakly rad-compatible* if there is some connected closed solvable subgroup A of G, containing $\operatorname{rad} G$ and normalized by Γ, such that ΓA is closed. This notion is due to Dani [7, Proposition 1.2] and Brezin and Moore [4, Section 4]. They use the term *admissible* instead of *weakly rad-compatible.*

LEMMA 4.10. *Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a Lie group G, and N is a closed normal subgroup of G contained in Γ. If the homogeneous space $(\Gamma/N)\backslash(G/N)$ is weakly rad-compatible, then $\Gamma\backslash G$ is weakly rad-compatible.*

Proof. There is a connected closed solvable subgroup A_1/N of G/N, containing $\operatorname{rad}(G/N)$ and normalized by Γ/N, such that ΓA_1 is closed in G. Since A_1/N is solvable, we have $A_1 = N \operatorname{rad} A_1$, and hence $\Gamma \operatorname{rad} A_1 = \Gamma A_1$ is closed in G. Setting $A = \operatorname{rad} A_1$, we see that $\Gamma\backslash G$ is weakly rad-compatible. □

THEOREM 4.11 (Auslander [17, Theorem 8.2.4, p. 149]). *If Γ is a lattice in a Lie group G, then Γ is weakly rad-compatible.* □

THEOREM 4.12. *Every finite-volume homogeneous space $\Gamma\backslash G$ of any Lie group G is weakly rad-compatible.*

Proof. We may assume $\Gamma\backslash G$ is faithful (see Lemma 4.10). Since Γ/Γ° is discrete, Auslander's theorem 4.11 (together with Lemma 4.10) asserts $\Gamma\backslash N_G(\Gamma^\circ)$ is weakly rad-compatible, so there is a connected closed solvable subgroup N of $N_G(\Gamma^\circ)$, containing $\operatorname{rad} N_G(\Gamma^\circ)$ and normalized by Γ, such that ΓN is closed.

Consider first the case where $N \subseteq \Gamma$. Then $\operatorname{rad} N_G(\Gamma^\circ) \subseteq \Gamma$. Since Corollary 4.3 asserts $\operatorname{nil} G \subseteq N_G(\Gamma^\circ)$, and because $\Gamma\backslash G$ is faithful, we conclude $\operatorname{nil} G = e$. Hence $\operatorname{rad} G = e$. So $\Gamma\backslash G$ is weakly rad-compatible (let $A = e$).

We may now assume $N \not\subseteq \Gamma$. Then $\dim(\Gamma N)^\circ > \dim \Gamma^\circ$ so, by induction on $\dim \Gamma\backslash G$, we may assume $(\Gamma N)\backslash G$ is weakly rad-compatible. Hence there is a connected closed solvable subgroup A_1 of G, containing $\operatorname{rad} G$ and normalized by ΓN, such that $\Gamma N A_1$ is closed. Since N normalizes A_1, and each of N and A_1 is a solvable subgroup normalized by Γ, the product NA_1 is solvable and normalized by Γ. Setting $A = NA_1$, we easily deduce that $\Gamma\backslash G$ is weakly rad-compatible. ■

By entirely different methods, R. J. Zimmer [26] has independently proved a generalization of Theorem 4.12. As a corollary of his work it follows that every homogeneous space $\Gamma\backslash G$ (*not necessarily of finite volume*) is weakly rad-compatible.

COROLLARY 4.13. *Suppose $\Gamma\backslash G$ is a locally faithful finite-volume homogeneous space of a Lie group G. If $\Gamma \cap G^\circ$ projects densely into the maximal compact semisimple factor of G°, and if $\Gamma G^\circ = G$, then $\Gamma\backslash G$ is strongly rad-compatible.*

Proof. A fundamental theorem of Mostow (cf. [17, Theorem 3.3, p. 46]) asserts that any locally faithful finite-volume homogeneous space of a connected solvable Lie group is strongly rad-compatible, so we need only show: (1) Γ is compatible with rad G; (2) $\Gamma^\circ \subseteq$ rad G; and (3) ($\Gamma \cap$ rad $G)\backslash$rad G is locally faithful.

Step 1. Γ *is compatible with* rad G. Theorem 4.12 shows Γ is weakly rad-compatible, so we may let A be a closed solvable subgroup of G normalized by Γ and containing rad G, such that ΓA is closed. Since A is normalized by both Γ and rad G, it follows from Corollary 4.5 that A is normal in G. Hence $A =$ rad G, so Γ is rad-compatible.

Step 2. $\Gamma^\circ \subseteq$ rad G (cf. [4, Theorem 4.6]). Corollary 4.5 implies Γ° rad $G \lhd G$. If $\Gamma^\circ \not\subseteq$ rad G, it follows that Γ° contains some normal subgroup L_1 of a Levi subgroup of G. The Zariski closure G^* of AdG in Aut($\mathcal{G}$) has a Malcev decomposition $G^* = (L^* \times T^*) \ltimes U^*$, where L^* is a Levi subgroup of G^*, T^* is a torus (reductive in G^*), and U^* is the unipotent radical. Now T^* centralizes L_1, and Dani's Theorem 4.1 implies L^*U^* normalizes Γ°. So

$$L_1^{G^*} = L_1^{T^*L^*U^*} = L_1^{L^*U^*} \subseteq \Gamma^\circ.$$

This contradicts the fact that $\Gamma\backslash G$ is locally faithful.

Step 3. ($\Gamma \cap$ rad $G)\backslash$rad G *is locally faithful*. Let us pretend there is a nontrivial connected normal subgroup N of rad G contained in Γ°. The normal closure N^Γ of N in Γ is a connected subgroup of Γ normalized by both Γ and rad G. We conclude from Collary 4.5 that $N^\Gamma \lhd G$. This contradicts the fact that $\Gamma\backslash G$ is locally faithful. ∎

4C. The structure of unipotent translations.

Standing assumptions (4.14). Throughout Section 4C, g is an ergodic translation on a locally faithful finite-volume homogeneous space $\Gamma\backslash G$ of a Lie group G. We assume $G = \Gamma G^\circ = G^\circ\langle g\rangle$.

