THE UNIVERSITY OF CHICAGO

MEASURABLE ISOMORPHISMS OF UNIPOTENT TRANSLATIONS ON HOMOGENEOUS SPACES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
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CHICAGO, ILLINOIS
JUNE, 1985

ACKNOWLEDGEMENTS

As is appropriate, the major influence on my mathematical development in the past few years has been that exerted by Robert J. Zimmer. Enthusiastically, I thank him for providing information, advice, and encouragement on numerous occasions. It has been very profitable, and (almost) always a pleasure, to be his student.

Of the many associates who have enriched my graduate career, I wish to mention Joseph A. Gallian and two of my fellow students: Scot Adams and José Pedrozo. I am indebted to John Rogawski for voting "thumbs up" at my thesis defense.

For helpful discussions on various aspects of my thesis research, I thank James Arthur, Louis Auslander, S. G. Dani, Calvin C. Moore, Dinakar Ramakrishnan, Marina Ratner, Melvin Rothenberg, Paul Sally, and Nolan Wallach. Also, thanks to Georgia Benkart and James Lepowsky, for suggesting I get help from Nolan Wallach.

My graduate work was generously supported by graduate fellowships from the NSF, the Robert R. McCormick Foundation, and the Alfred P. Sloan Foundation. Just so she won't think I've forgotten her, let me close by saying:

Hi Mom!

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CHAPTER 1

INTRODUCTION

Ergodic theory is the measure theoretic study of dynamical systems and of group actions. Irrational rotations of the circle present basic examples: Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the circle group; for any $\beta \in \mathbf{R}$, we have a transformation $T_\beta\colon \mathbf{T} \to \mathbf{T}\colon t \mapsto t + \beta$. The iterates $(T_\beta)^n$ $(n \in \mathbf{Z})$ of T_β form a dynamical system (with discrete time). If β is irrational, it is well known that every orbit of this dynamical system is uniformly distributed in $\mathbf{T}$—which is essentially what it means to say the dynamical system is "ergodic". Considering $\mathbf{T}$ as the quotient space $\mathbf{R}/\mathbf{Z}$, we see a natural family of generalizations: Let G be a connected Lie group and Γ a closed subgroup; any element $g \in G$ acts by translation T_g on the homogeneous space $\Gamma\backslash G$ (namely $T_g\colon \Gamma\backslash G \to \Gamma\backslash G\colon \Gamma x \mapsto \Gamma xg$). We will assume the homogeneous space $\Gamma\backslash G$ has finite volume; such examples include the classical geodesic and horocycle flows on a compact surface of constant negative curvature. The research of several authors culminated in 1981 in a joint paper of J. Brezin and C. C. Moore [3] where, among other things, it was determined which of these translations are ergodic. But much remains to be done; many basic properties of these examples still need to be investigated.

This paper deals with the so-called isomorphism question: Which translations are isomorphic? What are the isomorphisms between them? More concretely, consider ergodic translations T_g and T_h on finite-volume homogeneous spaces $\Gamma\backslash G$ and $\Lambda\backslash H$ of connected Lie groups G and H. When is there a measure preserving Borel isomorphism $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ with $T_g\psi = \psi T_h$? (Note that the author habitually writes his actions and maps on the right.)

If G and H are abelian, the answer to this question has been known for a long time: Suppose ψ is an isomorphism of T_g with T_h. Then the Lie groups $\Gamma\backslash G$ and $\Lambda\backslash H$ are isomorphic, and ψ (perhaps after being adjusted on a set of measure zero) is the composition of a translation with a Lie group isomorphism $\Gamma\backslash G \cong \Lambda\backslash H$. Thus ψ is an affine map (a.e.). In short, two ergodic translations on abelian groups are not measure theoretically isomorphic unless it is obvious from the algebraic setting that this is the case.

In 1971, W. Parry [17] proved a similar result for the case where G and H are nilpotent, and M. Ratner [19], in 1982, proved a theorem when $G = H = \mathrm{SL}_2(\mathbf{R})$. But Ratner needed to restrict the translations allowed—in her theorem it is necessary to assume g and h are unipotent matrices (i.e., 1 is the only eigenvalue). In fact this is the best possible result; every two-by-two matrix of determinant one is either semisimple (i.e., diagonalizable) or unipotent (or the negative of a unipotent matrix), and it had been shown by D. Ornstein and B. Weiss [16] in 1973 that any ergodic translation by a semisimple element is isomorphic to a so-called Bernoulli shift. In this paper we do not try to extend the work of Ornstein and Weiss, but we unify the theorems of Parry and Ratner to a general result valid for all connected Lie groups. For simplicity we state a slightly restricted version here (the general result appears in 10.1).

(1.1) *Definition.* An element u of a Lie group G is said to be *unipotent* if $\mathrm{Ad}_G u$ is a unipotent linear transformation of the Lie algebra of G.

(1.2) *Definition.* A homogeneous space $\Gamma\backslash G$ of a Lie group G is *faithful* if Γ contains no nontrivial normal subgroup of G.

(1.3) **Main Theorem.** *Let $\Gamma\backslash G$ and $\Lambda\backslash H$ be faithful finite-volume homogeneous spaces of connected real (finite dimensional) Lie groups G and H, and suppose T_u and T_v are ergodic translations on $\Gamma\backslash G$ and $\Lambda\backslash H$ by unipotent elements u and v of G and H respectively. If $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is a measure preserving Borel map which satisfies $T_u\psi = \psi T_v$ a.e., then ψ is an affine map (a.e.), i.e., there is a continuous homomorphism $G \to H\colon x \mapsto \tilde{x}$ and some $h \in H$ such that $\Gamma x\psi = \Lambda h \cdot \tilde{x}$ for a.e. $\Gamma x \in \Gamma\backslash G$.*

We can often construct ergodic actions of a connected Lie group G by embedding it in some other Lie group H: then G acts by translations on any finite-volume homogeneous space $\Lambda\backslash H$ of H. This construction is especially important when G is semisimple, because modifications of this supply all the known ergodic actions of G on manifolds. The original goal of this research was to settle the isomorphism question for these natural actions. The Main Theorem does this, and has other interesting consequences.

Application 1. Suppose G, H_1, H_2 are connected Lie groups, and let $\Lambda_i\backslash H_i$ be a faithful finite-volume homogeneous space of H_i. Embed G in H_1 and H_2. Assume G is connected, semisimple with no compact factors, and acts ergodically on $\Lambda_1\backslash H_1$. Then any measure theoretic isomorphism from the G-action on $\Lambda_1\backslash H_1$ to the G-action on $\Lambda_2\backslash H_2$ is an affine map (a.e.).

Definition. A discrete subgroup Γ of a Lie group G is a *lattice* if the homogeneous space $\Gamma\backslash G$ has finite volume.

Application 2. Suppose G, H_i, Λ_i are as in the preceding application, and let Γ be a lattice in G. Then any measure theoretic isomorphism of the Γ-actions on $\Lambda_1\backslash H_1$ and $\Lambda_2\backslash H_2$ is affine (a.e.).

Though little is known about the possible actions of a noncompact semisimple group on a manifold, the Main Theorem implies that the natural actions on a homogeneous space $\Lambda\backslash H$ can be picked out by the action of a single unipotent element:

Application 3. Let G and H be connected semisimple Lie groups, and Λ be a lattice in H. Assume each simple factor of G has real rank at least two. Suppose G acts measurably on $\Lambda\backslash H$ in an arbitrary way that preserves the finite measure. If some unipotent element of G acts by an ergodic translation of $\Lambda\backslash H$, then all elements of G act by translations (a.e.).

Entropy is an important (perhaps the most important) measure theoretic invariant of transformations preserving a finite measure. Since the translation by any unipotent element has zero entropy, it is natural to try to extend the Main Theorem to arbitrary translations of zero entropy. (Translation by $g \in G$ has zero entropy iff all eigenvalues of $\mathrm{Ad} g$ have absolute value one.) Unfortunately, an isomorphism of zero entropy ergodic translations need not be an affine map (a.e.). For example, certain nonabelian (solvable) groups give rise to the same translations as abelian groups; the isomorphism cannot be an affine map because the groups are not isomorphic. Even so, the methods of this paper should be sufficient to settle the isomorphism question for zero entropy ergodic translations on finite-volume homogeneous spaces of connected Lie groups.

CHAPTER 2

PRELIMINARIES IN LIE THEORY

The reader will need some familiarity with Lie theory and with algebraic groups defined over $\mathbf{R}$; as references the author suggests [22] and [11]. Save explicit mention to the contrary, all Lie groups and Lie algebras are real (and finite-dimensional).

(2.1) *Notation.* For a closed subgroup X of a Lie group G, we use $C_G(X)$, $N_G(X)$, and X°, respectively, to denote the centralizer, normalizer, and identity component (in the Hausdorff topology) of X. Thus

$$C_G(X) = \{\, g \in G \mid gx = xg \text{ for all } x \in X \,\} \qquad \text{and} \qquad N_G(X) = \{\, g \in G \mid gX = Xg \,\}.$$

We use $Z(G)$ to denote the center of G, i.e., $Z(G) = C_G(G)$. We use a corresponding script letter $\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$, ... to denote the Lie algebra of a Lie group $A, B, C, \ldots$. As is customary, we identify the Lie algebra of a subgroup of G with the corresponding subalgebra of $\mathcal{G}$.

(2.2) *Definition.* Two Lie groups G and H are *locally isomorphic* if they have isomorphic Lie algebras or, equivalently, if the universal cover of G° is isomorphic to the universal cover of H° (cf. [22, §2.8, pp. 72–74]).

(2.3) *Definitions.* A *real algebraic group* is a Lie group which is a subgroup of finite index in the real points of an algebraic group defined over $\mathbf{R}$. A Lie group is *locally algebraic* if it is locally isomorphic to some real algebraic group.

(2.4) *Definitions.* An element u of a Lie group G is *unipotent* if $\mathrm{Ad}_G u$ is a unipotent linear transformation on $\mathcal{G}$. A subgroup U of G is *unipotent* if each element of U is unipotent.

(2.5) *Caution.* The theory of algebraic groups provides a notion of unipotence for elements of a real algebraic group. To avoid hopeless confusion with the preceding definition, we refer in this context to elements (or subgroups) as being *algebraically* unipotent. (In a real algebraic group, every algebraically unipotent element is unipotent, but the converse fails if $Z(G)$ is not algebraically unipotent.)

(2.6) *Remark.* Any unipotent subgroup of a Lie group G is nilpotent (cf. Engel's Theorem [11, §V.2, pp. 63–67]). As a partial converse, any nilpotent normal subgroup of G is unipotent.

(2.7) *Definitions.* For any connected Lie group G, we let $\operatorname{rad} G$ (the *radical* of G) be the largest connected solvable normal subgroup of G, and $\operatorname{nil} G$ (the *nilradical* of G) be the largest connected nilpotent normal subgroup of G. Obviously $\operatorname{nil} G \subseteq \operatorname{rad} G$.

(2.8) **Lemma.** *Any Lie group G whose radical is nilpotent is locally isomorphic to an essentially unique connected real algebraic group whose radical and center are algebraically unipotent.*

Proof. The proof of [11, Theorem XVIII.1.1, p. 250] shows G is locally isomorphic to a connected real algebraic group whose radical and center are algebraically unipotent. It follows from (the proof of) [11, Theorem XVIII.2.2, p. 252] that a local isomorphism between any two such groups comes from an isomorphism of real algebraic groups. ∎

(2.9) *Definition.* Suppose G is a Lie group whose radical is nilpotent. If $\hat{G}$ is the universal cover of G, then there are covering homomorphisms $\pi\colon \hat{G} \to G$ and $\pi_a\colon \hat{G} \to G_a$, where G_a is a connected real algebraic group which is locally isomorphic to G, and $Z(G_a) \cdot \operatorname{nil} G_a$ is algebraically unipotent. A subset X of G is *Zariski closed* if X is a union of connected components (in the Hausdorff topology) of $C\pi_a^{-1}\pi$, where C is a closed subvariety of G_a.

(2.10) **Lemma.** *Let M be any finite-dimensional real or complex S-module, where S is a connected Lie group locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$. Let A be a split Cartan subgroup of S, and choose a maximal unipotent subgroup U of S normalized by A. The selection of U corresponds to an ordering of the weights of S (w.r.t. A), and this determines a decomposition $M = M^- \oplus M^0 \oplus M^+$ of M into the direct sum of its negative, zero, and positive weight spaces. Then:*

(a) $C_M(U) \subseteq M^0 + M^+$;

(b) *Any U-submodule of M contained in $M^- + M^0$ is contained in $C_M(S) \subseteq M^0$.*

Proof. Weyl's Theorem asserts M is completely reducible, so, by projecting to irreducible summands, we may assume M is irreducible. Any non-zero vector centralized by U is a maximal vector (cf. [12, §20.2, p. 108]). Since the highest weight of M is non-negative, this proves (a).

Any U-submodule contains a maximal vector of M. If the submodule is contained in $M^- + M^0$, this implies the highest weight of M is 0. Because an irreducible S-module is determined by its highest weight, we conclude that M is trivial, and (b) follows. ∎

(2.11) *Definition.* A Lie algebra $\mathcal{L}$ is *perfect* if it coincides with its derived algebra, i.e., if $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$. This is equivalent to the assertion that $\mathcal{L}$ has no nonzero abelian (or solvable) homomorphic images.

(2.12) *Definitions.* A *Borel* subalgebra of a complex Lie algebra $\mathcal{H}$ is a maximal solvable subalgebra, and any subalgebra containing a Borel subalgebra is said to be *parabolic*. A subalgebra $\mathcal{P}$ of a real Lie algebra $\mathcal{G}$ is *parabolic* if its complexification $\mathcal{P}\otimes\mathbf{C}$ is parabolic in $\mathcal{G}\otimes\mathbf{C}$. For a connected Lie group G, the normalizer $N_G(\mathcal{P})$ of any parabolic subalgebra of $\mathcal{G}$ is said to be a *parabolic subgroup* of G.

(2.13) **Lemma.** *For any Lie algebra $\mathcal{G}$, we have $C_{\mathcal{G}}(\operatorname{rad}\mathcal{G}) \subseteq Z(\mathcal{G}) + [\mathcal{G},\mathcal{G}]$.*

Proof. Let $\mathcal{L}$ be a Levi subalgebra of G. Since $\mathcal{L}$ is semisimple, Weyl's Theorem asserts every $\mathcal{L}$-module is completely reducible. Hence we may write $C_{\mathcal{G}}(\operatorname{rad}\mathcal{G}) = \mathcal{Z}\oplus V$, where $\mathcal{Z} = Z(\mathcal{G})$ is the centralizer of $\mathcal{L}$ in $C_{\mathcal{G}}(\operatorname{rad}\mathcal{G})$, and V is a sum of nontrivial irreducible $\mathcal{L}$-modules, so $V = [\mathcal{L}, V] \subseteq [\mathcal{G},\mathcal{G}]$. ∎

(2.14) **Lemma.** *If $\mathcal{P}$ is a parabolic subalgebra of a real or complex Lie algebra $\mathcal{G}$, then*

(i) *For any finite-dimensional $\mathcal{G}$-module V, we have $H^0(\mathcal{P},V) = H^0(\mathcal{G},V)$. In particular, $Z(\mathcal{P}) = Z(\mathcal{G})$;*

(ii) $[\mathcal{G},\mathcal{G}]\cap Z(\mathcal{G}) = [\mathcal{P},\mathcal{P}]\cap Z(\mathcal{P})$.

Proof. (i) We may assume $\mathcal{G}$ is a complex Lie algebra and $\mathcal{P}$ is Borel. Letting $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$, note that $\mathcal{P}\cap\mathcal{L}$ is Borel in $\mathcal{L}$. Since $H^0(\mathcal{P},V) = H^0(\mathcal{P}\cap\mathcal{L}, H^0(\operatorname{rad}\mathcal{G},V))$ and $H^0(\mathcal{G},V) = H^0(\mathcal{L}, H^0(\operatorname{rad}\mathcal{G},V))$, we may assume $\mathcal{G}=\mathcal{L}$ is semisimple. In this case, Weyl's Theorem asserts V is completely reducible, so we may assume V is irreducible.

Let $\mathcal{T}$ be a maximal toral subalgebra of $\mathcal{G}$ contained in $\mathcal{P}$. Then the Borel subalgebra $\mathcal{P}$ determines an ordering of the weights of $\mathcal{G}$ (w.r.t. $\mathcal{T}$). Letting λ be the maximal weight of V, we have $H^0(\operatorname{nil}\mathcal{P},V) = V_\lambda$. If $H^0(\mathcal{P},V)\neq 0$, we conclude that $\lambda = 0$, and hence V is the trivial $\mathcal{G}$-module.

(ii) Let $\overline{\mathcal{G}} = \mathcal{G}/[\operatorname{rad}\mathcal{G},\operatorname{rad}\mathcal{G}]$. Then $\operatorname{rad}\overline{\mathcal{G}}$ is abelian, so $[\overline{\mathcal{G}},\overline{\mathcal{G}}] = [\overline{\mathcal{G}},\overline{\mathcal{L}}]$ for any Levi subalgebra $\overline{\mathcal{L}}$ of $\overline{\mathcal{G}}$. Since $[\overline{\mathcal{G}},\overline{\mathcal{L}}]$ is the sum of the nontrivial irreducible $\overline{\mathcal{L}}$-submodules of $\overline{\mathcal{G}}$, then $\overline{\mathcal{L}}$ has trivial centralizer in $[\overline{\mathcal{G}},\overline{\mathcal{G}}]$. Therefore $[\overline{\mathcal{G}},\overline{\mathcal{G}}]\cap Z(\overline{\mathcal{G}}) = 0$, which implies $[\mathcal{G},\mathcal{G}]\cap Z(\mathcal{G}) \subseteq [\operatorname{rad}\mathcal{G},\operatorname{rad}\mathcal{G}]$. ∎

(2.15) *Definition.* A subgroup X of a Lie group G is *reductive* in G if the adjoint representation of X on $\mathcal{G}$ is completely reducible.

(2.16) *Definition.* Suppose U and U^- are maximal connected unipotent subgroups of a Lie group G whose radical is nilpotent. We say U^- is *opposite* to U if $(N_G(U^-)\cap N_G(U))/\operatorname{rad}G$ is reductive in $G/\operatorname{rad}G$.

(2.17) *Definition.* An involutive automorphism θ of a connected semisimple Lie group G is a *Cartan involution* if the fixed point set of θ is a maximal compact subgroup of G.

(2.18) **Proposition** (Mostow [18, §2.6, p. 11]). *An algebraic subgroup X of a connected semisimple real algebraic group G is reductive in G if and only if there is a Cartan involution $(*)$ of G with $X^* = X$.* ∎

(2.19) **Corollary.** *If U is a maximal connected unipotent subgroup of a connected semisimple real algebraic group G and $(*)$ is a Cartan involution of G, then U^* is opposite to U.*

Proof. Since U is connected, its normalizer is an algebraic subgroup of G, and the same goes for U^*. This means $N_G(U^*) \cap N_G(U)$ is an algebraic subgroup invariant under $(*)$. Therefore it is reductive. ∎

(2.20) *Definition.* Suppose g is an element of a Lie group G. Then

$$\{\, x \in G \mid g^{-n} x g^{n} \to e \text{ as } n \to +\infty \,\}$$

is a subgroup of G, called the *horospherical subgroup* associated to g.

(2.21) *Remark.* Any horospherical subgroup is unipotent. As a partial converse, any connected unipotent subgroup of a semisimple Lie group is contained in a horospherical subgroup.