Remark 4.15. The assumption that $\Gamma G^\circ = G$ is equivalent to requiring that $\Gamma\backslash G$ be connected. If $G^\circ\langle g\rangle$ were a proper subgroup of G, one could replace G with this subgroup, so there is no serious loss in assuming $G^\circ\langle g\rangle = G$.

Remark 4.16. We allow G to be disconnected because this provides an easy way to treat affine maps. Namely, suppose $\Gamma\backslash G$ is a connected homogeneous space, and let $A = \mathrm{Aff}(\Gamma\backslash G)$ be the group of volume-preserving invertible affine maps on $\Gamma\backslash G$. Then A is transitive on $\Gamma\backslash G$ (because $G \subset A$) so the action of an affine map $T_{g,\sigma}$ on $\Gamma\backslash G$ is equivalent to the action of $T_{g,\sigma}$ by translation on $\mathrm{Stab}_A(\Gamma)\backslash A$.

Notation (4.17) (cf. [4, Section 2]). If the Zariski closure G^* of $\mathrm{Ad}G$ in $\mathrm{Aut}(\mathcal{G})$ is connected, then it has a Malcev decomposition $G^* = (L^* \times T^*) \ltimes U^*$, where L^* is a Levi subgroup of G^*, T^* is a torus (reductive in G^*), and U^* is the unipotent radical. Let $\pi: G \to T^*$ be the map obtained by composing Ad with the projection of G^* onto T^*. [We implicitly assume throughout Section 4C that G^* is Zariski connected. As it requires no greater cost than passing to a subgroup of finite index in G, this assumption is essentially harmless.]

LEMMA 4.18. *Let B and T be subgroups of a compact semisimple Lie group K, and assume BT contains a dense subgroup of K°. If T is abelian, then B contains a dense subgroup of K°.*

Proof. We may assume B and T are closed (hence compact) and $TB = K$. Then $T^\circ/(T^\circ \cap B)$ is homeomorphic to $(K^\circ \cap B)\backslash K^\circ$. Because the former is a torus and the latter has a finite fundamental group, we conclude that $(K^\circ \cap B)\backslash K^\circ$ is a point, i.e., $K^\circ \subset B$. ∎

PROPOSITION 4.19 (cf. [4, Theorem 5.5]). *$\Gamma \cap G^\circ$ projects densely into the maximal compact semisimple factor of G°. Hence Γ is strongly rad-compatible.*

Proof. Replacing g by a conjugate if necessary, we may assume $\Gamma\langle g\rangle$ is dense in G. Then $\Gamma\langle g\rangle$ projects densely into the maximal compact semisimple factor K^* of G^*. Since $\langle g\rangle^*$ is abelian, we conclude from Lemma 4.18 that $(K^*)^\circ \subset \Gamma^*$. Since $(\Gamma \cap G^\circ)\backslash\Gamma$ is cyclic (hence abelian), Lemma 4.18 asserts that $(\Gamma \cap G^\circ)^*$ contains $(K^*)^\circ$ as desired. Corollary 4.13 shows the second conclusion follows from the first. ∎

PROPOSITION 4.20. *$\Gamma\pi$ is discrete (and hence closed) in T^*.*

Proof (adapted from the proof of [4, Theorem 2.1]). Replace Γ by a subgroup of finite index (if necessary) so that Γ^* is connected. Because Corollaries 4.2 and 4.4 imply Γ^* contains U^* and L^*, we may write $\Gamma^* = (L^* \times S^*) \ltimes U^*$, where $S^* = \Gamma^* \cap T^*$. Let $\mathfrak{N}$ and $\mathfrak{D}$ be the Lie algebras of nil G and Γ°. By [4, Lemma 2.2], we know S^* is faithful on $\mathfrak{N}/\mathfrak{D}$.

Let j be the natural homomorphism of Γ^* into $\mathrm{Aut}(\mathfrak{N}/\mathfrak{D})$. Now Γj normalizes the discrete cocompact subgroup $(\Gamma \cap \mathrm{nil}\, G)/\Gamma^\circ$ of $(\mathrm{nil}\, G)/\Gamma^\circ$, so it preserves the associated lattice in $\mathfrak{N}/\mathfrak{D}$ provided by the Malcev theorem [17, p. 34]. Hence Γ^*j is an algebraic group defined over the rational numbers, and with an appropriate choice of the torus S^*, the projection $\pi' : \Gamma^*j \to S^*j$ is also defined over $\mathbf{Q}$. Furthermore, Γ is contained in the group of integer points of Γ^*j. So $\Gamma^*j\pi'$ is contained in an arithmetic subgroup of S^*j [3, Corollary 7.13(3)]. Hence it is discrete. ■

COROLLARY 4.21 (cf. [4, Corollary 2.3]). *$G\pi$ is closed in T^*.*

Proof. Since any Levi subgroup of G° is in the kernel of π and $\Gamma G^\circ = G$, we have $G\pi = (\Gamma \cdot \operatorname{rad} G)\pi$. As $\Gamma\pi$ is discrete, and $\Gamma \cap \operatorname{rad} G$ is cocompact in rad G (see (4.19)), we conclude that $G\pi$ is closed. ■

PROPOSITION 4.22. *If $[G, G] \cdot \Gamma$ is dense in G, then* rad G *is nilpotent and $G = [G, G] \cdot Z(G)$.*

Proof. Proposition 4.20 asserts Γj is discrete in T^*. On the other hand, since $[G, G] \subseteq \ker j$, it follows that Γj is dense in Gj, which is connected. We conclude that $T^* = e$, so rad G is nilpotent (hence G is locally algebraic and Γ is discrete).

Let j be the natural homomorphism $G \to \operatorname{Aut}(\mathfrak{N})$, where $\mathfrak{N}$ is the Lie algebra of nil G. Because G is locally algebraic, Gj is Zariski closed in $\operatorname{Aut}(\mathfrak{N})$. As in the proof of Proposition 4.20, we see that Γj is contained in an arithmetic subgroup of Gj, and hence Γj is an arithmetic lattice in Gj. Hence $\Gamma j \cap [Gj, Gj]$ is a lattice in $[Gj, Gj]$, and therefore $([G, G] \cdot \Gamma)j$ is closed in $GL_n(\mathbf{R})$. Since $[G, G] \cdot \Gamma$ is dense in G, this implies $([G, G] \cdot \Gamma)j = Gj$, so $C_G(\operatorname{nil} G) \cdot [G, G] \cdot \Gamma = G$ because $\ker j = C_G(\operatorname{nil} G)$. As G is connected, we must have $C_G(\operatorname{nil} G)^\circ \cdot [G, G] = G$. Lemma 3.21 shows $C_G(\operatorname{rad} G) \subseteq Z(G) \cdot [G, G]$. Since rad G = nil G, we conclude $Z(G) \cdot [G, G] = G$ as desired. ■

PROPOSITION 4.23. *If g is unipotent, then $T^* = e$. Hence* rad G = nil G (*so G is locally algebraic*), Γ *is discrete*, *and* Γ *is compatible with* $Z(G)$.

Proof (cf. [4, Corollary 2.5, p. 578]). Replacing the ergodic unipotent translation g by a conjugate if necessary, we may assume $\Gamma\langle g\rangle$ is dense in G. Since $g \in \ker \pi$, this implies $\Gamma\pi$ is dense in $G\pi$. But $G\pi = (G^\circ\pi)(\langle g\rangle\pi) = G^\circ\pi \cdot e$ is connected, whereas $\Gamma\pi$ is discrete (see 4.20), so we must have $G\pi = \Gamma\pi = e$. Since $G\pi$ is Zariski dense in T^*, this implies $T^* = e$.

Now $T^* = e$, so rad G^* is nilpotent. Since $(\operatorname{rad} G)^* \subseteq \operatorname{rad} G^*$, we conclude that rad G is nilpotent. It follows from Corollary 4.6 that $\Gamma^\circ \lhd G$. Since $\Gamma\backslash G$ is locally faithful, this means $\Gamma^\circ = e$, i.e., Γ is discrete. Hence Corollary 4.6 implies Γ is compatible with $Z(G)$. □

4D. The Mautner phenomenon.

THEOREM 4.24 ("The Mautner phenomenon" [14, Theorem 1.1, p. 156]). *Let Γ be a lattice in a connected Lie group G. For any connected subgroup M of G, let N be the smallest connected normal subgroup of G such that M projects to an* Ad*-precompact subgroup of G/N. Then any M-invariant measurable function on $\Gamma\backslash G$ is N-invariant.* □

LEMMA 4.25. *Let Γ be a lattice in a connected Lie group G whose radical is nilpotent, and assume Γ projects densely into the maximal compact semisimple factor of G. Suppose ψ is a measurable function on $\Gamma\backslash G$ which is N-invariant, for some normal subgroup N of G. Then there is a normal subgroup N_1 of G containing N, such that ψ is N_1-invariant, and $N_1\Gamma$ is closed in G.*

Proof. Let N_0 be the identity component of the closure of $N\Gamma$. Since N_0 is a connected subgroup of G normalized by Γ, Corollary 4.6 implies N_0 is normal in G. Now ψ corresponds to a function ψ' on G which is (essentially) left-invariant under $N\Gamma$, so ψ' is left-invariant under N_0. Because N_0 is normal, then ψ' is also right-invariant under N_0, so ψ is N_0-invariant. Set $N_1 = NN_0$. ■

COROLLARY 4.26. *Suppose Γ is a lattice in a connected Lie group G whose radical is nilpotent, and assume Γ projects densely into the maximal compact semisimple factor of G. For V a connected unipotent subgroup of G, let N be the smallest closed normal subgroup of G containing V and such that $N\Gamma$ is closed. Then the N-orbits are the ergodic components of the action of V by translation on $\Gamma\backslash G$.*

Proof. Since $\Gamma\backslash G/N$ is countably separated, it suffices to show any V-invariant measurable function on $\Gamma\backslash G$ is (essentially) N-invariant. To this end, let f be a V-invariant measurable function. The Mautner Phenomenon (4.24) implies f is essentially N_0-invariant, where N_0 is the smallest normal subgroup of G such that V projects to an Ad-precompact subgroup of G/N_0. Since V is unipotent, this implies V projects to a central subgroup of G/N_0, and hence VN_0 is normal in G. Since VN_0 stabilizes f, Lemma 4.25 asserts N stabilizes f. ■

Corollary 4.27 ("Moore Ergodicity Theorem," cf. [25, Theorem 2.2.6, p. 19]). *Suppose Γ is an irreducible lattice in a connected semisimple Lie group G. If X is a connected subgroup of G, then either X is* Ad-*precompact, or X is ergodic on $\Gamma\backslash G$.* ∎

Corollary 4.28 (cf. [4, Theorem 6.1, p. 601]). *Suppose Γ is a lattice in a connected Lie group G. If X is a connected subgroup of G which is ergodic on both the maximal solvmanifold quotient and the maximal semisimple quotient of $\Gamma\backslash G$, then X is ergodic on $\Gamma\backslash G$.* □

5. Details to fill in the outline of the proof.

Remark 5.1. Suppose u is a unipotent element of a locally algebraic group G. Any algebraically unipotent element of the Zariski closure G^* of $\operatorname{Ad}G$ in $\operatorname{GL}(\mathfrak{g})$ lies in a unique one-parameter algebraically unipotent subgroup, so there is a one-parameter unipotent subgroup v^r ($r \in \mathbf{R}$) of G^* such that $v^1 = \operatorname{Ad}u$. If G° is simply connected, then for $y, c \in G^\circ$ and $r \in \mathbf{R}$, the expression $v^{-r}yv^r[v^r, c] \in G^\circ$ is well-defined. Thus, given $x, y \in G^\circ$, even though u itself may not lie in a one-parameter subgroup, there is no ambiguity when we write an expression such as $d(xu^r, yu^r[u^r, c])$, for $r \in \mathbf{R}$. (Where d is a left-invariant metric on G, cf. [24, Notation 2.10].)