(2.22) **Theorem** [24]. *Any path-connected subgroup of a Lie group is an immersed Lie subgroup.* ∎

(2.23) **Corollary.** *Suppose P is a parabolic subgroup of a connected Lie group G with $\mathbf{R}$-rank$(G/\operatorname{rad} G) = 1$. If V_1 is any one-parameter subgroup of G not contained in P, then $\langle P^\circ, V_1\rangle = G$.*

Proof. The Lie algebra of P is a parabolic subalgebra of $\mathcal{G}$. Since $\mathbf{R}$-rank$(G/\operatorname{rad} G) = 1$, this implies $\mathcal{P}$ is a maximal subalgebra of $\mathcal{G}$. Because $\mathcal{V}_1 \not\subseteq \mathcal{P}$, therefore $\langle \mathcal{V}_1, \mathcal{P}\rangle = \mathcal{G}$. Hence $\langle V_1, P^\circ\rangle = G$. ∎

(2.24) **Corollary.** *Suppose U and U^- are opposite maximal connected unipotent subgroups of a connected semisimple group G, and K is the maximal compact factor of G. Then $G = \langle U, U^-, K\rangle$.* ∎

(2.25) **Technical Lemma.** *Let K be a closed subgroup of a connected real Lie group G whose radical is nilpotent. Assume there is a closed normal subgroup N of G contained in K such that K projects to an Ad-precompact subgroup of G/N, and that K is normalized by the identity component P° of a parabolic subgroup of G. Then K is a normal subgroup of G.*

Proof. Passing to a quotient of G, we may assume K contains no normal subgroup of G. This implies $K \cap Z(G) = e$, and K is compact. Since $\operatorname{Ad}_G(K)$ (resp. $\operatorname{Ad}_G(\operatorname{rad} G)$) consists only of semisimple (resp. unipotent) elements, we have $K \cap \operatorname{rad} G = e$. Since K and $\operatorname{rad} G$ normalize each other, this implies $[K, \operatorname{rad} G] = e$.

Case 1. G is semisimple with trivial center.

Proof. Note that G is (isomorphic to) a real algebraic group [25, Proposition 3.1.6, p. 35]. Since $Z(G) = e$ and K is a closed Ad-precompact subgroup, K is compact, and hence K is an algebraic subgroup of G (cf. [25, p. 40]). Thus K is a reductive algebraic subgroup of G. This means there is a Cartan involution $(*)$ of G with $K^* = K$ (2.18). Then $N_G(K) = N_G(K)^*$. Since $P^\circ \subseteq N_G(K)$, then $N_G(K) \supseteq \langle P^\circ, (P^\circ)^* \rangle = G$.

Case 2. K is connected.

Proof. We wish to show $[\mathcal{G}, \mathcal{K}] \subseteq \mathcal{K}$. Since $\mathcal{K}$ is reductive in $\mathcal{G}$ and $[\mathcal{K}, \operatorname{rad} \mathcal{G}] = 0$, it suffices to show $[\mathcal{G}, \mathcal{K}] \subseteq \mathcal{K} + \operatorname{rad} \mathcal{G}$. Thus there is no loss in passing to the maximal semisimple quotient of G with trivial center. Then Case 1 applies.

Case 3. The general case.

Proof. Case 2 implies K° is normal in G, which implies K is discrete. Hence, showing K is normal is equivalent to showing K is central in G. We already know K centralizes $\operatorname{rad} \mathcal{G}$. Since K is reductive, then we need only show K centralizes $\mathcal{G} / \operatorname{rad} \mathcal{G}$. Thus we may assume G is semisimple (with trivial center), and Case 1 applies. ∎

CHAPTER 3

ISOMORPHISMS OF PARABOLIC SUBALGEBRAS

An isomorphism of parabolic subalgebras may or may not extend to an isomorphism of the ambient Lie algebras. (Proposition 3.1 shows that an extension, if it exists, is unique.) In this section we give a series of results which culminate in a criterion (Theorem 3.6) for the existence of an extension, under the assumption that one of the ambient Lie algebras is perfect. The section closes with a technical result (to be employed in the proof of the Main Theorem) on Lie algebras which are "perfect modulo the center."

(3.1) **Proposition.** *Let $\mathcal{P}$ be a parabolic subalgebra of a real or complex Lie algebra $\mathcal{G}$. Suppose $\sigma, \pi \colon \mathcal{G} \to \mathcal{H}$ are Lie algebra epimorphisms with the same restriction to $\mathcal{P}$, i.e., $\sigma|_{\mathcal{P}} = \pi|_{\mathcal{P}}$. Then $\sigma = \pi$.*

Proof. Let $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$, so it suffices to show $\sigma|_{\mathcal{L}} = \pi|_{\mathcal{L}}$. Set $\overline{\mathcal{H}} = \mathcal{H}/[\operatorname{rad}\mathcal{H}, \operatorname{rad}\mathcal{H}]$. Since $\overline{\mathcal{L}\pi}$ and $\overline{\mathcal{L}\sigma}$ are Levi subalgebras of $\overline{\mathcal{H}}$, they are conjugate via an inner automorphism $\alpha \in \operatorname{Exp}(\operatorname{ad}\ \operatorname{rad}\overline{\mathcal{H}})$ [22, Theorem 3.14.2, p. 226 (and Exercise 3.30, p. 252)]. Note that α centralizes $\overline{(\mathcal{L}\cap\mathcal{P})\pi}$. (For $p \in \mathcal{L}\cap\mathcal{P}$, we have $\overline{p\pi} - \overline{p\pi}\alpha \in \overline{\mathcal{L}\sigma}$ because $p\pi = p\sigma \in \mathcal{L}\sigma$ and $\overline{p\pi}\alpha \in \overline{\mathcal{L}\pi}\alpha = \overline{\mathcal{L}\sigma}$. On the other hand, $\alpha \in \operatorname{Exp}(\operatorname{ad}\ \operatorname{rad}\overline{\mathcal{H}})$ implies $\overline{p\pi} - \overline{p\pi}\alpha \in \operatorname{rad}\overline{\mathcal{H}}$. Therefore $\overline{p\pi} - \overline{p\pi}\alpha \in \overline{\mathcal{L}\sigma} \cap \operatorname{rad}\overline{\mathcal{H}} = 0$.) Since $\operatorname{rad}\overline{\mathcal{H}}$ is abelian and $\alpha \in \operatorname{Exp}(\operatorname{ad}\ \operatorname{rad}\overline{\mathcal{H}})$, we know that α also centralizes $\operatorname{rad}\overline{\mathcal{H}}$. Thus α centralizes $\overline{(\mathcal{L}\cap\mathcal{P})\pi} + \operatorname{rad}\overline{\mathcal{H}} = \overline{\mathcal{P}\pi}$. Writing $\alpha = \operatorname{Exp}(\operatorname{ad}\ r)$ with $r \in \operatorname{rad}\overline{\mathcal{H}}$, we conclude that r is in the center of the parabolic subalgebra $\overline{\mathcal{P}\pi}$ of $\mathcal{H}$, and hence r is in the center of $\mathcal{H}$ (see 2.14). Therefore α is trivial, and hence $\overline{\mathcal{L}\pi} = \overline{\mathcal{L}\sigma}$, i.e.,

$$\mathcal{L}\pi + [\operatorname{rad}\mathcal{H}, \operatorname{rad}\mathcal{H}] = \mathcal{L}\sigma + [\operatorname{rad}\mathcal{H}, \operatorname{rad}\mathcal{H}].$$

By induction on $\dim\mathcal{H}$, we conclude $\pi|_{\mathcal{L}} = \sigma|_{\mathcal{L}}$ as desired.

Perhaps a word should be said about the base case—when $\mathcal{H}$ is semisimple. By complexifying if necessary, we may assume $\mathcal{G}$ and $\mathcal{H}$ are complex Lie algebras, and then replacing $\mathcal{P}$ by a subalgebra if necessary, we may assume $\mathcal{P}$ is a Borel subalgebra of $\mathcal{G}$. It's easy to show $\operatorname{rad}\mathcal{G} \subseteq \ker\pi = \ker\sigma$, so the result follows from Chevalley's Theorem [12, Theorem 13.2, p. 75] on existence and uniqueness of extensions of isomorphisms of Borel subalgebras of semisimple Lie algebras. ∎

(3.2) **Lemma** (Wallach). *If $\mathcal{B}$ is parabolic in a real or complex semisimple Lie algebra $\mathcal{L}$, and if V is a finite-dimensional $\mathcal{L}$-module with no trivial submodules, then the first Lie algebra cohomology group of $\mathcal{B}$ with coefficients in V vanishes, i.e., $H^1(\mathcal{B}, V) = 0$.*

Proof. Weyl's Theorem asserts that any $\mathcal{L}$-module is completely reducible, so we may assume V is irreducible. Since $\mathcal{B}$ is parabolic in a semisimple Lie algebra, it is almost algebraic, in the sense that there is a subalgebra $\mathcal{M}$ complementary to the nilradical $\mathcal{N}$ of $\mathcal{B}$, and $\mathcal{M}$ is reductive in $\mathcal{L}$. We now apply the Hochschild-Serre spectral sequence [10, Exercise VIII.9.3, p. 305] to the decomposition $\mathcal{B} = \mathcal{M} + \mathcal{N}$ to determine $H^1(\mathcal{B}, V)$. In the E_2 term, there are two relevant groups: $E_2^{0,1} = H^0(\mathcal{M}, H^1(\mathcal{N}, V))$ and $E_2^{1,0} = H^1(\mathcal{M}, H^0(\mathcal{N}, V))$. We will show both of these groups vanish, for then the spectral sequence immediately yields $H^1(\mathcal{B}, V) = 0$.

The $\mathcal{M}$-module structure of $H^1(\mathcal{N}, V)$ is known [13, Theorem 5.14, p. 362]. A part of this structure is the fact that the highest weight α of any irreducible $\mathcal{M}$-submodule of $H^1(\mathcal{N}, V)$ is of the form

$$\alpha = (\lambda + \delta) w_\beta - \delta,$$

where δ is one-half the sum of the positive roots, λ is the highest weight of V, and w_β is the reflection corresponding to a simple root β. In particular, notice that 0 is not the highest weight of any $\mathcal{M}$-submodule. (Since $\delta w_\beta - \delta = -\beta = \beta w_\beta$ [12, Corollary to Lemma 10.2B, p. 50], $(\lambda + \delta) w_\beta - \delta = 0$ would imply $\lambda = -\beta$, and hence $<\lambda, \beta> < 0$. This would contradict the fact that λ, the highest weight of a finite-dimensional module, is dominant.) This means the trivial representation of $\mathcal{M}$ does not occur in $H^1(\mathcal{N}, V)$. In other words, $H^0(\mathcal{M}, H^1(\mathcal{N}, V)) = 0$, as desired.

Since $\mathcal{M}$ is reductive, we may write it as a direct sum $\mathcal{M} = \mathcal{S} \oplus \mathcal{A}$ of a semisimple and an abelian Lie algebra. We will apply the Hochschild-Serre spectral sequence to this decomposition to show $H^1(\mathcal{M}, H^0(\mathcal{N}, V)) = 0$. There are two relevant groups in the E_2 term: $E_2^{0,1} = H^0(\mathcal{A}, H^1(\mathcal{S}, \mathcal{X}))$ and $E_2^{1,0} = H^1(\mathcal{A}, H^0(\mathcal{S}, \mathcal{X}))$, where $\mathcal{X} = H^0(\mathcal{N}, V)$. Since Whitehead's Lemma asserts $H^1(\mathcal{S}, \mathcal{X}) = 0$, we need only consider the latter of these two groups. Now

$$H^0(\mathcal{S}, \mathcal{X}) = H^0(\mathcal{S}, H^0(\mathcal{N}, V)) = H^0(\mathcal{S} + \mathcal{N}, V).$$

Since V is a nontrivial irreducible $\mathcal{L}$-module and $\mathcal{B}$ is parabolic, we have $H^0(\mathcal{B}, V) = 0$ (2.14(i)). Therefore $H^0(\mathcal{S} + \mathcal{N}, V)$ has no trivial $\mathcal{A}$-submodules. Since $\mathcal{A} \subseteq \mathcal{M}$ is reductive in $\mathcal{L}$, this implies $H^0(\mathcal{S} + \mathcal{N}, V)$ is a direct sum of nontrivial irreducible $\mathcal{A}$-modules. Since $\mathcal{A}$ is abelian, it is then easy to see $H^1(\mathcal{A}, H^0(\mathcal{S} + \mathcal{N}, V)) = 0$. ∎

(3.3) **Lemma.** *Let $\mathcal{B}$ be a parabolic subalgebra of a real or complex semisimple Lie algebra $\mathcal{L}$. Suppose V and W are any finite dimensional $\mathcal{L}$-modules. If $\sigma: V \to W$ is a $\mathcal{B}$-module homomorphism, then σ is $\mathcal{L}$-equivariant.*

Proof. By Weyl's Theorem, we may assume V and W are irreducible $\mathcal{L}$-modules. By complexifying, we may assume $\mathcal{L}$ is a complex Lie algebra and $\mathcal{B}$ is Borel. Let $\mathcal{T} \subseteq \mathcal{B}$ be a Cartan subalgebra of $\mathcal{L}$. The Borel subalgebra $\mathcal{B}$ determines an ordering of the weights of $\mathcal{L}$ w.r.t. $\mathcal{T}$. Let λ_V and λ_W be the maximal weights of V and W respectively; and let

λ_V^- and λ_W^- be the maximal weights of V and W under the opposite ordering (the ordering associated to the opposite Borel $\mathcal{B}^-$). Note that μ is a weight of V if and only if it and all its conjugates μw under the Weyl group of $\mathcal{L}$ satisfy $\mu w \succ \lambda_V^-$ [12, Proposition 21.3, p. 114] (the inequality is reversed because λ_V^- is maximal with respect to the *opposite* ordering).

Since σ is $\mathcal{T}$-equivariant, it maps V_μ into W_μ, for any weight μ of V. We may assume $\sigma \neq 0$. Since $V_{\lambda_V^-}$ generates V as a $\mathcal{B}$-module, this implies $W_{\lambda_V^-} \neq 0$, and hence $\lambda_V^- \succ \lambda_W^-$. Of course λ_V is a weight of V, so it and all its conjugates $\lambda_V w$ satisfy $\lambda_V w \succ \lambda_V^-$. By transitivity, then $\lambda_V w \succ \lambda_W^-$, and hence λ_V is a weight of W. Therefore $\lambda_V \prec \lambda_W$.

Let $\mathcal{U} = \operatorname{nil} \mathcal{B}$. For some weight μ of V, we must have $V_\mu \sigma \neq 0$ and $V_\mu \mathcal{U} \sigma = 0$. This implies $V_\mu \sigma \mathcal{U} = 0$, so $\mu = \lambda_W$. Thus λ_W is a weight of V, and hence $\lambda_V \succ \lambda_W$.

The preceding two paragraphs jointly imply $\lambda_V = \lambda_W$, and hence V and W are isomorphic $\mathcal{L}$-modules. Now let $v \in V_{\lambda_V^-}$ (with $v \neq 0$). Then the unique $\mathcal{B}$-equivariant map (namely σ) taking v to $v\sigma$ is $\mathcal{L}$-equivariant. ∎

(3.4) **Lemma.** *Suppose $\mathcal{G}$ and $\mathcal{H}$ are real forms of a semisimple complex Lie algebra $\mathcal{L}$. If $\mathcal{P} = \mathcal{G} \cap \mathcal{H}$ is a (real) parabolic subalgebra of both $\mathcal{G}$ and $\mathcal{H}$, then $\mathcal{H} = \mathcal{G}$.*

Proof. Let $\mathcal{U} = \operatorname{nil} \mathcal{P}$ (so $\mathcal{P} = N_{\mathcal{G}}(\mathcal{U}) = N_{\mathcal{H}}(\mathcal{U})$). By Engel's Lemma there is a series of real vector subspaces $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_k$ of $\mathcal{G}$ with $C_{\mathcal{G}}(\mathcal{U}) = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_k = \mathcal{G}$ and $[\mathcal{G}_j, \mathcal{U}] \subseteq \mathcal{G}_{j-1}$. We will show by induction on j that

$$\mathcal{H} \cap (\mathcal{G}_j \otimes \mathbf{C}) \subseteq \mathcal{G}.$$

For $j = k$ this implies $\mathcal{H} = \mathcal{G}$ as desired.

The base case. Choose a maximal $\mathbf{R}$-split toral subalgebra $\mathcal{T}$ of $\mathcal{P}$. There is a (real) parabolic subalgebra $\mathcal{P}^-$ of $\mathcal{G}$ (resp. $\mathcal{Q}^-$ of $\mathcal{H}$) which contains $\mathcal{T}$ and is opposite to $\mathcal{P}$; let $\mathcal{U}^- = \operatorname{nil} \mathcal{P}^-$ and $\mathcal{V}^- = \operatorname{nil} \mathcal{Q}^-$. Now $\mathcal{U}^- \otimes \mathbf{C}$ and $\mathcal{V}^- \otimes \mathbf{C}$ are the sum of the same root spaces of $\mathcal{L}$ w.r.t. $\mathcal{T} \otimes \mathbf{C}$ (namely the negatives of the roots occurring in $\mathcal{U} \otimes \mathbf{C}$), so $\mathcal{V}^- \otimes \mathbf{C} = \mathcal{U}^- \otimes \mathbf{C}$. Therefore

$$\mathcal{H} = \mathcal{V}^- + \mathcal{P} \subseteq (\mathcal{V}^- \otimes \mathbf{C}) + \mathcal{P} = (\mathcal{U}^- \otimes \mathbf{C}) + \mathcal{P} = \mathcal{G} + i\mathcal{U}^-.$$

Since $\mathcal{G}_0 = C_{\mathcal{G}}(\mathcal{U}) \subseteq \mathcal{P}$, then we have

$$\mathcal{H} \cap (\mathcal{G} + i\mathcal{G}_0) \subseteq (\mathcal{G} + i\mathcal{U}^-) \cap (\mathcal{G} + i\mathcal{P}) = \mathcal{G}.$$

This implies $\mathcal{H} \cap (\mathcal{G}_0 \otimes \mathbf{C}) \subseteq \mathcal{G}$ as desired.