Notation. Let $\operatorname{Aff}_G(\psi) = \{g \in G\colon \psi \text{ is affine for } g\}$.

5A. Affine for the normalizer. For technical reasons it is easier to prove a strengthened form of Corollary 2.9.

Definition 5.2. Given an element x and a subgroup Y of a Lie group G. For $\delta > 0$, we set

$$C_G(x, Y; \delta) = \{c \in G \mid d(e, c) < \delta,\ [x^n, c] \in Y \text{ for all } n \in \mathbf{Z}\}.$$

The descending chain condition on connected Lie subgroups of G implies there is some $\delta_0 > 0$ such that $\langle C_G(x, Y; \delta)\rangle^\circ = \langle C_G(x, Y; \delta_0)\rangle^\circ$ whenever $0 < \delta \leq \delta_0$. We set $C_G(x, Y) = \langle C_G(x, Y; \delta_0)\rangle^\circ$, and call this the *centralizer* of x *relative* to Y in G. This terminology is motivated by the fact that if Y is normal in G (and connected), then $C_G(x, Y)/Y$ is the identity component of the centralizer of xY in G/Y.

Proposition 5.3 ("Affine for the Relative Centralizer"). *Suppose Γ (resp. Λ) is a lattice in a connected real Lie group G (resp. H) whose radical is nilpotent. Let u be an ergodic unipotent element of G and assume ψ*

is affine for u via a unipotent element $\tilde{u}$ of H. If U is any connected unipotent subgroup of G contained in $\mathrm{Aff}_G(\psi)$, *then ψ is affine for $C_G(u, U)$.*

Sketch of proof (cf. [18, Lemma 3.2] or Step 2 of the proof of [24, Lemma 3.1]). For most $n \in \mathbf{Z}$, we have $s\psi\tilde{u}^n = su^n\psi \doteq scu^n[u^n, c]\psi = sc\psi\tilde{u}^n[\widetilde{u^n, c}]$. By polynomial divergence, we conclude that this approximation holds for all $r \in \mathbf{R}$, and hence there is some $c^s \in H$ with $sc\psi = s\psi c^s$ and $[\tilde{u}, c^s] = [\widetilde{u, c}]$.

Since $[\tilde{u}, c^s] = [\widetilde{u, c}]$ is independent of s, we have $c^s(c^t)^{-1} \in C_H(\tilde{u})$ for $s, t \in \Gamma\backslash G$. Then, because

$$su\psi c^{su} = suc\psi = sc\psi\tilde{u}[\widetilde{u, c}] = s\psi c^s\tilde{u}[\widetilde{u, c}] = s\psi\tilde{u}c^s = su\psi c^s$$

and $C_H(\tilde{u})$ acts essentially freely on $\Lambda\backslash H$ [24, Lemma 2.8], we must have $c^{su} = c^s$ for a.e. s. Because u is ergodic on $\Gamma\backslash G$, this implies $s \mapsto c^s$ is essentially constant, as desired. □

It is straightforward to complete the above sketch to a full proof, except that it relies on the nonobvious assumption that $n \mapsto [\widetilde{u^n, c}]$ is a polynomial. Of course $n \mapsto [u^n, c]$ is a polynomial (cf. [25, Proposition 3.4.1]), but the homomorphism $\sim : U \to \tilde{U}$ is not a polynomial if $\tilde{U}$ is not unipotent. The following lemma is all that is required to patch up the proof.

LEMMA 5.4. *If U is any connected unipotent subgroup of G contained in* $\mathrm{Aff}_G(\psi)$, *then there is some connected subgroup U_0 of U with $C_G(u, U) = C_G(u, U_0)$ and such that $\tilde{U}_0$ is unipotent.*

Sketch of proof. It suffices to find some $\delta > 0$ such that $[\widetilde{u^r, c}]$ is unipotent for all $r \in \mathbf{R}$ and all $c \in C_G(u, U; \delta)$. (For then we set $U_0 = \langle [u^r, c] \mid r \in \mathbf{R}, c \in C_G(u, U; \delta) \rangle$.) Modding out rad H ($=$ nil H) and $Z(H)$, we may assume H is semisimple with trivial center (and hence H is a real algebraic group, not just locally algebraic).

Since $\tilde{U}$ is a connected nilpotent group, its Zariski closure can be written in the form $V \times T$, where V is unipotent and T is an algebraic torus. Because each element of $\tilde{U}$ has zero entropy, T is compact (see 6.1), so it is an algebraic subgroup of H [25, p. 40]. (Indeed, the variety H/T is quasi-affine.) Write $[\widetilde{u^r, c}] = [\widetilde{u^r, c}]_V \cdot [\widetilde{u^r, c}]_T$ with $[\widetilde{u^r, c}]_V \in V$ and $[\widetilde{u^r, c}]_T \in T$. An argument from polynomial divergence on H/T shows there is some $c^s \in H$ with $s\psi c^s = sc\psi$ and $\tilde{u}^r c^s = c^s\tilde{u}^r[\widetilde{u^r, c}]_V$. For most $r \in \mathbf{R}$, we have

$$s\psi \cdot \tilde{u}^r \doteq sc\psi \cdot \tilde{u}^r[\widetilde{u^r, c}] = s\psi \cdot c^s\tilde{u}^r[\widetilde{u^r, c}] = s\psi \cdot \tilde{u}^r c^s[\widetilde{u^r, c}]_T.$$

This implies the map $r \mapsto c^s[\widetilde{u^r, c}]_T$ is constant. Hence $[\widetilde{u^r, c}]_T = e$ for all $r \in \mathbf{R}$. □

5B. Reduction.

Notation 5.5. Let U be the identity component of a maximal unipotent subgroup of G containing u, and set $P = N_{G^\circ}(U)$, a minimal parabolic subgroup of G°. Let LEVI be a Levi subgroup of G°.