The induction step. Let $v \in \mathcal{H} \cap (\mathcal{G}_j \otimes \mathbf{C})$. Thus $v = x + iy$ with $x, y \in \mathcal{G}_j$. For all $u \in \mathcal{U}$:

$$\mathcal{H} \ni [v, u] = [x, u] + i[y, u] \in \mathcal{G}_{j-1} \otimes \mathbf{C}.$$

By induction $[x, u] + i[y, u] \in \mathcal{G}$. Hence $[y, u] = 0$. Since this holds for all $u \in \mathcal{U}$, we have $y \in C_{\mathcal{G}}(\mathcal{U}) = \mathcal{G}_0$, so $v \in \mathcal{H} \cap (\mathcal{G} + i\mathcal{G}_0)$. We showed in our proof of the base case that $\mathcal{H} \cap (\mathcal{G} + i\mathcal{G}_0) \subseteq \mathcal{G}$. ∎

(3.5) **Lemma.** *Suppose $\mathcal{G}$ and $\mathcal{H}$ are semisimple real or complex Lie algebras. Let $\mathcal{P}$ and $\mathcal{Q}$ be parabolic subalgebras of $\mathcal{G}$ and $\mathcal{H}$ respectively, and let $\sigma\colon \mathcal{P} \to \mathcal{Q}$ be a Lie algebra isomorphism. Then σ extends to an isomorphism of Lie algebras $\mathcal{G} \cong \mathcal{H}$.*

Proof. Suppose first that $\mathcal{G}$ and $\mathcal{H}$ are complex Lie algebras. Let $\mathcal{B} \subseteq \mathcal{P}$ be a Borel subalgebra of $\mathcal{G}$. The restriction $\sigma|_{\mathcal{B}}$ is an isomorphism of $\mathcal{B}$ with a maximal solvable subalgebra of $\mathcal{Q}$, i.e., with a Borel subalgebra of $\mathcal{H}$. Thus one knows that $\sigma|_{\mathcal{B}}$ extends uniquely to an isomorphism $\pi\colon \mathcal{G} \to \mathcal{H}$ [12, Theorem 14.2, p. 75]. It suffices to show $\sigma = \pi|_{\mathcal{P}}$. To this end, let $\mathcal{T} \subset \mathcal{B}$ be a Cartan subalgebra of $\mathcal{G}$. Now $\mathcal{P}\sigma$ and $\mathcal{P}\pi$ are the sum of the same weight spaces of $\mathcal{H}$ w.r.t. $\mathcal{T}\sigma$, because $\sigma|_{\mathcal{T}} = \pi|_{\mathcal{T}}$. So $\mathcal{P}\sigma = \mathcal{P}\pi$ and hence σ and $\pi|_{\mathcal{P}}$ are isomorphisms $\mathcal{P} \to \mathcal{P}\sigma$ that extend $\sigma|_{\mathcal{B}}$. Since Proposition 3.1 asserts this extension is unique we conclude $\sigma = \pi|_{\mathcal{P}}$. This completes the proof for complex Lie algebras.

We may now assume $\mathcal{G}$ and $\mathcal{H}$ are real Lie algebras. The previous paragraph shows there is an extension $\pi_{\mathbf{C}}\colon \mathcal{G} \otimes \mathbf{C} \to \mathcal{H} \otimes \mathbf{C}$ in the complexified setting. Then $\mathcal{G}\pi_{\mathbf{C}}$ and $\mathcal{H}$ are real forms of $\mathcal{H} \otimes \mathbf{C}$, and $\mathcal{G}\pi_{\mathbf{C}} \cap \mathcal{H}$ contains the real parabolic subalgebra $\mathcal{P}\sigma = \mathcal{Q}$. Therefore Lemma 3.4 asserts $\mathcal{G}\pi_{\mathbf{C}} = \mathcal{H}$, and hence the restriction of $\pi_{\mathbf{C}}$ to $\mathcal{G}$ is the desired extended isomorphism $\pi\colon \mathcal{G} \to \mathcal{H}$. ∎

(3.6) **Theorem.** *Let $\mathcal{G}$ and $\mathcal{H}$ be real or complex Lie algebras. Let $\mathcal{P}$ and $\mathcal{Q}$ be parabolic subalgebras of $\mathcal{G}$ and $\mathcal{H}$ respectively, and suppose $\sigma\colon \mathcal{P} \to \mathcal{Q}$ is a Lie algebra isomorphism such that $(\operatorname{rad}\mathcal{G})\sigma = \operatorname{rad}\mathcal{H}$. If $\mathcal{G}$ is perfect, then σ extends to a Lie algebra isomorphism $\pi\colon \mathcal{G} \to \mathcal{H}$.*

Proof. First recall the general fact that if $\mathcal{X}$ is a subalgebra of any nilpotent Lie algebra $\mathcal{R}$, with $\mathcal{X} + [\mathcal{R},\mathcal{R}] = \mathcal{R}$, then $\mathcal{X} = \mathcal{R}$. Suppose, to the contrary, that $\mathcal{X} < \mathcal{R}$, and let $\mathcal{M}$ be a maximal proper subalgebra of $\mathcal{R}$ which contains $\mathcal{X}$. Since $\mathcal{R}$ is nilpotent, we have $\mathcal{M} < N_{\mathcal{R}}(\mathcal{M})$, so the maximality of $\mathcal{M}$ implies $N_{\mathcal{R}}(\mathcal{M}) = \mathcal{R}$. Therefore $\mathcal{M}$ is an ideal in $\mathcal{R}$. Since $\mathcal{M}$ is maximal, $\dim \mathcal{R}/\mathcal{M} = 1$, so $\mathcal{R}/\mathcal{M}$ is abelian. Therefore $[\mathcal{R},\mathcal{R}] \subseteq \mathcal{M}$. Since also $\mathcal{X} \subseteq \mathcal{M}$, this contradicts the assumption that $\mathcal{X} + [\mathcal{R},\mathcal{R}] = \mathcal{R}$.

Using this fact, we can show that, for any Levi subalgebra $\mathcal{L}$ of $\mathcal{G}$, the $\mathcal{L}$-module $\operatorname{rad}\mathcal{G}$ has no trivial submodules. Since $\mathcal{G} = [\mathcal{G},\mathcal{G}] = \mathcal{L} + [\mathcal{G},\operatorname{rad}\mathcal{G}]$, we have $\operatorname{rad}\mathcal{G} = [\mathcal{G},\operatorname{rad}\mathcal{G}]$. As $[\mathcal{G},\operatorname{rad}\mathcal{G}] \subseteq \operatorname{nil}\mathcal{G}$ [22, Theorem 3.8.3(iii), p. 206] this implies $\operatorname{rad}\mathcal{G} = \operatorname{nil}\mathcal{G}$ is nilpotent. Since $\operatorname{rad}\mathcal{G} = [\mathcal{G},\operatorname{rad}\mathcal{G}] = [\mathcal{L},\operatorname{rad}\mathcal{G}] + [\operatorname{rad}\mathcal{G},\operatorname{rad}\mathcal{G}]$, this implies $\operatorname{rad}\mathcal{G} = [\mathcal{L},\operatorname{rad}\mathcal{G}]$. Because every $\mathcal{L}$-module is completely reducible this implies $\operatorname{rad}\mathcal{G}$ has no trivial $\mathcal{L}$-submodules, as desired.

Next we prove any two subalgebras complementary to $\operatorname{rad}\mathcal{G}$ in $\mathcal{P}$ are conjugate. To this end, set $\mathcal{B} = \mathcal{L} \cap \mathcal{P}$, a parabolic subalgebra of $\mathcal{L}$, and let $\mathcal{B}'$ be any other complement to $\operatorname{rad}\mathcal{G}$ in $\mathcal{P}$. We conclude from the preceding paragraph and Lemma 3.2 that $H^1(\mathcal{B}, W) = 0$ for every $\mathcal{L}$-submodule W of $\operatorname{rad}\mathcal{G}$. Setting $\overline{\mathcal{G}} = \mathcal{G}/[\operatorname{rad}\mathcal{G},\operatorname{rad}\mathcal{G}]$, we conclude $H^1(\mathcal{B}, \operatorname{rad}\overline{\mathcal{G}}) = 0$. Therefore $\overline{\mathcal{B}}$ is conjugate to $\overline{\mathcal{B}}'$ (via an inner automorphism of $\overline{\mathcal{P}}$). After replacing $\mathcal{B}'$ by a conjugate if necessary, we may then assume $\overline{\mathcal{B}} = \overline{\mathcal{B}}'$, i.e., $\mathcal{B} + [\operatorname{rad}\mathcal{G}, \operatorname{rad}\mathcal{G}] = \mathcal{B}' + [\operatorname{rad}\mathcal{G}, \operatorname{rad}\mathcal{G}]$. As $\mathcal{L} + [\operatorname{rad}\mathcal{G}, \operatorname{rad}\mathcal{G}]$ is a perfect proper subalgebra of $\mathcal{G}$, we conclude by induction on $\dim\mathcal{G}$ that $\mathcal{B}$ is conjugate to $\mathcal{B}'$ as desired.

Via the isomorphism σ, we conclude that any two subalgebras of $\mathcal{Q}$ complementary to $\operatorname{rad}\mathcal{H}$ are conjugate. In particular $\mathcal{B}\sigma$ is conjugate to $\mathcal{M} \cap \mathcal{Q}$, where $\mathcal{M}$ is any Levi subalgebra of $\mathcal{H}$ (and $\mathcal{B} = \mathcal{L} \cap \mathcal{P}$ as above). Replacing $\mathcal{M}$ by a conjugate if necessary, we may assume $\mathcal{B}\sigma = \mathcal{M} \cap \mathcal{Q}$ is contained in $\mathcal{M}$.

Now Lemma 3.5 implies $\sigma|_{\mathcal{B}}$ extends to an isomorphism $\sigma_1 : \mathcal{L} \to \mathcal{M}$. If we identify $\mathcal{L}$ with $\mathcal{M}$ under σ_1, then the restriction of σ to $\operatorname{rad}\mathcal{G}$ is $\mathcal{B}$-equivariant. Hence Lemma 3.3 asserts it is $\mathcal{L}$-equivariant. Therefore the linear map

$$\pi : \mathcal{G} \to \mathcal{H} : l + r \to l\sigma_1 + r\sigma \qquad (l \in \mathcal{L}, r \in \operatorname{rad}\mathcal{G})$$

is a Lie algebra isomorphism. ∎

(3.7) *Remark.* The assumption that $(\operatorname{rad}\mathcal{G})\sigma = \operatorname{rad}\mathcal{H}$ cannot be omitted. For example, if $\mathcal{P}$ is a proper parabolic subalgebra of a (perfect) Lie algebra $\mathcal{G}$, the isomorphism $\mathcal{P} \cong \mathcal{P}$ obviously does not extend to an isomorphism of $\mathcal{G}$ with $\mathcal{P}$.

(3.8) **Corollary.** *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are parabolic subalgebras of real or complex Lie algebras $\mathcal{G}$ and $\mathcal{H}$ respectively, and assume $[\mathcal{H}, \mathcal{H}] + Z(\mathcal{H}) = \mathcal{H}$. Then, for any isomorphism $\sigma : \mathcal{P} \to \mathcal{Q}$ with $(\operatorname{rad}\mathcal{G})\sigma = \operatorname{rad}\mathcal{H}$, there is an isomorphism $\pi : \mathcal{G} \to \mathcal{H}$, such that π and σ agree on $[\mathcal{P}, \mathcal{P}]$, and $p\pi \in p\sigma + Z(\mathcal{H})$ for all $p \in \mathcal{P}$.*

Proof. Let $\mathcal{Z}$ be complementary to $[\mathcal{P}, \mathcal{P}] \cap Z(\mathcal{P})$ in $Z(\mathcal{P})$; then $\mathcal{Z}\sigma$ is complementary to $[\mathcal{Q}, \mathcal{Q}] \cap Z(\mathcal{Q})$ in $Z(\mathcal{Q})$. Lemma 2.14 shows $\mathcal{Z}$ (resp. $\mathcal{Z}\sigma$) is complementary to $[\mathcal{G}, \mathcal{G}] \cap Z(\mathcal{G})$ in $Z(\mathcal{G})$ (resp. to $[\mathcal{H}, \mathcal{H}] \cap Z(\mathcal{H})$ in $Z(\mathcal{H})$). By hypothesis, $\mathcal{H} = [\mathcal{H}, \mathcal{H}] \oplus \mathcal{Z}\sigma$. Write $\sigma = \overline{\sigma} \oplus \sigma_{\mathcal{Z}}$, where $\mathcal{G}\overline{\sigma} = [\mathcal{H}, \mathcal{H}]$ and $\mathcal{G}\sigma_{\mathcal{Z}} = \mathcal{Z}\sigma$.

Since $[\mathcal{G}, \mathcal{G}] \cap \mathcal{Z} = 0$, we can choose a subalgebra $\overline{\mathcal{G}}$ complementary to $\mathcal{Z}$ in $\mathcal{G}$. Letting $\overline{\mathcal{G}}\pi_{\mathcal{Z}} = 0$ and $z\pi_{\mathcal{Z}} = z\sigma$ for $z \in \mathcal{Z}$, we obtain a homomorphism $\pi_{\mathcal{Z}} : \mathcal{G} \to \mathcal{Z}\sigma$ which agrees with $\sigma_{\mathcal{Z}}$ on $[\mathcal{P}, \mathcal{P}] + \mathcal{Z}$.

The kernel of $\overline{\sigma}$ is precisely $\mathcal{Z}$ (i.e., $g\sigma \in \mathcal{Z}\sigma$ iff $g \in \mathcal{Z}$), so $\overline{\sigma}$ factors through to an isomorphism $\overline{\sigma}'$ from $\mathcal{P} \cap \overline{\mathcal{G}}$ onto $\mathcal{Q} \cap [\mathcal{H}, \mathcal{H}]$. Now $[\mathcal{H}, \mathcal{H}]$ is perfect, because $\mathcal{H} = [\mathcal{H}, \mathcal{H}] + Z(\mathcal{H})$, so Theorem 3.6 asserts $\overline{\sigma}$ extends to an isomorphism $\overline{\pi}' : \overline{\mathcal{G}} \to [\mathcal{H}, \mathcal{H}]$. Extend this to a homomorphism $\overline{\pi} : \mathcal{G} \to [\mathcal{H}, \mathcal{H}]$ by setting $\mathcal{Z}\overline{\pi} = 0$.

The desired isomorphism $\pi : \mathcal{G} \to \mathcal{H}$ is $\pi = \overline{\pi} \oplus \pi_{\mathcal{Z}}$. ∎

CHAPTER 4

THE ALGEBRAIC STRUCTURE OF HOMOGENEOUS SPACES

By exploiting the structure theory for real algebraic groups, we now prove a number of technical results on finite-volume homogeneous spaces $\Gamma\backslash G$ (see [18] for an introduction to this subject). We work with two main ingredients: (1) Dani's theorem (4.4) on stabilizers of measures on homogeneous spaces, which we view as a powerful generalization of the Borel Density Theorem [18, Theorem 5.26(vi), pp. 87–88]; and (2) the method of splitting theorems and almost algebraic hulls, utilized so effectively by Auslander [1] in his study of homogeneous spaces of solvable groups. In particular, we show (by reducing to a known case) that every finite-volume homogeneous space is admissable; "the crux of this condition is that it allows us in some sense to decompose the analysis of [$\Gamma\backslash G$] into two pieces, one where G is semi-simple, the other where G is solvable" [3, p. 587].

We are mainly interested in the case where there is an ergodic translation on $\Gamma\backslash G$. We therefore often assume Γ projects densely into the maximal compact semisimple factor of G:

(4.1) **Theorem** [3, Theorem 5.5(1)]. *Suppose there is an ergodic translation on the finite-volume homogeneous space $\Gamma\backslash G$ of the connected Lie group G. Then Γ projects densely into the maximal compact semisimple factor of G.* ∎

(4.2) *Definitions.* Let $\Gamma\backslash G$ be a finite-volume homogeneous space of a Lie group G. We say $\Gamma\backslash G$ is *faithful* (resp. *a presentation*) if Γ contains no (resp. no connected) nontrivial normal subgroup of G.

Since one could mod out any normal subgroup contained in Γ, it usually causes no essential loss of generality to assume $\Gamma\backslash G$ is faithful. However, we often assume only that $\Gamma\backslash G$ is a presentation, so that we may also assume G is simply connected.

(4.3) *Definition.* A discrete subgroup Γ of a Lie group G is a *lattice* if $\Gamma\backslash G$ has a finite G-invariant measure.

(4.4) **Theorem** [8, Corollary 2.6]. *Suppose Γ^* is an algebraic subgroup of a real algebraic group G^*, and let ν be a finite measure on $\Gamma^*\backslash G^*$. Set*

$$G^*_\nu = \{\, g \in G^* \mid \textit{the } g\textit{-action on } \Gamma^*\backslash G^* \textit{ preserves } \nu \,\}$$

and

$$N^*_\nu = \{g \in G^* \mid sg = s \textit{ for all } s \in \operatorname{supp}\nu \,\}.$$

Then G^*_ν and N^*_ν are algebraic subgroups of G^*, and N^*_ν is a cocompact normal subgroup of G^*_ν. ∎

(4.5) **Corollary.** *Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a Lie group G, and $\pi\colon G \to \mathrm{GL}_n(\mathbf{R})$ is a finite-dimensional representation of G. Let G^* and Γ^* be the Zariski closures of $G\pi$ and $\Gamma\pi$, respectively, in $\mathrm{GL}_n(\mathbf{R})$. Then $\Gamma^*\backslash G^*$ has finite volume. Hence Γ^* contains a cocompact normal algebraic subgroup of G^*. In particular, Γ^* contains every algebraically unipotent element of G^*.*

Proof. The G-invariant probability measure μ on $\Gamma\backslash G$ pushes to a $G\pi$-invariant measure $\nu = \pi_*\mu$ on $\Gamma^*\backslash G^*$. Since $G\pi$ is Zariski dense, and because G^*_ν is algebraic, we must have $G^*_\nu = G^*$. Hence ν is a finite G^*-invariant measure on $\Gamma^*\backslash G^*$.

The support of the G^*-invariant probability measure ν is obviously all of $\Gamma^*\backslash G^*$; hence $N^*_\nu \subseteq \Gamma^*$. Any compact real algebraic group (e.g., $N^*_\nu\backslash G^*_\nu$) has no algebraically unipotent elements. Since $G^*_\nu = G^*$, this implies N^*_ν contains every algebraically unipotent element of G^*. ∎

(4.6) **Corollary.** *Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a Lie group G. Then every unipotent element of G normalizes (resp. centralizes) every connected subgroup of G normalized (resp. centralized) by Γ. In particular, $\mathrm{nil}\, G \subseteq N_G(\Gamma^\circ)$.*

Proof. If Γ normalizes a connected subgroup N, then $\mathrm{Ad}_G\Gamma$ stabilizes the Lie algebra of N. Hence the Zariski closure Γ^* of $\mathrm{Ad}_G\Gamma$ does so as well. If u is unipotent, the preceding corollary asserts $\mathrm{Ad}_G u \in \Gamma^*$, so we conclude that u normalizes N. ∎

(4.7) **Corollary.** *Suppose there is an ergodic unipotent translation on the finite-volume homogeneous space $\Gamma\backslash G$ of the Lie group G. Then Γ° is normal in G.*

Proof. Let u be an ergodic unipotent translation. Replacing u by a conjugate if necessary, we may assume $\Gamma\langle u\rangle$ is dense in G. Since $\Gamma \subseteq N_G(\Gamma^\circ)$ and because the preceding corollary asserts $u \in N_G(\Gamma^\circ)$, we have $\Gamma\langle u\rangle \subseteq N_G(\Gamma^\circ)$. Since $\Gamma\langle u\rangle$ is dense, this implies $G = N_G(\Gamma^\circ)$. ∎

(4.8) **Corollary.** *Let $\Gamma\backslash G$ be a finite-volume homogeneous space of a connected Lie group G, and assume Γ projects densely into the maximal compact factor of G. If H is a real algebraic group, and $\pi\colon G \to H$ is a continuous homomorphism, then the Zariski closure Γ^* of $\Gamma\pi$ contains every Levi subgroup of the Zariski closure G^* of $G\pi$ in H.*

Proof. We may assume $H = G^*$. Since any two Levi subgroups are conjugate by an element of the unipotent radical of G^*, and Corollary 4.5 implies Γ^* contains the unipotent radical, it suffices to find just one Levi subgroup of G^* contained in Γ^*. It therefore suffices to show $\Gamma^* \cdot \mathrm{rad}\, G^* = G^*$, so, passing to $G^*/\mathrm{rad}\, G^*$, we may assume $H = G^*$ is semisimple. Then $\mathrm{rad}\, G \subseteq \ker\pi$, so we may assume G is semisimple. Since connected semisimple

Lie subgroups of an algebraic group are algebraic [11, Theorem VIII.3.2, p. 112], we have $G\pi = G^*$. Now Γ is closed and contains both $\Gamma\pi$ and a cocompact normal subgroup of G^*, so, since Γ projects densely, we must have $\Gamma^* = G^*$. ∎

(4.9) **Corollary.** *Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a connected Lie group G whose radical is nilpotent, and assume Γ projects densely into the maximal compact semisimple factor of G. Then any closed connected subgroup of G normalized (resp. centralized) by Γ is normal (resp. central) in G. If Γ is discrete, this implies $\Gamma \cdot Z(G)$ is closed in G.*

Proof. Since $\operatorname{rad} G$ is nilpotent, Corollary 4.6 asserts the subgroup (call it N) is normalized (or centralized) by $\operatorname{rad} G$. In addition, it follows from Corollary 4.8 that every Levi subgroup of G normalizes (or centralizes) N. The Levi subgroups, together with $\operatorname{rad} G$, generate G.

Let H be the closure of $\Gamma \cdot Z(G)$ in G. Clearly $H \subseteq N_G(\Gamma)$. Hence H° is a connected subgroup of G which normalizes Γ. If Γ is discrete, this implies Γ centralizes H°, and hence $H^\circ \subseteq Z(G)$. We conclude $H^\circ = Z(G)^\circ$, from which it follows that $\Gamma \cdot Z(G)$ is closed. ∎

(4.10) *Definition.* Following [3, §4], we say that a finite-volume homogeneous space $\Gamma\backslash G$ is *admissable* if there is some connected closed solvable subgroup A of G, containing $\operatorname{rad} G$ and normalized by Γ, such that ΓA is closed.

(4.11) **Lemma.** *Suppose $\Gamma\backslash G$ is a finite-volume homogeneous space of a Lie group G, and N is a closed normal subgroup of G contained in Γ. If the homogeneous space $(\Gamma/N)\backslash(G/N)$ is admissable, then $\Gamma\backslash G$ is admissable.*

Proof. There is a connected closed solvable subgroup A_1/N of G/N, containing $\operatorname{rad}(G/N)$ and normalized by Γ/N, such that ΓA_1 is closed in G. Since A_1/N is solvable, we have $A_1 = N \operatorname{rad} A_1$, and hence $\Gamma \operatorname{rad} A_1 = \Gamma A_1$ is closed in G. Setting $A = \operatorname{rad} A_1$, we see that $\Gamma\backslash G$ is admissable. ∎

(4.12) **Theorem** (Auslander [18, Theorem 8.2.4, p. 149]). *If Γ is a lattice in a Lie group G, then $\Gamma\backslash G$ is an admissable finite-volume homogeneous space.*

(4.13) **Corollary.** *Every finite volume homogeneous space $\Gamma\backslash G$ of any Lie group G is admissable.*

Proof. Let $\Gamma\backslash G$ be a non-admissable finite-volume homogeneous space which is "minimal" in the following sense:

(1) $\dim G^\circ$ is minimal; and
(2) $\dim \Gamma^\circ$ is maximal, subject to (1).

Claim. The minimality of $\Gamma\backslash G$ implies:

(1) $\Gamma\backslash G$ is a presentation; and

(2) If N is a connected closed solvable subgroup of G normalized by Γ, with ΓN closed in G, then $N \subseteq \Gamma$.

Proof. (1) If Γ contains a connected normal subgroup N of G, then the minimality of $\dim G^\circ$ implies $(\Gamma/N)\backslash(G/N)$ is admissable. Hence Lemma 4.11 asserts $\Gamma\backslash G$ is admissable; a contradiction.

(2) If $N \not\subseteq \Gamma$, then $\dim(\Gamma N)^\circ > \dim \Gamma^\circ$, so the maximality of $\dim \Gamma^\circ$ implies $(\Gamma N)\backslash G$ is admissable. Hence there is a connected closed solvable subgroup A_1 of G, containing $\operatorname{rad} G$ and normalized by ΓN, such that $\Gamma N A_1$ is closed. Since N normalizes A_1, and each of N and A_1 is a solvable subgroup normalized by Γ, the product NA_1 is solvable and normalized by Γ. Setting $A = NA_1$, we easily deduce that $\Gamma\backslash G$ is admissable; a contradiction.

We now return to the proof of the corollary. Since Γ/Γ° is discrete, Auslander's theorem (together with Lemma 4.11) asserts $\Gamma\backslash N_G(\Gamma^\circ)$ is admissable; there is a connected closed solvable subgroup N of $N_G(\Gamma^\circ)$, containing $\operatorname{rad} N_G(\Gamma^\circ)$ and normalized by Γ, such that ΓN is closed. Part 2 of the claim asserts $N \subseteq \Gamma$. Therefore $\operatorname{rad} N_G(\Gamma^\circ) \subseteq \Gamma$. Since Corollary 4.6 asserts $\operatorname{nil} G \subseteq N_G(\Gamma^\circ)$, and because $\Gamma\backslash G$ is a presentation (part 1 of the claim), we conclude $\operatorname{nil} G = e$. Hence $\operatorname{rad} G = e$. So $\Gamma\backslash G$ is a non-admissable finite-volume homogeneous space of the *semisimple* group G. This is preposterous! (Let $A = e$.) ∎

(4.14) **Corollary.** *Let Γ be a lattice in a connected real Lie group G, and assume Γ projects densely into the maximal compact semisimple factor of G. Then:*

(1) $\Gamma \cap \operatorname{rad} G$ *is a lattice in* $\operatorname{rad} G$*; and*

(2) $\Gamma \cap \operatorname{nil} G$ *is a lattice in* $\operatorname{nil} G$.

Proof. (1) Corollary 4.8 implies $N = \overline{\Gamma \cdot \operatorname{rad} G}^\circ$ is normal in G. Since N is solvable (cf. Corollary 4.13), we conclude $N = \operatorname{rad} G$. Therefore Γ projects discretely into $G/\operatorname{rad} G$, and hence $\Gamma \cap \operatorname{rad} G$ is a lattice in $\operatorname{rad} G$ [18, Theorem 1.13, p. 23].

(2) Given (1), [18, Theorem 3.3, p. 46] implies $\Gamma \cap \operatorname{nil} G$ is a lattice in $\operatorname{nil} G$. ∎

(4.15) *Definition.* Let G be a simply connected Lie group. The Zariski closure $(\mathrm{Ad}G)^*$ of $\mathrm{Ad}G$ in $\mathrm{Aut}(\mathcal{G})$ has a Malcev decomposition $(\mathrm{Ad}G)^* = (L^* \times T^*)U^*$, where U^* is the unipotent radical, T^* is a torus (reductive in G^*), and L^* is a Levi subgroup of $(\mathrm{Ad}G)^*$. Let π be the map one gets by composing Ad with the projection of $(\mathrm{Ad}G)^*$ onto T^*, and set $T = G\pi$. Because G is simply connected, we can identify $\mathrm{Aut}\, G$ with $\mathrm{Aut}(\mathcal{G})$ and hence view T as a group of automorphisms of G. The resulting semi-direct product $G^\sharp = T[G]$ is the *big almost algebraic hull* of G. Note that $G^\sharp$ has the (semidirect product) decomposition $G^\sharp = (L \times T)[\operatorname{nil} G^\sharp]$, where L is a Levi subalgebra, and T is a (reductive) torus. Hence $G^\sharp$ is "almost algebraic." (This presentation is based on [3, §2]. For a more complete discussion, see [2].)

(4.16) *Remark.* If $\operatorname{rad} G$ admits a lattice, then $T = G\pi$ is closed in T^*.

Proof. Since any Levi subgroup of G is contained in the kernel of π, we have $G\pi = (\operatorname{rad} G)\pi$. Thus we may assume G is a solvable group, for which case we refer to [3, Corollary 2.3, p. 578]. ∎

(4.17) **Lemma.** *Let Γ be a lattice in a connected Lie group G that projects densely into the maximal compact semisimple factor of G. Then $\Gamma \cap (\mathrm{LEVI} \cdot \operatorname{nil} G)$ is a lattice in $\mathrm{LEVI} \cdot \operatorname{nil} G$ for any Levi subgroup LEVI of G. (Since $\mathrm{LEVI} \cdot \operatorname{nil} G$ is normal in G, this is equivalent to the assertion that Γ projects to a lattice in $G/(\mathrm{LEVI} \cdot \operatorname{nil} G)$.)*

Proof. Set $Z = Z(\operatorname{nil} G)$. Since $\operatorname{nil}(\mathrm{LEVI} \cdot C_G(Z)/Z) = (\operatorname{nil} G)/Z$, the result will follow by induction on the nilpotence class of $\operatorname{nil} G$ if we show $\Gamma \cap (\mathrm{LEVI} \cdot C_G(Z))$ is a lattice in $\mathrm{LEVI} \cdot C_G(Z)$. Equivalently, we will prove $\mathrm{LEVI} \cdot C_G(Z) \cdot \Gamma$ is closed in G.

We may assume G is simply connected, so Z, being a simply connected abelian group, is isomorphic to a Euclidean space $\mathbf{R}^n$. Thus the action of G by conjugation on Z yields a representation $\pi\colon G \to \mathrm{GL}_n(\mathbf{R})$. Since the kernel of π is precisely $C_G(Z)$, we need only show $(\mathrm{LEVI} \cdot \Gamma)\pi$ is closed in $\mathrm{GL}_n(\mathbf{R})$.

For any subgroup X of G, we write X^* for the Zariski closure of $X\pi$ in $\mathrm{GL}_n(\mathbf{R})$. From $\operatorname{nil} G \subseteq C_G(Z)$, it follows that $(\operatorname{rad} G)^*$ is abelian, and LEVI^* centralizes $(\operatorname{rad} G)^*$. Therefore LEVI^* is the only Levi subgroup of G^*, and $[\mathrm{LEVI}, \mathrm{LEVI}]^* = [G,G]^*$. Since Corollary 4.8 asserts $\mathrm{LEVI}^* \subseteq \Gamma^* \subseteq G^*$, we have

$$\mathrm{LEVI}^* = [\mathrm{LEVI}, \mathrm{LEVI}]^* \subseteq [\Gamma, \Gamma]^* \subseteq [G,G]^* = \mathrm{LEVI}^*.$$

We conclude that $\Gamma\pi \cap \mathrm{LEVI}^*$ is Zariski dense in LEVI^*.

There is no loss in assuming the isomorphism $Z \cong \mathbf{R}^n$ to be taken so that $\Gamma \cap Z$ corresponds to $\mathbf{Z}^n \subseteq \mathbf{R}^n$. As Γ normalizes $\Gamma \cap Z$, this implies $\Gamma\pi \subseteq \mathrm{GL}_n(\mathbf{Z})$. (In particular, $\Gamma\pi$ is discrete.) So the preceding paragraph implies LEVI^* contains a Zariski-dense set of $\mathbf{Z}$-points, which in turn implies LEVI^* is defined over $\mathbf{Q}$. Because LEVI^* is semisimple, then a theorem of Borel and Harish-Chandra [18, Theorem 13.28, p. 214] asserts $\mathrm{GL}_n(\mathbf{Z}) \cap \mathrm{LEVI}^*$ is a lattice in LEVI^*. It follows that $\mathrm{LEVI}^* \cdot \mathrm{GL}_n(\mathbf{Z})$ is closed in $\mathrm{GL}_n(\mathbf{R})$ (cf. proof of [18, Theorem 1.13, p. 23]). Since $\mathrm{LEVI}\pi$ is of finite index in LEVI^* and $\Gamma\pi \subseteq \mathrm{GL}_n(\mathbf{Z})$, we conclude that $\mathrm{LEVI}\pi \cdot \Gamma\pi$ is closed. ∎

(4.18) **Proposition.** *Suppose Γ is a lattice in a simply connected Lie group G which projects densely into the maximal compact semisimple factor of G. Let $G^\sharp = T[G] = (L \times T)[M]$ be the big almost algebraic hull of G, and let $\pi\colon G^\sharp \to T$ be the natural homomorphism with kernel LM. Then $\Gamma\pi$ is closed (and discrete) in T.*

Proof. Corollary 4.14 asserts $\Gamma_1 = \Gamma \cap \operatorname{rad} G$ is a lattice in $\operatorname{rad} G$, and it follows from Lemma 4.17 that $\Gamma_2 = \Gamma \cap (L \cdot \operatorname{nil} G)$ projects to a lattice in $G/\operatorname{rad} G$. Therefore $\Gamma_0 = \Gamma_1 \cdot \Gamma_2$ is a lattice in G. Replacing Γ by a subgroup of finite index (namely Γ_0), we may assume $\Gamma = \Gamma_1 \cdot \Gamma_2$. Then $\Gamma\pi = (\Gamma_1\pi) \cdot (\Gamma_2\pi) = \Gamma_1\pi$, since $\Gamma_2 \subset L \cdot \operatorname{nil} G \subseteq LM = \ker \pi$. Now Γ_1 is a lattice in the solvable group $\operatorname{rad} G$, so [3, Theorem 2.1, p. 576] asserts $\Gamma_1\pi$ is discrete. ∎

(4.19) **Theorem.** *Suppose* Γ *is a lattice in a connected Lie group* G*. If there is an ergodic unipotent translation on* $\Gamma\backslash G$*, then* $\operatorname{rad} G$ *is a unipotent subgroup of* G*. Hence* G *is locally algebraic.*

Proof (cf. [3, Corollary 2.5, p. 578]). We may assume G is simply connected. Let $G^\sharp = (L \times T)[M]$ be the big almost algebraic hull of G, and let $\pi\colon G^\sharp \to T$ be the natural homomorphism with kernel LM. Replacing the ergodic unipotent translation u by a conjugate if necessary, we may assume $\Gamma\langle u\rangle$ is dense in G. Since $u \in \ker\pi$, this implies $\Gamma\pi$ is dense in $G\pi$. But $G\pi$ is connected, and $\Gamma\pi$ is discrete (see 4.18), so we must have $G\pi = \Gamma\pi = 1$. Since $G\pi = T$, this implies $T = 1$, which means R is a nilpotent subgroup of G. ∎

(4.20) **Proposition.** *Let* Γ *be a lattice in a connected Lie group* G*. If* $(\operatorname{rad} G) \cdot \Gamma$ *is closed, and* $[G,G] \cdot \Gamma$ *is dense in* G*, then* $G = [G,G] \cdot Z(G)$*.*

Proof. In the notation of Proposition 4.18, we know $\Gamma\pi$ is descrete in T. On the other hand, since $[G,G] \subseteq \ker\pi$, it follows that $\Gamma\pi$ is dense in $G\pi$, which is connected. We conclude that $T = e$, and hence $\operatorname{rad} G$ is nilpotent.

The Lie algebra $\mathcal{R}$ of $\operatorname{rad} G$ is isomorphic to a Euclidean space $\mathbf{R}^n$, and thus the adjoint representation of G on $\mathcal{R}$ yields a representation $\pi\colon G \to \mathrm{GL}_n(\mathbf{R})$. We may assume G is simply connected, so that $\operatorname{rad} G$ is a simply connected nilpotent group, and hence the exponential map is a homeomorphism of $\mathcal{R}$ with $\operatorname{rad} G$. Since $\Gamma \cap \operatorname{rad} G$ is a lattice in $\operatorname{rad} G$, then the $\mathbf{Z}$-span of $\operatorname{Exp}^{-1}(\Gamma \cap \operatorname{rad} G)$ is a lattice in the vector space $\mathcal{R}$ (cf. [18, Theorem 2.12, p. 34]). There is no loss in assuming the isomorphism $\mathcal{R} \cong \mathbf{R}^n$ to be taken so that this lattice is identified with $\mathbf{Z}^n \subseteq \mathbf{R}^n$. Since Γ normalizes $\Gamma \cap \operatorname{rad} G$, this implies $\Gamma\pi \subseteq \mathrm{GL}_n(\mathbf{Z})$.

As $\operatorname{rad} G$ is nilpotent, we know G is locally algebraic, and hence $G\pi$ is Zariski closed in $\mathrm{GL}_n(\mathbf{R})$. Because Γ is a lattice in G and $\Gamma\pi \subseteq \mathrm{GL}_n(\mathbf{Z})$, we conclude that $\Gamma\pi$ is an arithmetic lattice in $G\pi$. Hence $\Gamma\pi \cap [G\pi, G\pi]$ is a lattice in $[G\pi, G\pi]$, and therefore $([G,G]\cdot\Gamma)\pi$ is closed in $\mathrm{GL}_n(\mathbf{R})$. Since $[G,G]\cdot\Gamma$ is dense in G, this implies $([G,G]\cdot\Gamma)\pi = G\pi$, so $C_G(\operatorname{rad} G) \cdot [G,G] \cdot \Gamma = G$ because $\ker\pi = C_G(\operatorname{rad} G)$. As G is connected, we must have $C_G(\operatorname{rad} G)^\circ \cdot [G,G] = G$. Lemma 2.13 shows $C_G(\operatorname{rad} G) \subseteq Z(G) \cdot [G,G]$, so we conclude $Z(G) \cdot [G,G] = G$ as desired. ∎

(4.21) *Definition.* If a lattice Γ in a semisimple Lie group G projects densely into G/N, for every closed noncompact normal subgroup N of G, then Γ is *irreducible.*

(4.22) **Proposition** [18, Theorem 5.22 (and 5.26(vii))]. *Let* Γ *be a lattice in a connected semisimple Lie group* G*, and suppose* Γ *projects densely into the maximal compact factor of* G*. Then there are closed normal subgroups* $G_1, G_2, \ldots, G_n$ *of* G *such that:*

(1) *The product homomorphism*

$$G_1 \times \cdots \times G_n \to G\colon (g_1, \ldots, g_n) \mapsto g_1 g_2 \ldots g_n$$

is surjective and has finite kernel;

(2) $\Gamma \cap G_i = \Gamma_i$ *is an irreducible lattice in* G_i*; and*

(3) $\prod_1^n \Gamma_i$ *is a subgroup of finite index in* Γ. ∎

(4.23) **Lemma.** *Any faithful lattice in a connected Lie group has a torsion-free subgroup of finite index.*

Proof. Any lattice Γ in a connected Lie group G is finitely generated [18, Remark 6.18, pp. 99–100]. Therefore $\mathrm{Ad}_G \Gamma$ is a finitely generated subgroup of the linear group $\mathrm{Ad} G$, and hence $\mathrm{Ad}_G \Gamma$ has a torsion-free subgroup of finite index [18, Theorem 6.11, p. 93]. If Γ is faithful, we have $\Gamma \cong \mathrm{Ad}_G \Gamma$, so Γ, like $\mathrm{Ad}_G \Gamma$, has a torsion-free subgroup of finite index. ∎

(4.24) **Lemma** [5, Lemma 2.1, p. 258]. *Let* Γ *be a lattice in a connected Lie group* G. *If* CPCT *is any compact subset of* $\Gamma \backslash G$*, then no small element of* G *has a fixed point in* CPCT. *(More precisely, there is a neighborhood* B *of* e *in* G *such that* $g^{-1} \Gamma g \cap B = \emptyset$ *whenever* $\Gamma g \in \mathrm{CPCT}$*.)* ∎

CHAPTER 5

PRELIMINARIES ON AFFINE MAPS

(5.1) *Definitions.* Let $\Gamma\backslash G$ and $\Lambda\backslash H$ be finite-volume homogeneous spaces of Lie groups G and H. Suppose $g \in G$, $h \in H$, and $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is measure preserving. We say ψ is *affine for* g (via h) if, for a.e. $s \in \Gamma\backslash G$, we have $sg\psi = s\psi h$. If $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is affine for g via h, and $\Lambda\backslash H$ is faithful, then the element h is uniquely determined by g. In note of this we often write $h = \tilde{g}$. Set

$$\operatorname{Aff}_G(\psi) = \{\, g \in G \mid \psi \text{ is affine for } g \,\}.$$

We often say ψ is *affine for* X when X is a subset of $\operatorname{Aff}_G(\psi)$.