THEOREM 5.6. *ψ is affine for P°.*

Proof. We set $U_0 = e$ and recursively define $U_{i+1} = C_G(u, U_i) \cap U$. By induction on i, Affine for the Relative Centralizer (5.3) implies ψ is affine for U_i for all i. Because $\langle u, U \rangle$ is nilpotent, we have $U_i = U$ when i is sufficiently large, so ψ is affine for U. Therefore Affine for the Relative Centralizer (5.3) asserts ψ is affine for $N_{G^\circ}(U) \subseteq C_G(u, U)$, as desired. □

COROLLARY 5.7. *$(U \cap \mathrm{LEVI})^{\sim}$ is a unipotent subgroup of H.*

Proof. There is some $a \in P^\circ$ such that $U \cap$ LEVI is contained in the horospherical subgroup associated to a (cf. 3.10). Then $(U \cap \mathrm{LEVI})^{\sim}$ is contained in the horospherical subgroup associated to $\tilde{a}$, so $(U \cap \mathrm{LEVI})^{\sim}$ is unipotent. □

PROPOSITION 5.8. *We may assume every nontrivial connected unipotent subgroup of* LEVI *is ergodic.*

Proof. Assume the contrary.

Step 1. LEVI is a product LEVI $= N_1 \cdot N_2$ of two of its nonergodic connected normal subgroups.

Proof. We may assume LEVI is ergodic on $\Gamma \backslash G$ (else the assertion of Step 1 is obvious), and hence the maximal solvmanifold quotient of $\Gamma \backslash G$ is trivial. Set $\bar{G} = G/\mathrm{rad}\, G$, so that $\bar{\Gamma} \backslash \bar{G}$ is the maximal semisimple quotient of $\Gamma \backslash G$. If V is a nontrivial connected nonergodic unipotent subgroup of LEVI, then, since $\Gamma \backslash G$ has no solvmanifold quotient, we know V must be nonergodic on $\bar{\Gamma} \backslash \bar{G}$ (see 4.28). Since $\bar{V}$ is not Ad-precompact, we conclude that $\bar{\Gamma}$ is a reducible lattice in $\bar{G}$ (see 4.27). Therefore $\bar{G}$ can be decomposed into a product of two nonergodic normal subgroups (cf. [17, Theorem 5.22]).

Step 2. We may assume that if V is any connected unipotent nonergodic subgroup of LEVI $\cap$ U, then ψ is affine for a nonergodic normal subgroup of G containing V.

Proof. The Mautner Phenomenon (4.26) implies the ergodic components of V are the orbits of some normal subgroup N of G° which contains V. Since $\tilde{V}$ is unipotent, ψ maps N-orbits to $\tilde{N}$-orbits, where $\tilde{N}$ is some normal subgroup of H° with closed orbits on $\Lambda\backslash H$. By induction on dim G, we may assume ψ is affine for N via $\tilde{N}$ on each N-orbit. Thus, for each N-orbit θ, we have a (local) epimorphism $\sigma_\theta\colon N \to \tilde{N}$. Since ψ is affine for P°, all the σ_θ agree on $(P \cap N)^\circ$. Since $P \cap N$ is a parabolic subgroup of N, we conclude from Theorem 3.13(iii) that all the σ_θ are equal. Hence ψ is affine for N as desired.

Step 3. Completion of proof.

Proof. Since ψ is known to be affine for P°, the main theorem will be proven if we show ψ is affine for a cocompact normal subgroup of LEVI. Thus it suffices to show ψ is affine for a cocompact normal subgroup of each N_i in the decomposition of Step 1. Let $V = N_i \cap U$, a maximal connected unipotent subgroup of N_i. Since N_i is semsimple, any closed normal subgroup containing V is cocompact. Thus Step 2 completes the proof. ∎

COROLLARY 5.9. *We may assume* **R**-rank$(G/\text{rad } G) > 0$ *and the maximal solvmanifold quotient of* $\Gamma\backslash G$ *is trivial.*

Proof. Since ψ is affine for P°, we may assume G has a proper parabolic subgroup, which means $G^\circ/\text{rad } G$ is noncompact (i.e., **R**-rank$(G^\circ/\text{rad } G) > 0$). Therefore LEVI has a nontrivial connected unipotent subgroup, which the Proposition asserts we may assume is ergodic. Then LEVI is ergodic, and hence $\Gamma\backslash G$ has no solvmanifold quotient. ∎

COROLLARY 5.10. *We may assume that if X is any connected subgroup of G which does not project to an* Ad*-precompact subgroup of* $G/\text{rad } G$, *then X is ergodic on* $\Gamma\backslash G$.

Proof. The Moore Ergodicity Theorem (4.27) (in conjunction with Proposition 5.8) asserts X is ergodic on the maximal semisimple quotient of $\Gamma\backslash G$. Since $\Gamma\backslash G$ has no solvmanifold quotient (5.9), then the Mautner Phenomenon (4.28) implies X is ergodic on $\Gamma\backslash G$. ∎

We could easily reduce further to the case where **R**-rank$(G^\circ/\text{rad } G) = 1$ (cf. [24, Section 4]), but this is not necessary.

5C. $G = H$.

LEMMA 5.11. *We may assume that if $g \in \text{Aff}_G(\psi)$ and $\tilde{g} = e$, then $g = e$.*

Proof. Let ker be the kernel of the homomorphism $\sim : \mathrm{Aff}_G(\psi) \to H$. We wish to show that, by passing to a factor group of G, we may assume $\ker = \{e\}$. Since ψ is (essentially) ker-invariant, it will suffice to show ker is a normal subgroup such that $\ker \cdot \Gamma$ is closed in G. Note that P° normalizes ker, because $P^\circ \subseteq \mathrm{Aff}_G(\psi)$ and the kernel of a homomorphism is normal. Since ker is precisely the essential stabilizer of ψ, the Mautner phenomenon (4.24) implies there is a closed normal subgroup N of G contained in ker such that ker projects to an Ad-precompact subgroup of G/N. Therefore Lemma 3.23 asserts ker is normal in G. Now Lemma 4.25 implies $\ker \cdot \Gamma$ is closed in G. ∎

LEMMA 5.12. *There is an isomorphism* $\wedge: G \to H$ *such that* $\hat{p} = \tilde{p}$ *for all* $p \in [P^\circ, P^\circ]$ (*in particular, for* $p \in U \cap$ LEVI), *and* $\hat{p} \in \tilde{p} \cdot Z(G)$ *for all* $p \in P^\circ$.