(5.2) **Lemma.** *Suppose $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is measure preserving, and $\Lambda\backslash H$ is faithful. Then $\operatorname{Aff}_G(\psi)$ is an immersed Lie subgroup of G, and the function $\sim\colon \operatorname{Aff}_G(\psi) \to H\colon g \mapsto \tilde{g}$ is a continuous homomorphism (where $\operatorname{Aff}_G(\psi)$ is given its Lie group topology).*

Proof. It's instructive to view $\operatorname{Aff}_G(\psi)$ from another perspective: Let $F(\Gamma\backslash G, \Lambda\backslash H)$ be the space of measurable functions $\Gamma\backslash G \to \Lambda\backslash H$, two functions being identified if they agree almost everywhere. Convergence in measure defines a topology on $F(\Gamma\backslash G, \Lambda\backslash H)$ which is metrizable by a complete separable metric, and it is easy to check that $G \times H$ acts continuously on $F(\Gamma\backslash G, \Lambda\backslash H)$ via the action $(x)[\zeta \cdot (g,h)] = xg^{-1}\zeta h$ for $\zeta \in F(\Gamma\backslash G, \Lambda\backslash H)$, $g \in G$, $h \in H$. (The above is excerpted from [25, §3.3, pp. 49–50].) Letting $\operatorname{Stab}(\psi)$ be the stabilizer of $\psi \in F(\Gamma\backslash G, \Lambda\backslash H)$ under this action of $G \times H$, we see that $\operatorname{Stab}(\psi)$ is the graph of $(\sim)$.

Since stabilizers are closed, $\operatorname{Stab}(\psi)$ is a closed subgroup of $G \times H$, and hence is a Lie group. The projection $G \times H \to G$ restricts to an immersion of $\operatorname{Stab}(\psi)$ into G whose range is the domain of $(\sim)$, i.e., whose range is $\operatorname{Aff}_G(\psi)$. Therefore $\operatorname{Aff}_G(\psi)$ is an immersed Lie subgroup. Lifted to $\operatorname{Stab}(\psi)$, the function $\sim\colon \operatorname{Stab}(\psi) \to H$ is simply the restriction of the projection $G \times H \to H$. This is a continuous homomorphism. ∎

(5.3) *Definition.* Suppose Ω is a g-invariant Borel subset of $\Gamma\backslash G$, where $g \in G$. We say $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is *strictly affine* for g (via $\tilde{g}$) on Ω if $sg\psi = s\psi\tilde{g}$ for every $s \in \Omega$ (not just a.e.).

(5.4) *Remark.* The map $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ can be modified on a null set to become strictly affine for $\operatorname{Aff}_G(\psi)$ on a conull subset of $\Gamma\backslash G$ [25, Proposition B.5, p. 198]. More precisely there is a conull $\mathrm{Aff}_G(\psi)$-invariant subset Ω of $\Gamma\backslash G$ and a measurable $\psi_0\colon \Gamma\backslash G \to \Lambda\backslash H$ such that $\psi = \psi_0$ a.e. and ψ_0 is strictly affine for each $g \in \mathrm{Aff}_G(\psi)$ on Ω. In particular if $\mathrm{Aff}_G(\psi) = G$, then ψ_0 is an affine map.

CHAPTER 6

PRELIMINARIES IN ERGODIC THEORY

(6.1) **Lemma** [6, p. 136]. *Let g be a translation on $\Gamma\backslash G$, where Γ is a lattice in the connected Lie group G. Then the translation has zero entropy iff all eigenvalues of* Adg *have absolute value one.* ∎

(6.2) **Theorem** ("The Mautner Phenomenon" [15, Theorem 1.1, p. 156]). *Let Γ be a lattice in a connected Lie group G. For any connected subgroup M of G, let N be the smallest connected normal subgroup of G such that M projects to an* Ad*-precompact subgroup of G/N. Then any M-invariant measurable function on $\Gamma\backslash G$ is N-invariant.* ∎

(6.3) **Lemma.** *Let Γ be a lattice in a connected Lie group G whose radical is nilpotent, and assume Γ projects densely into the maximal compact semisimple factor of G. Suppose ψ is a measurable function on $\Gamma\backslash G$ which is N-invariant, for some normal subgroup N of G. Then there is a normal subgroup N_1 of G containing N, such that ψ is N_1-invariant, and $N_1\Gamma$ is closed in G.*

Proof. Let N_0 be the identity component of the closure of $N\Gamma$. Since N_0 is a connected subgroup of G normalized by Γ, Corollary 4.9 implies N_0 is normal in G. Now ψ corresponds to a function ψ' on G which is (essentially) left-invariant under $N\Gamma$, so ψ' is left-invariant under N_0. Because N_0 is normal, then ψ' is also right-invariant under N_0, so ψ is N_0-invariant. Set $N_1 = NN_0$. ∎

(6.4) **Corollary.** *Suppose Γ is a lattice in a connected Lie group G whose radical is nilpotent, and assume Γ projects densely into the maximal compact semisimple factor of G. For V a connected unipotent subgroup of G, let N be the smallest closed normal subgroup of G containing V and such that $N\Gamma$ is closed. Then the N-orbits are the ergodic components of the action of V by translation on $\Gamma\backslash G$.*

Proof. Since $\Gamma\backslash G/N$ is countably separated, it suffices to show any V-invariant measurable function on $\Gamma\backslash G$ is (essentially) N-invariant. To this end, let f be a V-invariant measurable function. The Mautner Phenomenon (6.2) implies f is essentially N_0-invariant, where N_0 is the smallest normal subgroup of G such that V projects to an Ad-precompact subgroup of G/N_0. Since V is unipotent, this implies V projects to a central subgroup of G/N_0, and hence VN_0 is normal in G. Since VN_0 stabilizes ψ, Lemma 6.3 asserts N stabilizes ψ. ∎

(6.5) **Corollary** ("Moore Ergodicity Theorem", cf. [25, Theorem 2.2.6, p. 19]). *Suppose Γ is an irreducible lattice in a connected semisimple Lie group G. If X is a connected subgroup of G, then either X is* Ad*-precompact, or X is ergodic on $\Gamma\backslash G$.* ∎

(6.6) **Corollary** (cf. [3, Theorem 6.1, p. 601]). *Suppose Γ is a lattice in a connected Lie group G. If X is a connected subgroup of G which is ergodic on both the maximal solvmanifold quotient and the maximal semisimple quotient of $\Gamma\backslash G$, then X is ergodic on $\Gamma\backslash G$.* ∎

(6.7) **Lemma.** *Suppose Λ is a faithful lattice in a connected locally algebraic group H. If h is an ergodic translation on $\Lambda\backslash H$, then $C_H(h)$ is essentially free on $\Lambda\backslash H$. I.e. there is a conull $C_H(h)$-invariant subset $\Omega \subseteq \Lambda\backslash H$ such that if $sc = s$ with $s \in \Omega$ and $c \in C_H(h)$, then $c = e$.*

Proof. For each $\lambda \in \Lambda - \{e\}$, set $[\lambda] = \{\, x \in H | x^{-1}\lambda x \in C_H(h)\,\}$. Since $C_H(h)$ is free off the union of the countably many sets $\Lambda[\lambda] \subseteq \Lambda\backslash H$, it suffices to show each $[\lambda]$ is a null set. Now $[\lambda]$ is a countable union of Zariski closed subsets of H (since $C_H(h)^\circ$ is Zariski closed), so if non-null, we have $[\lambda] = H$. Then every conjugate of λ belongs to $C_H(h)$, so $C_H(h)$ contains a closed normal subgroup N of H that intersects Λ. Now $C_H(N)$ is a normal subgroup of H that is ergodic on $\Lambda\backslash H$ (since $h \in C_H(N)$). Therefore $\Lambda C_H(N)$ is dense in H. Because $\Lambda C_H(N)$ normalizes $\Lambda \cap N$, this implies $\Lambda \cap N$ is normal in H—contradicting the assumption that Λ is a faithful lattice. ∎

CHAPTER 7

POLYNOMIAL DIVERGENCE OF ORBITS

(7.1) *Notation.* Suppose Γ is a lattice in a Lie group G. Choose a left-invariant topological metric d on G, and project to a metric on $\Gamma\backslash G$: for $x, y \in G$ we have

$$d(\Gamma x, \Gamma y) = \min_{\gamma \in \Gamma} d(x, \gamma y).$$

If T is a compact subgroup of G, we may assume d is T-invariant. Then we have a metric on G/T:

$$d(xT, yT) = \min_{t \in T} d(x, yt).$$

(7.2) *Notation.* If G_a is a real algebraic group, then there is a non-negative polynomial function $G_a \to \mathbf{R}\colon g \mapsto \|g\|$ such that the $\|\cdot\|$-balls form a basis for the Hausdorff topology at e. For example, we could embed G_a in some special linear group $\mathrm{SL}_n(\mathbf{R})$ and let $\|g\|$ be the sum of the squares of the coordinates of $g - \mathrm{Id}$ considered as a vector in n^2-space.

We now present a similar construction when G is a simply connected locally algebraic Lie group. Let G_a be a connected real algebraic group locally isomorphic to G (with $\operatorname{nil} G_a$ algebraically unipotent), and let $\pi\colon G \to G_a$ be a covering map. Abusing notation, we write $\|g\|$ for $\|g\pi\|$ when $g \in G$. Since the $\|\cdot\|$-balls form a basis in G_a, there is a neighborhood B of e in G small enough that the family of sets $\{\, g \in G\colon \|g\| < \epsilon \,\} \cap B$ forms a basis for the topology at e in G.

(7.3) *Remark.* Suppose u is a unipotent element of a connected locally algebraic group G. Any unipotent element of the real algebraic group $\mathrm{Ad}G \cong G/Z(G)$ lies in a unique one-parameter unipotent subgroup, so there is a one-parameter unipotent subgroup $v^r \quad (r \in \mathbf{R})$ of G such that $v^1 \in u \cdot Z(G)$. Furthermore, if $\overline{v}^r$ is any other such subgroup, then $\overline{v}^r \in v^r \cdot Z(G)$ for all $r \in \mathbf{R}$. So, for $y, c \in G$ and $r \in \mathbf{R}$, the expression $v^{-r} y v^r [v^r, c]$ does not depend on the choice of the one-parameter subgroup v^r. Thus, given $x, y \in G$, even though u itself may not lie in a one-parameter subgroup, there is no ambiguity when we write an expression such as $d(xu^r, yu^r[u^r, c])$, for $r \in \mathbf{R}$.

(7.4) *Remark.* For convenience, we often write $d_p(x, y)$ for $\|x^{-1}y\|$. The non-metric d_p is useful to showcase the *polynomial divergence of orbits*: if u^r is a unipotent element of G, then for any fixed $x, y, c \in G$, $d_p(xu^r, yu^r[u^r, c])$ is a polynomial function of r. Notice the degree of the polynomial is bounded by a constant which is independent of x, y, and c—the bound depends only on the unipotent element u. This is because the matrix entries of a one-parameter unipotent subgroup of a real algebraic group are polynomial [25, Proposition 3.4.1, p. 53] and $d_p(a,b)$ is a polynomial function of the matrix entries of a and b.

(7.5) **Lemma.** *Suppose T is a connected compact subgroup of a real algebraic group H. Then H/T is a quasi-affine variety, so there is a polynomial $H/T \to \mathbf{R}: hT \mapsto \|h\|_T$ such that the $\|\cdot\|_T$-balls form a basis for the Hausdorff topology at the point eT in H/T.*

Proof. Since T is compact, it is an algebraic subgroup of H [25, p. 40]. Thus there is an $\mathbf{R}$-representation $\pi: H \to \mathrm{GL}(V)$ of H such that T is the stabilizer of a one-dimensional subspace $\mathbf{R}v$ of V. A connected compact group has no $\mathbf{R}$-characters, so T centralizes $\mathbf{R}v$. Therefore T is the centralizer of v in H, and hence the variety H/T is isomorphic to the orbit of v in V, which is a locally closed subset of V. Put any inner product on V, and let $\|h\|_T$ be the square of the norm of $hv - v$. ∎

(7.6) *Remark.* Let DEG be a natural number. The collection of (real) polynomials of degree at most DEG which satisfy $|r \cdot f(r)| \leq 1$ for all $r \in [0,1]$ is a compact set. So there is a constant $M = M(\mathrm{DEG})$ such that each of these polynomials satisfies $|f(r)| \leq M$ on $[0,1]$. By translation and rescaling, we see that if f is any polynomial of degree at most DEG which satisfies $|r \cdot f(r)| \leq \epsilon$ for $r \in [R, R+\delta]$ (some $R \in \mathbf{R}$ and $\epsilon, \delta > 0$), then $|f(r)| \leq M\epsilon/r_0$ for $r \in [R, R+\delta]$, where M is a constant depending only on DEG. In the sequel, arguments like this will be referred to as *by compactness*. Such results are important because any one-parameter unipotent flow has polynomial divergence of orbits.

(7.7) *Definition.* The *upper density* of a set E of positive reals is

$$\limsup_{A\to\infty} \frac{\mu(E \cap [0,A])}{A}.$$

The *lower density* is a similar $\liminf$.

(7.8) *Notation.* We use the physicist's notation " $\doteq$ " for "close to"; i.e., $x \doteq y$ if $d(x,y) < \epsilon$ for some implicitly assumed $\epsilon > 0$. A statement $P(n)$ is true for *most* integers n if $\{\, n \in \mathbf{Z} \mid P(n) \text{ is true } \}$ has lower density $> 1/2$.

(7.9) **Lemma** ("Ratner Covering Lemma"). *Suppose $\{E_1, E_2, \dots\}$ is a collection of subsets of the real line, I is an interval on the line, D is a natural number, and $\beta > 0$. Assume the following conditions are satisfied:*

(1) *Each E_i has $\leq D$ connected components.*
(2) *The union of the sets $E_1, E_2, \dots$ has relative measure greater than β on I.*
(3) *The sets $(2D/\beta) \cdot E_1, (2D/\beta) \cdot E_2, \dots$ are pairwise disjoint, where*

$$\rho \cdot E = \bigcup \{\, [a - \rho w, a + \rho w] \mid [a - w, a + w] \subseteq E \,\}.$$

Then some set $(2D/\beta) \cdot E_{i_0}$ *covers the whole interval* I.

Proof (cf. [19, Lemma 2.1]). Let μ be Lebesgue measure on I (thus $\mu(E) = \mu(E \cap I)$ for any subset E of $\mathbf{R}$). We may assume I is bounded (so $\mu(I) < \infty$). The union of the E_i has relative measure greater than β on I and the sets $(2D/\beta) \cdot E_1$, $(2D/\beta) \cdot E_2, \ldots$ are pairwise disjoint, so there is some $E = E_{i_0}$ with $\mu((2D/\beta) \cdot E) < \mu(E)/\beta$. Let J be the largest component of $E \cap I$, so $\mu(J) \geq \mu(E)/D$. If $(2D/\beta) \cdot J$ does not cover I, then $\mu((2D/\beta) \cdot J) \geq (D/\beta)\mu(J)$. Hence $\mu((2D/\beta) \cdot E) \geq \mu((2D/\beta) \cdot J) \geq (D/\beta)\mu(J) \geq (D/\beta)\mu(E)/D = \mu(E)/\beta$. This is a contradiction. ∎

CHAPTER 8

THE MAIN LEMMA: AFFINE FOR THE RELATIVE CENTRALIZER

(8.1) *Definition.* Given an element x and a subgroup Y of a Lie group G. For $\delta > 0$, we set

$$C_G(x, Y; \delta) = \{\, c \in G \mid d(e, c) < \delta, [x^n, c] \in Y \text{ for all } n \in \mathbf{Z} \,\}.$$

The descending chain condition on connected Lie subgroups of G implies there is some $\delta_0 > 0$ such that $\langle C_G(x, Y; \delta)\rangle^\circ = \langle C_G(x, Y; \delta_0)\rangle^\circ$ whenever $0 < \delta < \delta_0$. We set $C_G(x, Y) = \langle C_G(x, Y; \delta_0)\rangle^\circ$, and call this the *centralizer* of x *relative* to Y in G. This terminology is motivated by the fact that if Y is normal in G (and connected), then $C_G(x, Y)/Y$ is the identity component of the centralizer of xY in G/Y.

(8.2) **Lemma** ("Affine for the Relative Centralizer"). *Suppose Γ (resp. Λ) is a lattice in a connected real Lie group G (resp. H) whose radical is nilpotent. Let u be an ergodic unipotent element of G and assume ψ is affine for u via a unipotent element $\tilde{u}$ of H. If U is any connected unipotent subgroup of G contained in $\mathrm{Aff}_G(\psi)$, then ψ is affine for $C_G(u, U)$.*

The remainder of this section constitutes a proof of Lemma 8.2.

(8.3) **Assumption.** *Let us assume $\tilde{U}$ is a unipotent subgroup of H. (Proposition 8.4 will eliminate this hypothesis.)*

Notation. Let CPCT be a large compact subset of $\Lambda\backslash H$ (say $\mu(\mathrm{CPCT}) > .9$), and choose some $\epsilon_0 > 0$ so small that if $d(h, \lambda h) < 2\epsilon_0$, with $\lambda \in \Lambda$ and $\Lambda h \in \mathrm{CPCT}$, then $\lambda = e$ (see 4.24). In the notation of §7, choose $\epsilon_1 > 0$ so small that the component of $\{\, y \in H \mid \|y\| < \epsilon_1 \,\}$ containing e has diameter less than ϵ_0, and choose $\epsilon_2 > 0$ so that the ball of radius ϵ_2 about e is contained in $\{\, y \in H \mid \|y\| < \epsilon_1 \,\}$. The homomorphism $\sim\colon U \to \tilde{U}$ is a polynomial (since U and $\tilde{U}$ are unipotent), and the $\tilde{u}$-flow has polynomial divergence of orbits (see 7.4), so there is a constant DEG such that, for any $x, y \in H$ and $c \in C_G(u, U; 1)$, there is a polynomial f of degree $\le$ DEG such that $d_p(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r, c]}) = f(r)$. By compactness (see 7.6) there is some $\epsilon_3 > 0$ such that if $x, y \in H$ and $c \in C_G(u, U; 1)$ with $d_p(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r, c]}) \le \epsilon_3$ for all r in some interval E on the real line, then $d_p(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r, c]}) < \epsilon_1$ for $r \in (8\mathrm{DEG}) \cdot E$. If, in addition, $d(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r, c]}) < \epsilon_2$ for some $r \in E$, this implies $d(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r, c]}) < \epsilon_0$ for all $r \in (8\mathrm{DEG}) \cdot E$. Choose ϵ_4 so small that if $d(x,y) < \epsilon_4$ and $c \in C_G(u,U;\epsilon_4)$, then $d_p(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r,c]}) < \epsilon_3$ for $r \in [0,1]$. Since

$$\tilde{u}^{-(R+r)}x^{-1}y\tilde{u}^{R+r}\widetilde{[u^{R+r},c]} = \tilde{u}^{-r}(\tilde{u}^{-R}x^{-1}y\tilde{u}^R\widetilde{[u^R,c]})\tilde{u}^r\widetilde{[u^r,c]},$$

it follows that, for any $R \in \mathbf{R}$, if $d(x\tilde{u}^R, y\tilde{u}^R\widetilde{[u^R,c]}) < \epsilon_4$ and $c \in C_G(u,U;\epsilon_4)$, we have $d(x\tilde{u}^r, y\tilde{u}^r\widetilde{[u^r,c]}) < \epsilon_3$ for $r \in [R, R+1]$. By Lusin's Theorem [19, Lemma 3.1] there is a large compact set CONT (say $\mu(\mathrm{CONT}) > .9$) on which ψ is uniformly continuous. Then there is some $\delta > 0$ such that if $s,t \in$ CONT and $d(s,t) < \delta$, then $d(s\psi, t\psi) < \epsilon_4$. There is no loss in assuming $(\mathrm{CONT})\psi \subseteq \mathrm{CPCT}$.

Fix $c \in C_G(u,U;\min(\delta,\epsilon_4))$; we will prove ψ is affine for c. This suffices because the collection of all such elements c generates $C_G(u,U)$.

Step 1. For any $s \in \Gamma\backslash G$ and $n \in \mathbf{Z}$ with $su^n, su^nc \in$ CONT, we have $s\psi\tilde{u}^n, sc\psi\tilde{u}^n\widetilde{[u^n,c]} \in$ CPCT and $d(s\psi\tilde{u}^n, sc\psi\tilde{u}^n\widetilde{[u^n,c]}) < \epsilon_4$.
Proof. Because $su^n \doteq su^nc = scu^n[u^n,c]$ (namely c is so small that $d(su^n, scu^n[u^n,c]) < \delta$) and ψ is uniformly continuous on CONT it follows that

$$s\psi\tilde{u}^n = su^n\psi \doteq scu^n[u^n,c]\psi = sc\psi\tilde{u}^n\widetilde{[u^n,c]}.$$

The other conclusion follows from $(\mathrm{CONT})\psi \subseteq \mathrm{CPCT}$.

Step 2. For a.e. $s \in \Gamma\backslash G$ there is some $c^s \in H$ with $sc\psi = s\psi c^s$ and $[\tilde{u}, c^s] = \widetilde{[u,c]}$.
Proof. Since $\mu(\mathrm{CONT}) > .9$, the Pointwise Ergodic Theorem implies that, for almost any $s \in \Gamma\backslash G$, both su^n and su^nc belong to CONT, for most $n \in \mathbf{Z}$. Fix any such s. Choose $x, y \in H$ with $s\psi = \Lambda x$ and $sc\psi = \Lambda y$. For each $\lambda \in \Lambda$, set

$$E'_\lambda = \{\, r \in \mathbf{R} \mid d_p(x\tilde{u}^r, \lambda y\tilde{u}^r\widetilde{[u^r,c]}) < \epsilon_3 \,\},$$

and let E_λ be the union of those components of E'_λ which contain an integer n such that both su^n and su^nc are in CONT. Let us verify the hypotheses of the Ratner Covering Lemma (7.9) for this family (with $I = \mathbf{R}$, $D = \mathrm{DEG}$, and $\beta = 1/2$). First note that each E_λ has at most DEG components, since there is a polynomial f of degree $\leq$ DEG such that $d_p(x\tilde{u}^r, \lambda y\tilde{u}^r\widetilde{[u^r,c]}) < \epsilon_3$ iff $f(r) < \epsilon_3$. Secondly, Step 1 implies the union of the E_λ covers most of the real line because $d_p(\Lambda x\tilde{u}^r, \Lambda y\tilde{u}^r\widetilde{[u^r,c]}) < \epsilon_3$ for $r \in [R, R+1]$ whenever $d(\Lambda x\tilde{u}^R, \Lambda y\tilde{u}^R\widetilde{[u^R,c]}) < \epsilon_4$. Finally we show that if $(4\mathrm{DEG})\cdot E_{\lambda_1} \cap (4\mathrm{DEG})\cdot E_{\lambda_2} \neq \emptyset$, then $\lambda_1 = \lambda_2$. Let E_i $(i = 1,2)$ be components of E_{λ_i}, and suppose $(4\mathrm{DEG})\cdot E_1 \cap (4\mathrm{DEG})\cdot E_2 \neq \emptyset$. We may assume without loss that E_1 is at least as long as E_2, so $E_2 \subseteq (8\mathrm{DEG})\cdot E_1$. Choose $n \in E_2 \cap \mathbf{Z}$ with $su^n, su^nc \in$ CONT. Now ϵ_3 is so small that $n \in (8\mathrm{DEG})\cdot E_1$ implies $d(x\tilde{u}^n, \lambda_1 y\tilde{u}^n\widetilde{[u^n,c]}) < \epsilon_0$. Hence $\lambda_1 y\tilde{u}^n\widetilde{[u^n,c]} \doteq x\tilde{u}^n \doteq \lambda_2 y\tilde{u}^n\widetilde{[u^n,c]}$. Namely

$d(\lambda_1 y \tilde{u}^n [\widetilde{u^n, c}], \lambda_2 y \tilde{u}^n [\widetilde{u^n, c}]) < \epsilon_0 + \epsilon_4 < 2\epsilon_0$. But $2\epsilon_0$ is so small that this implies $\lambda_2^{-1}\lambda_1 = e$ as desired.

We may now apply the Ratner Covering Lemma (7.9) to conclude there is some $\lambda \in \Lambda$ with (4DEG) $\cdot E_\lambda = \mathbf{R}$. Hence $d(e, \tilde{u}^{-r} x^{-1} \lambda y \tilde{u}^r [\widetilde{u^r, c}]) < \epsilon_0$ for all $r \in \mathbf{R}$, which means the polynomial $\mathbf{R} \to H_a : r \mapsto \tilde{u}^{-r} x^{-1} \lambda y \tilde{u}^r [\widetilde{u^r, c}]$ is bounded and therefore constant. So $[\tilde{u}^r, x^{-1}\lambda y] = [\widetilde{u^r, c}]$. Set $c^s = x^{-1}\lambda y$.

Step 3. ψ is affine for c.

Proof. We have shown that for a.e. $s \in \Gamma\backslash G$, there is some $c^s \in H$ with $sc\psi = s\psi c^s$ and $[\tilde{u}, c^s] = [\widetilde{u, c}]$. We wish to prove c^s is independent of s. Since $[\tilde{u}, c^s] = [\widetilde{u, c}]$ is independent of s, we have $c^s (c^t)^{-1} \in C_H(\tilde{u})$ for $s, t \in \Gamma\backslash G$. Because

$$su\psi c^{su} = suc\psi = sc\psi \tilde{u}[\widetilde{u, c}] = s\psi c^s \tilde{u}[\widetilde{u, c}] = s\psi \tilde{u} c^s = su\psi c^s$$

and $C_H(\tilde{u})$ acts essentially freely on $\Lambda\backslash H$ (see 6.7), we must have $c^{su} = c^s$ for a.e. s. Because u is ergodic on $\Gamma\backslash G$, this implies $s \mapsto c^s$ is essentially constant, as desired. ∎

This completes the proof of Lemma 8.2 modulo the assumption that $\tilde{U}$ is unipotent.

(8.4) **Proposition.** *We may assume $\tilde{U}$ is unipotent.*

Proof. After a pair of preliminary reductions, the argument is similar to that in 8.3.

Step 1. It suffices to find some $\delta > 0$ such that $[\widetilde{u^r, c}]$ is unipotent for all $r \in \mathbf{R}$ and all $c \in C_G(u, U; \delta)$.

Proof. Set $U_0 = \langle [u^r, c] \mid r \in \mathbf{R}, c \in C_G(u, U; \delta) \rangle$. Then $C_G(u, U_0) = C_G(u, U)$, and $\tilde{U}_0$ is a connected unipotent subgroup of H (because $\tilde{U}_0$ is a connected nilpotent subgroup generated by unipotent elements of H). ∎

Step 2. We may assume H is semisimple with trivial center (and hence H is a real algebraic group).

Proof. Since $\operatorname{rad} H$ is unipotent (see 4.19), any connected subgroup of H that projects to a unipotent subgroup of $H/\operatorname{rad} H$ is itself a unipotent subgroup of H. Hence we may replace H by the semisimple group $H/\operatorname{rad} H$. (Recall that Corollary 4.14 asserts Γ projects to a lattice in $H/\operatorname{rad} H$.) Similarly (cf. 4.9), we may furthermore replace H by $H/Z(H)$, a group with trivial center. Any connected semisimple Lie group with trivial center is (isomorphic to) a real algebraic group [25, Proposition 3.1, p. 35]. ∎

Notation. Since $\tilde{U}$ is a connected nilpotent group, its Zariski closure can be written in the form $V \times T$, where V is unipotent and T is an algebraic torus. Because each element of $\tilde{U}$ has zero entropy, T is compact (see 6.1). Let $\hat{T}$ be the universal cover of T (thus $\hat{T}$ is a (unipotent) abelian real algebraic group, and let $\exp\colon \hat{T} \to T$ be the covering homomorphism. We may choose $\epsilon_1 > 0$ so small that if $f\colon \mathbf{R} \to \hat{T}$ is any nonconstant polynomial and $h \in H$, then the upper density of $\{\, r \in \mathbf{R} \mid d(h, \exp[f(r)]) < \epsilon_1 \,\}$ on $\mathbf{R}$ is less than $1/(2|\Lambda : \Lambda'|)$, where Λ' is a torsion-free subgroup of finite index in Λ (4.23).

Let $\mathrm{CPCT} \subseteq \Lambda\backslash H$ be a large compact set (say $\mu(\mathrm{CPCT}) > .9$). Since Λ' is torsion-free and T is compact, Λ' acts properly discontinuously on H/T. Indeed we may assume ϵ_1 is so small that if $d(hT, \lambda hT) < 2\epsilon_1$, with $\lambda \in \Lambda'$ and $\Lambda h \in \mathrm{CPCT}$, then $\lambda = e$.

There is a constant DEG satisfying: for any $y \in H$ and $c \in C_G(u, U; 1)$, there is a polynomial f of degree $\le$ DEG such that $\|\tilde{u}^{-r} y \tilde{u}^r \widetilde{[u^r, c]}\|_T = f(r)$. By compactness (see 7.6) there is some $\epsilon_2 > 0$ such that if $y \in H$ and $c \in C_G(u, U; 1)$ with $\|\tilde{u}^{-r} y \tilde{u}^r \widetilde{[u^r, c]}\|_T \le \epsilon_2$ for all $r \in [R, R + \delta]$, then $d(T, \tilde{u}^{-r} y \tilde{u}^r \widetilde{[u^r, c]} T) < \epsilon_1$ for $r \in [R, R + 8|\Lambda : \Lambda'|\mathrm{DEG} \cdot \delta]$. Choose $\epsilon_3 > 0$ (with $\epsilon_3 < \epsilon_1$) so that $d(e, h) < \epsilon_3$ implies $\|h\|_T < \epsilon_2$. There is some $\epsilon_4 > 0$ so small that if $d(x, y) < \epsilon_4$ and $c \in C_G(u, U; \epsilon_4)$, then $d(x\tilde{u}^r, y\tilde{u}^r \widetilde{[u^r, c]}) < \epsilon_3$ for $r \in [0, 1]$. By Lusin's Theorem [19, Lemma 3.1] there is a large compact set CONT (say $\mu(\mathrm{CONT}) > .9$) on which ψ is uniformly continuous. Then there is some $\delta > 0$ such that if $s, t \in \mathrm{CONT}$ and $d(s, t) < \delta$, then $d(s\psi, t\psi) < \epsilon_4$. We may assume $(\mathrm{CONT})\psi \subseteq \mathrm{CPCT}$.

Step 3. For $c \in C_G(u, U; \min(\delta, 1))$, we have $\widetilde{[u^r, c]} \in V$ for all $r \in \mathbf{R}$.
Proof (cf. Steps 1 and 2 of 8.3). Fix any $s \in \Gamma\backslash G$ such that, for most integers n, both su^n and $su^n c$ are in CONT (this is true for a.e. $s \in \Gamma\backslash G$). Since the natural map $\Lambda'\backslash H \to \Lambda\backslash H$ is a covering map with fibers of cardinality $|\Lambda : \Lambda'|$, there are x and y in H, with $s\psi = \Lambda x$ and $sc\psi = \Lambda y$, such that the upper density of

$$\{\, n \in \mathbf{Z} \mid d(\Lambda' x \tilde{u}^n, \Lambda' y \tilde{u}^n \widetilde{[u^n, c]}) < \epsilon_3 \text{ and } \Lambda x \tilde{u}^n, \Lambda y \tilde{u}^n \widetilde{[u^n, c]} \in \mathrm{CPCT} \,\}$$

on $\mathbf{Z}$ is greater than $1/(2|\Lambda : \Lambda'|)$. For each $\lambda \in \Lambda'$, set

$$E'_\lambda = \{\, r \in \mathbf{R} \mid \|x^{-1} \lambda y \tilde{u}^r \widetilde{[u^r, c]}\|_T < \epsilon_2 \,\},$$

and let E_λ be the union of those components of E'_λ that contain some $n \in \mathbf{Z}$ such that both su^n and $su^n c$ are in CONT.

As in Step 2 of 8.3, we verify the hypotheses of the Ratner Covering Lemma (7.9) for this family (with $I = \mathbf{R}, D = \mathrm{DEG}$, and $\beta = 1/(2|\Lambda : \Lambda'|)$). Note that each E_λ has at most DEG components, and the union of the E_λ has relative measure greater than $1/(2|\Lambda : \Lambda'|)$ on the real line. Finally, if $r \in E_{\lambda_1} \cap (8|\Lambda : \Lambda'|\mathrm{DEG}) \cdot E_{\lambda_2}$, with $\Lambda x \tilde{u}^r, \Lambda y \tilde{u}^r \widetilde{[u^r, c]} \in \mathrm{CPCT}$, then $\lambda_1 y \tilde{u}^r \widetilde{[u^r, c]} T \doteq x \tilde{u}^r T \doteq \lambda_2 y \tilde{u}^r \widetilde{[u^r, c]} T$—namely $d(\lambda_1 y \tilde{u}^r T, \lambda_2 y \tilde{u}^r T) < 2\epsilon_1$. But ϵ_1 is so small that this implies $\lambda_2^{-1} \lambda_1 = e$ as desired.

We may now apply the Ratner Covering Lemma (7.9) to conclude there is some $\lambda_0 \in \Lambda'$ with $(4|\Lambda : \Lambda'|\mathrm{DEG}) \cdot E_{\lambda_0} = \mathbf{R}$. Hence $d(T, \tilde{u}^{-r} x^{-1} \lambda_0 y \tilde{u}^r \widetilde{[u^r, c]} T) < \epsilon_1$ for all $r \in \mathbf{R}$. Writing $\widetilde{[u^r, c]} = \widetilde{[u^r, c]}_V \cdot \widetilde{[u^r, c]}_T$ with $\widetilde{[u^r, c]}_V \in V$ and $\widetilde{[u^r, c]}_T \in T$, this implies the polynomial $\mathbf{R} \to H \colon r \to \tilde{u}^{-r} x^{-1} \lambda_0 y \tilde{u}^r \widetilde{[u^r, c]}_V$ is bounded and hence constant. This means the element $c^s = x^{-1}\lambda_0 y$ of H satisfies $\tilde{u}^{-r} c^s \tilde{u}^r \widetilde{[u^r, c]}_V = c^s$ for all $r \in \mathbf{R}$. Furthermore $d(c^s T, T) < \epsilon_2$, and $s\psi c^s = sc\psi$.

Suppose $\lambda \in \Lambda'$ such that there is some $r \in \mathbf{R}$ with $d(x\tilde{u}^r, \lambda x c^s \tilde{u}^r \widetilde{[u^r, c]}) < \epsilon_3$ and $\Lambda x \tilde{u}^r, \Lambda x c^s \tilde{u}^r \widetilde{[u^r, c]} \in \mathrm{CPCT}$. Then $x\tilde{u}^r T \doteq \lambda x c^s \tilde{u}^r \widetilde{[u^r, c]} T = \lambda x \tilde{u}^r c^s T \doteq \lambda x \tilde{u}^r T$. (More precisely $d(x\tilde{u}^r T, \lambda x \tilde{u}^r T) < 2\epsilon_1$.) Now ϵ_1 is so small that this implies $\lambda = e$. Hence

$$
\begin{aligned}
\{\, r \in \mathbf{R} \mid d(\Lambda' x \tilde{u}^r, \Lambda' y \tilde{u}^r \widetilde{[u^r, c]}) < \epsilon_3 \quad &\text{and} \quad \Lambda x \tilde{u}^r, \Lambda y \tilde{u}^r \widetilde{[u^r, c]} \in \mathrm{CPCT} \,\} \\
&\subseteq \{\, r \in \mathbf{R} \mid d(x\tilde{u}^r, x c^s \tilde{u}^r \widetilde{[u^r, c]}) < \epsilon_3 \,\} \\
&= \{\, r \in \mathbf{R} \mid d((c^s)^{-1}, (c^s)^{-1} \tilde{u}^{-r} c^s \tilde{u}^r \widetilde{[u^r, c]}) < \epsilon_3 \,\} \\
&= \{\, r \in \mathbf{R} \mid d((c^s)^{-1}, \widetilde{[u^r, c]}_T) < \epsilon_3 \,\}.
\end{aligned}
$$

The first of these is known to have upper density greater than $1/(2|\Lambda : \Lambda'|)$ on $\mathbf{R}$. Since $\epsilon_3 < \epsilon_1$, this implies the map $\mathbf{R} \to T \colon r \mapsto \widetilde{[u^r, c]}_T$ is constant. Therefore $\widetilde{[u^r, c]}_T = e$ for all $r \in \mathbf{R}$. This completes the proof. ∎

CHAPTER 9

THE RATNER PROPERTY

(9.1) **Theorem** ("The Ratner Property"). *Let u be an unipotent element of a Lie group G. Given any neighborhood Q of e in $C_G(u)$, there is a compact subset ∂Q of $Q - \{e\}$ such that: For any $\epsilon > 0$ and $M > 0$, there are $\alpha = \alpha(u, Q, \epsilon) > 0$ and $\delta = \delta(u, Q, \epsilon, M) > 0$ such that, for any lattice Γ in G, if $s, t \in \Gamma \backslash G$ with $d(s,t) < \delta$, then either $s = tc$ for some $c \in C_G(u)$ with $d(e, c) < \delta$, or there are $N > 0$ and $q \in \partial Q$ such that $d(su^n, tu^n q) < \epsilon$ whenever $N \le n \le N + \max(M, \alpha N)$.*

Essentially this statement of the Ratner property was suggested by Marina Ratner in a discussion with the author. The simplicity of the proof is obscured by the notation. The basic idea is as follows: Consider two points $s, t \in \Gamma \backslash G$ that are close together and suppose their orbits are not parallel (i.e., there is no small $c \in C_G(u)$ with $s = tc$). Then the two points wander apart. The first thing to show is that s and t move apart much faster in the direction of the $C_G(u)$-orbits than in other directions. Therefore, letting ∂Q be the unit sphere in $C_G(u)$, it follows that s passes near $t \cdot \partial Q$ as the points wander apart. Secondly, polynomial divergence implies the points disperse slowly. So there is some $q \in \partial Q$ such that s spends a long time near tq. (Where "a long time" is a duration proportional to the length of time required for s to approach tq.) Pass to the Lie algebra of G to make the argument precise.