Proof. We have shown that ψ is affine for P°, where $P = N_{G^\circ}(U)$ is a parabolic subgroup of G° (see 5.6). So $P_0 = \mathrm{Aff}_G(\psi)^\circ$ is the identity component of a parabolic subgroup of G°. Similarly, since ψ has finite fibers over $\Lambda\backslash H$ (see 2.15), we know $Q_0 = \mathrm{Aff}_H(\psi^{-1})$ is a parabolic subgroup of H. Lemma 5.11 implies $\sim : P_0 \to Q_0$ is a (local) isomorphism.

Since $\Gamma\backslash G$ has no solvmanifold quotient, we know $[G, G] \cdot \Gamma$ is dense in G. Therefore Proposition 4.22 asserts $G = [G, G] \cdot Z(G)$. Hence Theorem 3.16 will imply the desired conclusion if we show $\widetilde{\mathrm{rad}\, G} = \mathrm{rad}\, H$. Let K_G and K_H be the connected normal subgroups of G and H, such that $K_G/\mathrm{rad}\, G$ and $K_H/\mathrm{rad}\, H$ are the maximal compact factors of $G/\mathrm{rad}\, G$ and $H/\mathrm{rad}\, H$, respectively. We will show K_G is the unique maximal connected nonergodic normal subgroup of P°. Of course, K_H can then be similarly characterized in Q°, from which it follows that $\tilde{K}_G = K_H$. Since $\mathrm{rad}\, K_G = \mathrm{rad}\, G$ and $\mathrm{rad}\, K_H = \mathrm{rad}\, H$, this implies $\widetilde{\mathrm{rad}\, G} = \mathrm{rad}\, H$, as desired.

All that remains is to prove K_G is the unique maximal connected nonergodic normal subgroup of P°. So let K be some other such. Since K is nonergodic, it projects to an Ad-precompact subgroup of $G/\mathrm{rad}\, G$ (5.10). Therefore $K \cdot K_G$ projects to an Ad-precompact subgroup of $G/\mathrm{rad}\, G$, and hence $K \cdot K_G$ is nonergodic. Then the maximality of K implies $K_G \subseteq K$. Since K is normalized by P° and projects to an Ad-precompact subgroup of $G/\mathrm{rad}\, G$, Lemma 3.23 asserts K is normal in G. Because $K_G \subseteq K$ and $K/\mathrm{rad}\, G$ is Ad-precompact, this implies $K = K_G$ as desired. ∎

6. Zero-entropy translations. This section can be viewed as a continuation of Section 4C. We show that, modulo finite covers, the study of ergodic zero-entropy translations can be reduced to the study of unipotent translations.

Assumptions. Assumption 4.14 and Notation 4.17 are in effect.

LEMMA 6.1 (Dani). *The translation by g has zero entropy iff every eigenvalue of* Adg *is of absolute value* 1.

Proof. ($\Leftarrow$) See [7, Appendix]. ($\Rightarrow$) This follows from [5, Theorem 3.5]. □

PROPOSITION 6.2. *If g has zero entropy, then T^* is compact. Therefore $\Gamma\pi$ is finite.*

Proof (cf. [4, Corollary 2.5, p. 578]). Let T_K be the maximal compact subgroup of T^*, and let $\bar{\pi}: G \to T^*/T_K$ be the composition of π with the natural homomorphism $T^* \to T/T_K$. Replacing the ergodic zero-entropy translation g by a conjugate if necessary, we may assume $\Gamma\langle g\rangle$ is dense in G. Since $g \in \ker \bar{\pi}$ (see 6.1), this implies $G\bar{\pi} \subseteq \Gamma\bar{\pi}$. But $G\bar{\pi}$ is connected because $g \in \ker \bar{\pi}$, and $\Gamma\bar{\pi}$ is discrete (see 4.12), so we must have $G\bar{\pi} = \Gamma\bar{\pi} = 1$. Since $G\pi = T$, this implies $T = T_K$ is compact, as desired. ■

When G is connected and solvable, Auslander [1, Theorem C] went beyond Proposition 6.2 by showing $\Gamma\pi = e$. The following example shows this stronger result is unfortunately not true for arbitrary (nonsolvable) groups.

Example 6.3. Let $\mathrm{Mat}_2(\mathbf{R})$ be the additive group of 2×2 real matrices ($\cong \mathbf{R}^4$). Let

$$G = \mathrm{Mat}_2(\mathbf{R}) \rtimes (\mathrm{SL}_2(\mathbf{R}) \times \mathrm{SO}(2)),$$

where $\mathrm{SL}_2(\mathbf{R})$ acts by left multiplication and SO(2) acts by multiplication on the right. For $\alpha \in \mathbf{R}$, let $R_\alpha \in \mathrm{SO}(2)$ be the rotation through $2\pi\alpha$ radians. Set

$$\Gamma = \mathrm{Mat}_2(\mathbf{Z}) \rtimes (\mathrm{SL}_2(\mathbf{Z}) \times \langle R_{1/4}\rangle) \subset G.$$

Note Γ is a lattice in G, and we have $G^* \cong G$, $T^* = \mathrm{SO}(2)$, and $\Gamma\pi = \langle R_{1/4}\rangle \neq e$. However, for any $g \in \mathrm{SL}_2(\mathbf{R})$ which is ergodic on $\mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})$, and for any irrational α, the translation by $e \rtimes (g \times R_\alpha)$ is ergodic (e.g., by the Brezin-Moore criterion (4.28) because the maximal solvmanifold quotient is SO(2).)

PROPOSITION 6.4 (cf. Theorem 4.1 of [1, Part II]). *If $\Gamma\pi = e$, then some nonzero power of g is isomorphic to a translation on a homogeneous space of a group whose radical is nilpotent.*

Proof (cf. [4, pp. 575–576]). Passing to a covering group of G if necessary, we may assume G° is simply connected, and $G = G^\circ \rtimes \langle g \rangle$. Possibly replacing G by a subgroup of finite index (and g by some power) we may assume G^* is connected. Replacing Γ by a conjugate subgroup if necessary, we may assume $\Gamma\langle g \rangle$ is dense in G. Since $\Gamma\pi = e$, we know $\langle g \rangle \pi$ is Zariski dense in T^*, so we may choose T^* to be a subgroup of $\langle g \rangle^*$. Hence T^* commutes with Adg.

Since G° is simply connected, we can identify Aut G° with Aut($\mathcal{G}$) and hence view T^* as a group of automorphisms of G. Since $[T^*, \mathrm{Ad}g] = e$, we can form the semidirect product $G \rtimes T^* = G^\circ \rtimes (\langle g \rangle \times T^*)$. Of course, we have the embedding $G \to G \rtimes T^*: x \mapsto x \rtimes e$.

Define the map $\varphi: G \to G \rtimes T^*: x \mapsto x \rtimes (x^{-1}\pi)$. Though φ is (usually) not a group homomorphism, its image is a subgroup of $G \rtimes T^*$:

$$x\varphi \cdot y\varphi = [x \rtimes (x^{-1}\pi)] \cdot [y \rtimes (y^{-1}\pi)] = xy^{x\pi} \rtimes (x^{-1}y^{-1}\pi) = (xy^{x\pi})\varphi,$$

since $y^{x\pi}\pi = y\pi$ because $G/\ker \pi$ is abelian. Since φ is a homeomorphism onto its image, it follows that the image is closed. Note that φ is affine for g via $g\varphi$, because $g^{T^*} = g$ implies $g^{x\pi} = g$ for all $x \in G$. Since $\Gamma\pi = e$, φ factors through to a map $\bar{\varphi}: \Gamma\backslash G \to (\Gamma \rtimes e)\backslash G\varphi$ which is affine for g. It is not difficult to verify that $\mathrm{rad}(G\varphi)(=(\mathrm{rad}\, G)\varphi)$ is nilpotent. □

COROLLARY 6.5. *Suppose g has zero entropy. Then, for some nonzero power g^n of g, there is a finite-volume homogeneous space $\Gamma'\backslash G'$ of some Lie group G' whose radical is nilpotent, and a continuous map ψ: $\Gamma'\backslash G' \to \Gamma\backslash G$ which is affine for some translation $g' \in G'$ via g^n. Furthermore, every fiber of ψ is finite.*

Proof. Let Γ' be a subgroup of finite index in Γ, with $\Gamma'\pi = e$ (cf. 6.2). The natural map $\Gamma'\backslash G \to \Gamma\backslash G$ has finite fibers, and Proposition 6.4 shows some power of g on $\Gamma'\backslash G$ is isomorphic to a translation on a group whose radical is nilpotent. □

The following lemma is well-known if G is algebraic, but seems to require a bit of additional work in the general case, especially if G/G° is infinite.

LEMMA 6.6. *Suppose* $\Gamma\backslash G$ *is locally faithful, g has zero entropy, and* rad G *is nilpotent. Then, for some nonzero power* g^n *of g, there is a unipotent element u of G and some* $k \in G^\circ$ *satisfying*: $g^n = uk = ku$, *and* Adk *generates a precompact subgroup of* GL($\mathfrak{G}$). *Furthermore, if* $\Gamma\backslash G$ *is faithful, then k generates a precompact subgroup of G.*

Proof. Replacing G by a subgroup of finite index if necessary, we may assume that either $g \in G^\circ$ or $G = G^\circ \rtimes \langle g \rangle$. (We may also assume G° is simply connected.) Since rad G is nilpotent (so G° is locally algebraic), in either case it is easy to construct a real algebraic group G_{alg} and a homomorphism $\sigma: G \to G_{\text{alg}}$ such that ker σ is a discrete subgroup of G°. For any subgroup X of G, we write $\bar{X}$ for the Zariski closure of $X\sigma$ in G_{alg}. Because $\Gamma\pi$ is finite we may replace Γ and G by subgroups of finite index to assume $\Gamma\pi = e$, so that G_{alg} can be constructed with $\bar{G}/\bar{G}^\circ$ unipotent.

Write $g\sigma = \bar{u}\bar{k} = \bar{k}\bar{u}$, with $\bar{u}$ algebraically unipotent and $\bar{k}$ semisimple, in $\bar{G}$. Since $\bar{G}/\bar{G}^\circ$ is unipotent, $\bar{k} \in \bar{G}^\circ$. Because G° is locally algebraic, $|\bar{G}^\circ: G^\circ\sigma| < \infty$. So, perhaps replacing g by a power g^n, we may assume $\bar{k} \bullet G^\circ\sigma$. Indeed we may assume the Zariski closure of $\langle \bar{k} \rangle$ is connected. Then there is a one-parameter Lie subgroup $\bar{k}^r$ of $G^\circ\sigma$ through $\bar{k}$, and $[\bar{u}, \bar{k}^r] = e$ for all $r \in \mathbf{R}$. Lift $\bar{k}^r$ to a one-parameter subgroup k^r of G° and set $u = gk^{-1}$, so $u\sigma = \bar{u}$. Now $[u, k^r]\sigma = [u\sigma, (k\sigma)^r] = [\bar{u}, \bar{k}^r] = e$ for all $r \in \mathbf{R}$. Since ker σ is discrete, this implies $[u, k] = e$.