We now proceed to the details.

(9.2) **Lemma.** *Suppose T is a nilpotent endomorphism of a finite dimensional vector space V, let $\pi \colon V \to \ker T$ be any projection onto the kernel of T, and let $\pi^* = \mathrm{Id} - \pi$ be the complementary projection. Given compact neighborhoods B_ρ and B_ϵ of 0 in V. For all sufficiently large $N > 0$, if w is any element of V with $w \operatorname{Exp}(rt)\pi \in B_\rho$ for all $r \in [0, N]$, then $w \operatorname{Exp}(rT)\pi^* \in B_\epsilon$ for all $r \in [0, N]$.*

Proof. Put an inner product on V and assume without loss that B_ρ and B_ϵ are balls about 0 in this metric. Thus $\|w \operatorname{Exp}(rT)\pi\| \le \rho$ for $r \in [0, N]$. Letting n be minimal with $T^{n+1} = 0$, we have

$$w \operatorname{Exp}(rT)\pi = w r^n T^n / n! + \text{lower order terms}.$$

By compactness (see 7.6) this implies

$$\|wT^n\| \le C/N^n$$

where $C = C(n, \rho)$ is independent of w and N. Writing $w = v + w'$ with $v \perp \ker T^n$ and $w' \in \ker T^n$, we therefore have

$$\|v\| \leq C'/N^n$$

where C' is independent of w and N.

It follows from the inequalities above that if N is sufficiently large (independent of w), then

$$\|v \operatorname{Exp}(rT)\pi^*\| \leq \epsilon$$

and

$$\|w' \operatorname{Exp}(rT)\pi\| \leq \|w \operatorname{Exp}(rT)\pi\| + \|v \operatorname{Exp}(rT)\pi\| \leq \rho + (C/n! + 1) = \rho'.$$

By induction on n, the latter implies $\|w' \operatorname{Exp}(rT)\pi^*\| \leq \epsilon$ for N sufficiently large. Therefore $\|w \operatorname{Exp}(rT)\pi^*\| \leq \|v \operatorname{Exp}(rT)\pi^*\| + \|w' \operatorname{Exp}(rT)\pi^*\| \leq 2\epsilon$. ∎

(9.3) **Proposition.** *Let T be a nilpotent endomorphism of a finite dimensional vector space V, and put any norm on V. Given $\rho, \epsilon, M > 0$, there are $\alpha = \alpha(T, \rho, \epsilon) > 0$ and $\delta = \delta(T, \rho, \epsilon, M) > 0$ such that if $v \notin \ker T$ and $\|v\| < \delta$, then there are $N > 0$ and $q \in \ker T$ with $\|q\| = \rho$ such that $\|v \operatorname{Exp}(rT) - q\| < \epsilon$ whenever $N \leq r \leq N + \max(M, \alpha N)$.*

Proof. Choose n with $T^n = 0$. By compactness (see 7.6) there is $\alpha = \alpha(n, \rho, \epsilon) > 0$ such that if f is a polynomial of degree at most n, with $|f(r)| < \rho$ for $r \in [0, N]$, then $|f(r) - f(N)| < \epsilon$ for $r \in [N/(1 + \alpha), N]$.

For $v \in V$, set

$$N = N(v) = \inf\{\, r \in [0, \infty) \mid \|v \operatorname{Exp}(rT)\pi\| = \rho \,\},$$

with π as in Lemma 9.2. Let $\delta = \delta(T, \rho, \epsilon, M)$ be so small that $N = N(v)$ is "sufficiently large" (see Lemma 9.2) and satisfies $\alpha N > M$, whenever $\|v\| < \delta$.

Put $q = v \operatorname{Exp}(NT)\pi$. Then, for $r \in [N/(1 + \alpha), N]$, we have $\|v \operatorname{Exp}(rT) - q\| \leq \|v \operatorname{Exp}(rT) - v \operatorname{Exp}(NT)\| + \|v \operatorname{Exp}(NT)\pi^*\| \leq \epsilon + \epsilon$. ∎

(9.4) *Proof of the Ratner Property.* Consider coordinates in a neighborhood of e in G given by the exponential map from the Lie algebra $\mathcal{G}$ (so-called "canonical coordinates"). Apply the proposition with $T = \operatorname{ad} \underline{u}$, where $\operatorname{Exp}(\underline{u}) = u$, to conclude that there is a topological metric $\bar{d}$ in a neighborhood of e in G such that: Given $\rho > 0$ there is a compact subset ∂Q of the ball of radius ρ in $C_G(u)$ with $e \notin \partial Q$ and such that: For any $\epsilon > 0$ and $M > 0$ there are $\alpha = \alpha(u, \rho, \epsilon)$ and $\delta = \delta(u, \rho, \epsilon, M)$ such that if $\bar{d}(e, g) < \delta$ and $g \notin C_G(u)$, then there is $N > 0$ and $q \in \partial Q$ such that $\bar{d}(u^{-r} g u^r, q) < \epsilon$ whenever $r \in [N, N + \max(M, \alpha N)]$. This is a topological statement so it is also true with the metric d in place of $\bar{d}$. Projecting to the homogeneous space $\Gamma \backslash G$, we deduce the Ratner property. ∎

CHAPTER 10

PROOF OF THE MAIN THEOREM

We are now prepared to commence the proof of this paper's main result. The proof consists of a sequence of results that occupy this entire section. For ease of reference, we restate the theorem here.

(10.1) **Main Theorem.** *Suppose $\Gamma\backslash G$ and $\Lambda\backslash H$ are finite-volume homogeneous spaces of connected Lie groups G and H. If $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ is a measure preserving Borel map which is affine for an ergodic unipotent translation on $\Gamma\backslash G$, via a unipotent translation on $\Lambda\backslash H$, then ψ is affine for G.*

(10.2) **Lemma.** *We may assume Γ and Λ are lattices.*

Proof. Passing to factor groups of G and H, we may assume $\Gamma\backslash G$ and $\Lambda\backslash H$ are presentations. Then, because there are ergodic unipotent translations on $\Gamma\backslash G$ and $\Lambda\backslash H$, Corollary 4.7 asserts Γ and Λ are lattices. ∎

(10.3) *Remarks.* (1) Theorem 4.1 asserts Γ projects densely into the maximal compact semisimple factor of G. Hence $\Gamma \cap \operatorname{rad} G$ is a lattice in $\operatorname{rad} G$ (4.14). (2) Theorem 4.19 asserts $\operatorname{rad} G$ is a unipotent subgroup of G, and hence G is locally algebraic.

(10.4) *Notation.* Let u be the ergodic unipotent translation for which ψ is known to be affine (with $\tilde{u}$ unipotent). Let U be the identity component of a maximal unipotent subgroup of G containing u, and set $P = N_G(U)$, a minimal parabolic subgroup of G. Let LEVI be a Levi subgroup of G.

(10.5) **Theorem.** *ψ is affine for P°.*

Proof. We set $U_0 = e$ and recursively define $U_{i+1} = C_G(u, U_i) \cap U$. By induction on i, Affine for the Relative Centralizer (8.2) implies ψ is affine for U_i for all i. Because $\langle u, U\rangle$ is nilpotent, we have $U_i = U$ when i is sufficiently large, so ψ is affine for U. Therefore Affine for the Relative Centralizer (8.2) asserts ψ is affine for $N_G(U)^\circ \subseteq C_G(u, U)$, as desired. ∎

(10.6) **Corollary.** *$(U \cap \mathrm{LEVI})^\sim$ is a unipotent subgroup of H.*

Proof. There is some $a \in P^\circ$ such that $U \cap \mathrm{LEVI}$ is contained in the horospherical subgroup associated to a (cf. 2.21). Then $(U \cap \mathrm{LEVI})^\sim$ is contained in the horospherical subgroup associated to $\tilde{a}$, so $(U \cap \mathrm{LEVI})^\sim$ is unipotent. ∎

(10.7) **Proposition.** *We may assume every nontrivial connected unipotent subgroup of* LEVI *is ergodic.*

Proof. Assume the contrary.

Step 1. LEVI is a product LEVI $= N_1 \cdot N_2$ of two of its nonergodic connected normal subgroups.

Proof. We may assume LEVI is ergodic on $\Gamma\backslash G$ (else the assertion is obvious), and hence the maximal solvmanifold quotient of $\Gamma\backslash G$ is trivial. Set $\overline{G} = G/\operatorname{rad} G$, so that $\overline{\Gamma}\backslash\overline{G}$ is the maximal semisimple quotient of $\Gamma\backslash G$. If V is a nontrivial connected nonergodic unipotent subgroup of LEVI, then, since $\Gamma\backslash G$ has no solvmanifold quotient, we know V must be nonergodic on $\overline{\Gamma}\backslash\overline{G}$ (see 6.6). Since $\overline{V}$ is not Ad-precompact, we conclude from the Moore Ergodicity Theorem (6.5) that $\overline{\Gamma}$ is a reducible lattice in $\overline{G}$. Therefore $\overline{G}$ can be decomposed into a product of two nonergodic normal subgroups (cf. (4.22)).

Step 2. We may assume that if V is any connected unipotent nonergodic subgroup of $\mathrm{LEVI} \cap U$, then ψ is affine for a nonergodic normal subgroup of G containing V.

Proof. The Mautner Phenomonen (6.4) implies the ergodic components of V are the orbits of some normal subgroup N of G which contains V. Since $\tilde{V}$ is unipotent, ψ maps N-orbits to $\tilde{N}$-orbits, where $\tilde{N}$ is some normal subgroup of G with closed orbits on $\Lambda\backslash G$. By induction on $\dim G$, we may assume ψ is affine for N via $\tilde{N}$ on each N-orbit. So, for each N-orbit θ, we have a (local) epimorphism $\sigma_\theta \colon N \to \tilde{N}$. Since ψ is affine for P°, all the σ_θ agree on $(P \cap N)^\circ$. Since $P \cap N$ is parabolic in N, we conclude from Proposition 3.1 that all the σ_θ are equal. Hence ψ is affine for N as desired.

Step 3. Completion of proof.

Proof. Since ψ is known to be affine for P°, it suffices to show ψ is affine for a cocompact normal subgroup of LEVI. Thus it suffices to show ψ is affine for a cocompact normal subgroup of each N_i in the decomposition of Step 1. Let $V = N_i \cap U$, a maximal connected unipotent subgroup of N_i. Since N_i is semisimple, any closed normal subgroup containing V is cocompact. Thus Step 2 completes the proof. ∎

(10.8) **Corollary.** *We may assume* $\mathbf{R}$-rank$(G/\operatorname{rad} G) > 0$ *and the maximal solvmanifold quotient of* $\Gamma\backslash G$ *is trivial.*

Proof. Since ψ is affine for P°, we may assume G has a proper parabolic subgroup, which means $G/\operatorname{rad} G$ is noncompact (i.e., $\mathbf{R}$-rank$(G/\operatorname{rad} G) > 0$). Therefore LEVI has a nontrivial connected unipotent subgroup, which the Proposition asserts we may assume is ergodic. Then LEVI is ergodic, and hence $\Gamma\backslash G$ has no solvmanifold quotient. ∎

(10.9) **Corollary.** *We may assume that if X is any connected subgroup of G which does not project to an* Ad*-precompact subgroup of $G/\operatorname{rad} G$, then X is ergodic on $\Gamma\backslash G$.*

Proof. The Moore Ergodicity Theorem (6.5) asserts X is ergodic on the maximal semisimple quotient of $\Gamma\backslash G$. Since $\Gamma\backslash G$ has no solvmanifold quotient (10.8), then the Mautner Phenomenon (6.6) implies X is ergodic on $\Gamma\backslash G$. ∎

(10.10) **Lemma.** *We may assume $\mathbf{R}\text{-rank}(G/\operatorname{rad} G) = 1$.*

Proof. Suppose, to the contrary, that $\mathbf{R}$-rank(LEVI) ≥ 2. Thus the root system of LEVI has (at least) two simple $\mathbf{R}$-roots α and β. Then $\alpha - \beta$ is not a root (because it is neither positive nor negative) so any element $u_0 \in U \cap \text{LEVI}$ belonging to the root α centralizes any element v_0 of $U^- \cap \text{LEVI}$ belonging to $-\beta$ (where U^- is an opposite unipotent subgroup of G). Therefore Affine for the Relative Centralizer (8.2) asserts ψ is affine for $v_0 \in C_G(u_0, e)$. Note that $\tilde{v}_0$ is unipotent, because it belongs to the horospherical subgroup associated to some $\tilde{a} \in \tilde{P}^\circ$. Then Theorem 10.5 (with v_0 in the place of u and U^- in the place of U) asserts ψ is affine for $N_G(U^-)^\circ$. Hence $\operatorname{Aff}_G(\psi) \supseteq \langle U^-, P^\circ \rangle = G$. ∎

(10.11) **Lemma.** *We may assume that if $g \in \operatorname{Aff}_G(\psi)$ and $\tilde{g} = e$, then $g = e$.*

Proof. Let ker be the kernel of the homomorphism $\sim\colon \operatorname{Aff}_G(\psi) \to H$. We wish to show that, by passing to a factor group of G, we may assume ker $= \{e\}$. Since ψ is (essentially) ker-invariant, it will suffice to show ker is a normal subgroup such that ker $\cdot\Gamma$ is closed in G. Note that P° normalizes ker, because $P^\circ \subseteq \operatorname{Aff}_G(\psi)$ and the kernel of a homomorphism is normal. Since ker is precisely the essential stabilizer of ψ, the Mautner phenomenon (6.2) implies there is a closed normal subgroup N of G contained in ker such that ker projects to an Ad-precompact subgroup of G/N. Therefore Lemma 2.25 asserts ker is normal in G. Now Lemma 6.3 implies ker $\cdot\Gamma$ is closed in G. ∎

(10.12) **Lemma.** *Let* CONT *be a large compact set (say $\mu(\text{CONT}) > .9$) on which ψ is continuous, let u be an ergodic unipotent element of $\operatorname{Aff}_G(\psi)$, and let ∂Q be a compact subset of $C_G(u)^\circ - \{e\}$ as specified in the Ratner property (9.1). We may assume there is some $\epsilon > 0$ such that if $s, t \in \text{CONT}$ with $s\psi = t\psi$, then $d(s, tq) > \epsilon$ for all $q \in \partial Q$.*

Proof (cf. proof of [21, Theorem 3]). Let

$$\text{GRAPH} = \{\, (s, s\psi) \in \Gamma\backslash G \times \Lambda\backslash H \mid s \in \text{CONT} \,\}.$$

We may assume no non-identity element of $C_H(\tilde{u})$ has a fixed point in (CONT)ψ (see 6.7), and that $\tilde{q} \neq e$ for all $q \in \partial Q$ (see 10.11). Thus $\tilde{q}$ has no fixed point in (CONT)ψ. For $q \in \partial Q$ this implies that if $(s, s\psi) \in \text{GRAPH}$, then $(sq, s\psi) \notin \text{GRAPH}$.

The group G acts continuously on $\Gamma\backslash G \times \Lambda\backslash H$ via $(s,t) \cdot g = (sg, t)$. The preceding paragraph shows GRAPH and GRAPH $\cdot\, q$ are disjoint for any $q \in \partial Q$, so GRAPH and

GRAPH $\cdot \partial Q$ are disjoint compact subsets of $\Gamma\backslash G \times \Lambda\backslash H$. Thus there is some distance between them: for $s,t \in$ CONT and $q \in \partial Q$, we have $d((s,s\psi),(tq,t\psi)) > \epsilon$. In particular $d(s,tq) > \epsilon$ whenever $s,t \in$ CONT with $s\psi = t\psi$. ∎

(10.13) **Theorem.** *We may assume there is a conull subset Ω of $\Gamma\backslash G$ such that, for each $t \in \Lambda\backslash H$, the fiber $t\psi^{-1} \cap \Omega$ is finite (and all these fibers are of the same cardinality).*

Proof (cf. [20, Lemma 3.1]). It suffices to find a non-null set on which all fibers are count- [20, Proposition 1.1]. To do this we will apply the Ratner property (9.1) to exhibit a non-null set $\Omega \subseteq \Gamma\backslash G$ and some $\delta > 0$ such that if $s,t \in \Omega$ and $s\psi = t\psi$, then $d(s,t) > \delta$.

Let $\epsilon > 0$ be as specified by Lemma 10.12. By judicious application of the Ratner property and the Pointwise Ergodic Theorem (see the proof of [20, Lemma 3.1]): there is $\delta > 0$ and a non-null set $\Omega \subseteq$ CONT such that if $s,t \in \Omega$ with $d(s,t) < \delta$, then either $s = tc$ for some $c \in C_G(u)^\circ$, or, for some $n \in \mathbf{Z}$ and some $q \in \partial Q$, we have $d(su^n, tu^n q) < \epsilon$ and $su^n, tu^n \in$ CONT.

We claim that if $s,t \in \Omega$ with $s\psi = t\psi$, then $d(s,t) \leq \delta$. Suppose not. Now $s\psi = t\psi$, so if $s = tc$ for some $c \in C_G(u)^\circ$, then $s\psi\tilde{c} = t\psi\tilde{c} = tc\psi = s\psi$—contrary to the assumption that no non-identity element of $C_H(\tilde{u})$ has a fixed point in (CONT)ψ. Hence the preceding paragraph implies there is some $n \in \mathbf{Z}$ and $q \in \partial Q$ with $su^n, tu^n \in$ CONT and $d(su^n, tu^n q) < \epsilon$. Since $s\psi = t\psi$, we have $su^n\psi = tu^n\psi$, and therefore the choice of ϵ implies $d(su^n, tu^n q) > \epsilon$. This is a contradiction. ∎

(10.14) **Theorem.** *For a certain closed subgroup Γ' of G containing Γ, the translation by g on $\Gamma'\backslash G$ is (measure theoretically) isomorphic to the translation by $\tilde{g}$ on $\Lambda\backslash H$ in such a way that ψ corresponds (a.e.) to the natural G-map $\Gamma\backslash G \to \Gamma'\backslash G$.*

The proof of this theorem consists of showing that the partition of $\Gamma\backslash G$ into fibers of ψ is G-invariant (a.e.). More precisely we wish to show that, for each $g \in G$, there is a conull subset $\Omega \subseteq \Gamma\backslash G$ such that if $s,t \in \Omega$ with $s\psi = t\psi$, then $sg\psi = tg\psi$.

Perhaps we should offer an elaboration of the preceding paragraph. The so-called measure algebra $B(\Gamma\backslash G)$ of $\Gamma\backslash G$ is the space of measurable subsets of $\Gamma\backslash G$, two sets being identified if they differ by a null set. The map $\psi\colon \Gamma\backslash G \to \Lambda\backslash H$ yields an injective map $\psi^*\colon B(\Lambda\backslash H) \to B(\Gamma\backslash G)$ whose image $\psi^* B(\Lambda\backslash H)$ is a closed Boolean sub-σ-algebra of $B(\Gamma\backslash G)$. To prove the proposition it suffices to show $\psi^* B(\Lambda\backslash H)$ is an invariant subspace of $B(\Gamma\backslash G)$ (see the remarks surrounding the statement of Theorem 8.1.4 in [25]). Thus we wish to show $\psi^* B(\Lambda\backslash H)$ is g-invariant for each $g \in G$. Clearly it suffices to do so for the elements g of a generating set for G. For any $g \in G$, the subalgebra $\psi^* B(\Lambda\backslash H)$ is g-invariant if the partition of $\Gamma\backslash G$ into fibers of ψ is g-invariant (a.e.). This is because a subset $C \subseteq \Gamma\backslash G$ belongs to $\psi^* B(\Lambda\backslash H)$ iff C is a union of fibers of ψ (modulo a null set). Thus the remarks in the preceding paragraph are justified.