Now suppose $\Gamma\backslash G$ is faithful. Since Γ is compatible with $Z(G)$, then $Z(G)$ is compact, so the map Ad: $G \to$ GL($\mathfrak{G}$) is proper. □

THEOREM 6.7. *Suppose g and h are invertible ergodic zero-entropy affine maps on faithful finite-volume homogeneous spaces* $\Gamma\backslash G$ *and* $\Lambda\backslash H$ *of connected Lie groups G and H. Assume* rad G *and* rad H *are nilpotent, and let* ψ *be any ergodic joining of* $(g, \Gamma\backslash G)$ *with* $(h, \Lambda\backslash H)$ *with finite fibers over* $\Gamma\backslash G$. *Then* ψ *is a twisted affine joining. I.e., there is a finite cover* G' *of G, a lattice* Γ' *in* G', *homomorphisms* $\alpha: G' \to C_G(g)$ *and* β: $G' \to C_H(h)$ *with* $\Gamma'\alpha = e = \Gamma'\beta$, *and a measure-preserving affine map* $\phi: \Gamma'\backslash G' \to \Lambda\backslash H$ *such that, under the natural map* $\Gamma'\backslash G' \times \Lambda\backslash H \to \Gamma\backslash G \times \Lambda\backslash H$, *the joining on* $\Gamma'\backslash G' \times \Lambda\backslash H$ *associated to the map* $\Gamma's \mapsto (\Gamma s \cdot \alpha_s)\psi \cdot \beta_s$ *projects to* ψ.

Proof. By Lemma 6.6 we may write $g = uk = ku$ and $h = vl = lv$ where u and v are unipotent, while $K = \langle \bar{k} \rangle \subset G^\circ$ and $L = \langle \bar{l} \rangle \subset G^\circ$ are compact. We may assume K and L are connected. Any ergodic component of the action of (g, k, l) on $\Gamma\backslash G \times K \times L$ is isomorphic to the action of (g, t) on $\Gamma'\backslash G \times T$ for some Γ' of finite index in Γ, and some torus T.

The projections $G \times K \times L \to K$ and $G \times K \times L \to L$ restrict to homomorphisms $\alpha: G \times T \to K$ and $\beta: G \times T \to L$ with $(g, t)\alpha = k$ and $(g, t)\beta = l$ and $\Gamma'\alpha = e = \Gamma'\beta$. Thus we may twist on both α and β. For simplicity of notation let us assume ψ is a map rather than a general joining. The resulting map

$$\phi: \Gamma' \backslash G \times T \to \Lambda \backslash H: (\Gamma' s, t) \mapsto (\Gamma s \cdot \alpha_{s,t})\psi \cdot \beta_{s,t}$$

is affine for u via v, so Theorem 2.1 asserts it is affine. Set $t = e$ to conclude that

$$\Gamma' \backslash G \to \Lambda \backslash H: \Gamma' s \mapsto (\Gamma s \cdot \alpha_s)\psi \cdot \beta_s$$

is affine. This implies ψ is twisted affine. □

COROLLARY 6.8. *Suppose g and h are weak-mixing invertible ergodic zero-entropy affine maps on faithful finite-volume homogeneous spaces of connected Lie groups G and H. If $\psi: \Gamma \backslash G \to \Lambda \backslash H$ is affine for g via h, then ψ is an affine map (a.e.).*

Proof. Proposition 4.22 implies rad G and rad H are nilpotent, so Theorem 6.7 applies. Since $(g, \Gamma \backslash G)$ is weak-mixing, also the finite cover $(g, \Gamma' \backslash G)$ is weak-mixing, so there can be no twist. Thus ψ is affine. □

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REFERENCES

[1] L. Auslander, An exposition of the structure of solvmanifolds I, II, *Bull. Amer. Math. Soc.*, **79** (1973), 227–261, 262–285.
[2] G. M. Bergman, Epimorphisms of Lie algebras (preprint).
[3] A. Borel, *Introduction aux Groupes Arithmétiques*, Hermann, Paris, 1969.
[4] J. Brezin and C. C. Moore, Flows on homogeneous spaces: a new look, *Amer. J. Math.*, **103** (1981), 571–613.
[5] S. G. Dani, Kolmogorov automorphisms on homogeneous spaces, *Amer. J. Math.*, **98** (1976), 119–163.
[6] ———, Bernoulli translations and minimal horospheres on homogeneous spaces, *J. Indian. Math. Soc.*, **39** (1976), 245–284.

[7] ______, Spectrum of an affine transformation, *Duke Math. J.*, **44** (1977), 129–155.
[8] ______, On ergodic quasi-invariant measures of group automorphism, *Israel J. Math.*, **43** (1982), 62–74.
[9] M. H. Hall, Jr., *The Theory of Groups*, Macmillan, New York, 1959.
[10] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer, New York, 1971.
[11] G. P. Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*, Springer, New York, 1981.
[12] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1972.
[13] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. of Math.*, **74** (1961), 329–387.
[14] C. C. Moore, The Mautner phenomenon for general unitary representations, *Pacific J. Math.*, **86** (1980), 155–169.
[15] D. Ornstein and B. Weiss, Geodesic flows are Bernoulli, *Israel J. Math.*, **14** (1973), 184–198.
[16] W. Parry, Metric classification of ergodic nilflows and unipotent affines, *Amer. J. Math.*, **93** (1971), 819–828.
[17] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, New York, 1972.
[18] M. Ratner, Rigidity of horocycle flows, *Ann. of Math.*, **115** (1982), 597–614.
[19] ______, Factors of horocycle flows, *Ergodic Theory Dyn. Syst.*, **2** (1982), 465–489.
[20] ______, Horocycle flows, joinings and rigidity of products, *Ann. of Math.*, **118** (1983), 277–313.
[21] ______, Ergodic theory in hyperbolic space, *Contemporary Math.*, **26** (1984), 309–334.
[22] V. A. Rohlin, On the fundamental ideas of measure theory, *Mat. Sb.*, **25** (1949), 107–150 = *Amer. Math. Soc. Transl.* (1), **10** (1962), 1–54.
[23] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer, New York, 1984.
[24] D. Witte, Rigidity of some translations on homogeneous spaces, *Invent. Math.*, **81** (1985), 1–27.
[25] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.
[26] ______, Amenable actions and dense subgroups of Lie groups (preprint).