Because ψ is affine for P°, we know the partition is also invariant under P°. We now proceed to show that the partition of $\Gamma\backslash G$ is invariant under a one-parameter subgroup not contained in P°. These subgroups generate G (2.23), so the theorem follows.

(10.15) **Lemma.** *There is a one-parameter subgroup V_1 of the intersection of* LEVI *with a maximal connected unipotent subgroup U^- opposite to U such that the partition of $\Gamma\backslash G$ into fibers of ψ is invariant under V_1 (a.e.).*

Proof (cf. [20, Lemma 4.4]). Let U_1 be a one-parameter subgroup of $U \cap \mathrm{LEVI}$ and let V_1 be any one-parameter unipotent subgroup of LEVI such that $\langle U_1, V_1\rangle$ is locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$. For $v \in V_1$, we will show there is a conull set $\Omega \subseteq \Gamma\backslash G$ such that if $s, t \in \Omega$ and $s\psi = t\psi$, then $sv\psi = tv\psi$.

Since ψ has finite fibers (a.e.) there are pairwise disjoint subsets $X_1, X_2, \ldots, X_f$ whose union is conull, such that each X_i intersects (almost) every fiber in a single point (see the remarks following [20, Proposition 1.1]). Let $\pi_i\colon \Gamma\backslash G \to X_i$ be the projection, viz. $s\psi = s\pi_i\psi$ for $s \in \Gamma\backslash G$. (Here and in the rest of the proof we ruthlessly ignore null sets—thus many statements are true only if s and t are restricted to some conull subset of $\Gamma\backslash G$.) We wish to show that if $s, t \in \Gamma\backslash G$ with $s\psi = t\psi$, then $sv = tv\pi_j$ for some j. It suffices to do this for every sufficiently small $v \in V_1$.

Let U^- be the maximal connected unipotent subgroup of G containing V_1, let $A \subseteq N_G(U) \cap N_G(U^-)$ be the identity component of a maximal $\mathbf{R}$-split torus in G, and let A^+ be the sub-semigroup of expanding automorphisms of $U \cap \mathrm{LEVI}$ (so A^+ contracts $U^- \cap \mathrm{LEVI}$; this means that if $v \in U^- \cap \mathrm{LEVI}$ and $a \in A^+$, then $a^{-n} v a^n \to e$ as $n \to \infty$).

Step 1. Given $\delta > 0$ and any compact interval $N \subseteq U_1$. If $v \in V_1$ is sufficiently small, then there are an unbounded subset $[A^+] \subseteq A^+$ and some π_j satisfying: for each $a \in [A^+]$ there is some $x_a \in G$ with $sv = tv\pi_j x_a$ and such that $d(ua, x_a ua) < \delta$ for all $u \in N$.
Proof. By polynomial divergence of orbits and the Ratner Covering Lemma (see Step 2 of the proof of Lemma 8.2), it suffices (after shrinking δ somewhat) to find $[A^+]$ and π_j such that, for each $a \in [A^+]$, the relative measure of $\{\, u \in N \mid d(svua, tv\pi_j ua) < \delta \,\}$ on N is at least $1/(2f)$, where f is a constant independent of δ (namely f is the cardinality of the fibers of ψ).

For any $\delta_0 > 0$, if $v \in V_1$ is sufficiently small, then there is a map $U_1 \to U_1\colon u \mapsto \overline{u}$ such that $d(vua, \overline{u}a) < \delta_0$ for all $u \in N$ and all sufficiently large $a \in A^+$. (Namely, let B be the ball of radius $\delta_0/2$ in $V_1 A$ and choose $\overline{u} \in U_1$ with $vu \in \overline{u}B$. Note that if v is small, then the derivative of $u \mapsto \overline{u}$ is very close to 1 for all $u \in N$.) Let CONT be a large (say $\mu(\mathrm{CONT}) > .9$) compact set on which $\pi_1, \ldots, \pi_f$ are uniformly continuous. Since A^+ expands U_1, the Pointwise Ergodic Theorem implies (for almost all $t \in \Gamma\backslash G$) there is an unbounded subset $[A^+] \subseteq A^+$ such that for each $a \in [A^+]$, we have $t\overline{u}a, tvua \in \mathrm{CONT}$ for most $u \in N$. For each a and u, there is some $i = i(a,u)$ with $s\overline{u}a \in X_i$. And for each

$a \in [A^+]$, there is some $j = j_a$ such that the relative measure of

$$\{ u \in N \mid t\overline{u}a, tvua \in \mathrm{CONT} \quad \text{and} \quad tvua\pi_i(ua)^{-1} \in X_j \}$$

(where $i = i(a,u)$) on N is at least $1/(2f)$. Replace $[A^+]$ by an unbounded subset on which j_a is constant.

Now the following string of approximations establishes the desired conclusion if v is sufficiently small:

$$\begin{array}{lll} svua & \doteq s\overline{u}a & \text{(since } vua \doteq \overline{u}a\text{)} \\ & = s\overline{u}a\pi_i & \text{(where } s\overline{u}a \in X_i\text{)} \\ & = t\overline{u}a\pi_i & \text{(since the partition is invariant under } \overline{u} \text{ and } a\text{)} \\ & \doteq tvua\pi_i & \text{(whenever } t\overline{u}a, tvua \in \mathrm{CONT}\text{)} \\ & = tv\pi_j ua & \text{(whenever } tvua\pi_i(ua)^{-1} \in X_j\text{).} \end{array}$$

Step 2. Given $\epsilon > 0$. If $v \in V_1$ is sufficiently small, then there is some $x \in C_G(A)$ with $d(e,x) < \epsilon$ and $sv = tv\pi_j x$ for some j.

Proof. Let N be a bounded neighborhood of e in U_1. The Lie algebra $\mathcal{G}$ is a $\langle U_1, V_1 \rangle$-module, and thus $\mathcal{G}$ splits into a direct sum $\mathcal{G}^- \oplus \mathcal{G}^0 \oplus \mathcal{G}^+$ of negative, zero, and positive weight spaces as in Lemma 2.10. Let $\mathcal{B}^-, \mathcal{B}^0, \mathcal{B}^+$ be compact convex neighborhoods of 0 in $\mathcal{G}^-, \mathcal{G}^0, \mathcal{G}^+$, respectively.

If $v \in V_1$ is sufficiently small, then there exist $[A^+]$ and x_a as in Step 1, corresponding to the neighborhood $\mathrm{Exp}(\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+)$ of e in G: $sv = tv\pi_j x_a$ and $(ua)^{-1}x_a ua \in \mathrm{Exp}(\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+)$ for all $u \in N$ and all $a \in [A^+]$.

We may assume the neighborhoods are small enough that Exp is one-to-one on $\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+$. Thus there is a unique $\underline{x}_a \in \mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+$ with $\mathrm{Exp}(\underline{x}_a) = x_a$, and

$$\underline{x}_a.(\mathrm{Ad}u) \in [\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+].(\mathrm{Ad}a^{-1}) \subseteq \mathcal{G}^- + \mathcal{B}^0 + \mathcal{B}^+.$$

Therefore $\underline{x}_a.(\mathrm{Ad}u)\pi \in \mathcal{B}^0 + \mathcal{B}^+$, where π is projection onto $\mathcal{G}^0 + \mathcal{G}^+$. Since $C_{\mathcal{G}}(u) \subseteq \mathcal{G}^0 + \mathcal{G}^+$ (see Lemma 2.10(a)), the projection into $C_{\mathcal{G}}(u)$ is even smaller. We conclude from Lemma 9.2 that if N is large enough, then $d(e, u^{-1}x_a u) < \epsilon$ for $u \in N$.

Letting $u = e$, in particular we have $d(e, x_a) < \epsilon$ for all $a \in [A^+]$. Because $sv = tv\pi_j x_a$ and no small element of G has a fixed point near s (see 4.24), this implies $x_a = x$ is independent of a (if ϵ and v are sufficiently small). Thus $\underline{x}.(\mathrm{Ad}u) \in [\mathcal{B}^- + \mathcal{B}^0 + \mathcal{B}^+].(\mathrm{Ad}a^{-1})$ for all $u \in N$ and all $a \in [A^+]$. Letting $a \to \infty$, we get $\underline{x}.(\mathrm{Ad}u) \in \mathcal{G}^- + \mathcal{B}^0$ for all $u \in N$. Hence $\underline{x}.(\mathrm{Ad}u) \in \mathcal{G}^- + \mathcal{G}^0$ for all $u \in U_1$ (since N is Zariski dense in U_1). By the structure of $\mathrm{SL}_2(\mathbf{R})$-modules (see Lemma 2.10(b)), this implies $\underline{x} \in \mathcal{G}^0$. Therefore $x = \mathrm{Exp}(\underline{x}) \in C_G(A)^\circ$.

Step 3. $x = e$.

Proof. The Pointwise Ergodic Theorem implies there is a sequence $\{a_n\} \to \infty$ in A^+ with $sa_n, sva_n, ta_n, tva_n \in \mathrm{CONT}$. Since A^+ contracts U^-, we have $d(sa, sva) \to 0$ as $a \to +\infty$ in A^+. Because ψ is uniformly continuous on CONT, then $d(sa_n\psi, sva_n\psi) \to 0$ and $d(ta_n\psi, tva_n\psi) \to 0$ as $a_n \to +\infty$. We have $sva_n = tv\pi_j xa_n$, so $sva_n\psi = tva_n\psi\tilde{x}$. Hence

$$sa_n\psi \doteq sva_n\psi = tva_n\psi\tilde{x} \doteq ta_n\psi\tilde{x} = sa_n\psi\tilde{x}.$$

More precisely $d(sa_n\psi, sa_n\psi\tilde{x}) \to 0$ as $a_n \to +\infty$. Since $(\mathrm{CONT})\psi$ is compact, then $\tilde{x}$ has a fixed point on $(\mathrm{CONT})\psi$. If ϵ is sufficiently small, this implies $\tilde{x} = e$ (see 4.24). Therefore $x = e$ (see 10.11). ∎

This completes the proof of Theorem 10.14.

(10.16) **Theorem.** *We may assume ψ is a Borel isomorphism.*

Proof. We wish to show $\psi: \Gamma\backslash G \to \Lambda\backslash H$ is affine for G. By Theorem 10.14 it suffices to prove the isomorphism $\Gamma'\backslash G \to \Lambda\backslash H$ is affine for G. This establishes the desired reduction, because Γ' is a lattice in G. (The map $\Gamma\backslash G \to \Gamma'\backslash G$ has finite fibers (see 10.13), so Γ has finite index in Γ'. Thus Γ', like Γ, is a lattice.) ∎

(10.17) **Lemma.** *There is a local isomorphism $\wedge: G \to H$ such that $\hat{p} = \tilde{p}$ for all $p \in [P^\circ, P^\circ]$ (in particular, for $p \in U \cap \mathrm{LEVI}$), and $\hat{p} \in \tilde{p} \cdot Z(G)$ for all $p \in P^\circ$.*

Proof. We have shown that ψ is affine for P°, where $P = N_G(U)$ is a parabolic subgroup of G (see 10.5). Since $\mathbf{R}\text{-rank}(G/\operatorname{rad} G) = 1$ (see 10.10), P° is a maximal connected subgroup of G, and hence we may assume $\mathrm{Aff}_G(\psi)^\circ = P^\circ$. Similarly, since ψ is invertible (see 10.16), we may assume $\mathrm{Aff}_H(\psi^{-1})^\circ = Q^\circ$, where Q is a parabolic subgroup of H. Therefore $\sim: P^\circ \to Q^\circ$ is a (local) isomorphism.

Since $\Gamma\backslash G$ has no solvmanifold quotient, we know $[G, G] \cdot \Gamma$ is dense in G. Therefore Proposition 4.20 asserts $G = [G, G] \cdot Z(G)$. Hence Corollary 3.8 will imply the desired conclusion if we show $\widetilde{\operatorname{rad} G} = \operatorname{rad} H$. Let K_G and K_H be the connected normal subgroups of G and H, such that $K_G/\operatorname{rad} G$ and $K_H/\operatorname{rad} H$ are the maximal compact factors of $G/\operatorname{rad} G$ and $H/\operatorname{rad} H$, respectively. We will show K_G is the unique maximal connected nonergodic normal subgroup of P°. Of course, K_H can then be similarly characterized in Q°, from which it follows that $\tilde{K}_G = K_H$. Since $\operatorname{rad} K_G = \operatorname{rad} G$ and $\operatorname{rad} K_H = \operatorname{rad} H$, this implies $\widetilde{\operatorname{rad} G} = \operatorname{rad} H$, as desired.

All that remains is to prove K_G is the unique maximal connected nonergodic normal subgroup of P°. So let K be some other such. Since K is nonergodic, it projects to an Ad-precompact subgroup of $G/\operatorname{rad} G$ (10.9). Therefore $K \cdot K_G$ projects to an Ad-precompact subgroup of $G/\operatorname{rad} G$, and hence $K \cdot K_G$ is nonergodic. Then the maximality of K implies

$K_G \subseteq K$. Since K is normalized by P° and projects to an Ad-precompact subgroup of $G/\operatorname{rad} G$, Lemma 2.25 asserts K is normal in G. Because $K_G \subseteq K$ and $K/\operatorname{rad} G$ is Ad-precompact, this implies $K = K_G$ as desired. ∎

(10.18) **Theorem.** *There is a one-parameter subgroup V_1 of the intersection of* LEVI *with a maximal connected unipotent subgroup U^- opposite to U such that ψ is affine for V_1.*

Proof (cf. [19, Lemma 3.4]). The proof is very similar to that of Lemma 10.15 so we give only a sketch of the argument. Let U_1 be a one-parameter subgroup of $U \cap$LEVI and let V_1 be any one-parameter unipotent subgroup of LEVI such that $\langle U_1, V_1 \rangle$ is locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$. Let U^- be the maximal connected unipotent subgroup of G containing V_1, let $A \subseteq N_G(U) \cap N_G(U^-)$ be the identity component of a maximal $\mathbf{R}$-split torus in LEVI, and let A^+ be the sub-semigroup of expanding automorphisms of $U \in$ LEVI. For every sufficiently small $v \in V_1$, we wish to show $sv\psi = s\psi\hat{v}$ for a.e. $s \in \Gamma\backslash G$.

Step 1. Given $\delta > 0$ and any compact interval $N \subseteq U_1$. If $v \in V_1$ is sufficiently small, then (for a.e. $s \in \Gamma\backslash G$) there is an unbounded subset $[A^+] \subseteq A^+$ satisfying: for each $a \in [A^+]$, there is some $x_a \in H$ with $sv\psi = s\psi\hat{v}x_a$ such that $d(\tilde{u}\tilde{a}, x_a\tilde{u}\tilde{a}) < \delta$ for all $u \in N$.

Sketch of Proof. Let CONT be a large compact set (say $\mu(\text{CONT}) > .9$) on which ψ is continuous. There is a map $U_1 \to U_1 \colon u \mapsto \overline{u}$ such that $vua \doteq \overline{u}a$ for all $u \in N$ and all sufficiently large $a \in A^+$. Therefore $\hat{v}\tilde{u}\tilde{a} \doteq \tilde{\overline{u}}\tilde{a}$ for all $u \in N$. (We have $\tilde{u} = \hat{u}, \tilde{\overline{u}} = \hat{\overline{u}}$, and $\tilde{a} \in \hat{a} \cdot Z(G)$, so $\tilde{a}^{-1}\tilde{\overline{u}}^{-1}\hat{v}\tilde{u}\tilde{a} = \hat{a}^{-1}\hat{\overline{u}}^{-1}\hat{v}\hat{u}\hat{a} = (a^{-1}\overline{u}^{-1}vua)^\wedge \doteq e$.)

Thus

$$sv\psi\tilde{u}\tilde{a} = svua\psi \doteq s\overline{u}a\psi = s\psi\tilde{\overline{u}}\tilde{a} \doteq s\psi\hat{v}\tilde{u}\tilde{a}$$

whenever $svua, s\overline{u}a \in \text{CONT}$. By polynomial divergence of orbits and the Ratner Covering Lemma, this is sufficient to establish the claim.

Step 2 [see Step 2 of Lemma 10.15]. Given $\epsilon > 0$. If $v \in V_1$ is sufficiently small, then for a.e. $s \in \Gamma\backslash G$, we have $sv\psi = s\psi\hat{v}c^s$ for some $c^s \in C_H(\tilde{A})$ with $d(e, c^s) < \epsilon$.

Step 3. $c^s = e$.

Sketch of Proof. Since $d(sa, sva) \to 0$ as $a \to \infty$ in A^+, we have

$$sa\psi \doteq sva\psi = s\psi\hat{v}c^s\tilde{a} = s\psi\hat{v}\tilde{a}c^s \doteq s\psi\tilde{a}c^s = sa\psi c^s$$

whenever $sa, sva \in \text{CONT}$. Therefore $c^s = e$.

This completes the proof of Theorem 10.18. ∎

We now know ψ is affine for P° (see 10.15) and for a one-parameter subgroup V_1 not contained in P° (see 10.18). Since $\langle P^\circ, V_1 \rangle = G$ (see 2.23), we conclude that ψ is affine for G. This completes the proof of the Main Theorem (1.3 or 10.1). ∎

CHAPTER 11

APPLICATIONS TO ACTIONS OF SEMISIMPLE GROUPS

(11.1) **Theorem.** *Suppose G, H_1, H_2 are connected Lie groups, and let $\Lambda_i \backslash H_i$ be a faithful finite-volume homogeneous space of H_i. Embed G in H_1 and H_2. Assume G is semisimple with no compact factors, and acts ergodically on $\Lambda_1 \backslash H_1$. Then any measure theoretic isomorphism from the G-action on $\Lambda_1 \backslash H_1$ to the G-action on $\Lambda_2 \backslash H_2$ is an affine map (a.e.).*

Proof. Let π_i be the embedding of G in H_i and let $\psi: \Lambda_1 \backslash H_1 \to \Lambda_2 \backslash H_2$ be a measure preserving Borel map which is affine for $G\pi_1$ via $G\pi_2$. The Moore Ergodicity Theorem (6.5) implies there is some unipotent $u \in G$ which acts ergodically on $\Lambda_1 \backslash H_1$ and $\Lambda_2 \backslash H_2$. Because $u\pi_1$ and $u\pi_2$ are unipotent elements of H_1 and H_2 [25, Proposition 3.4.2, p. 54], the Main Theorem asserts ψ is affine for H_1. ∎

(11.2) **Corollary.** *Suppose G, H_i, Λ_i are as in the preceding corollary, and let Γ be a lattice in G. Then any measure theoretic isomorphism of the Γ-actions on $\Lambda_1 \backslash H_1$ and $\Lambda_2 \backslash H_2$ is affine (a.e.).*

Proof. Let π_i be the embedding of G in H_i, and let $\psi: \Lambda_1 \backslash H_1 \to \Lambda_2 \backslash H_2$ be a measure preserving Borel map that is affine for $\Gamma\pi_1$. A straightforward check shows that the map

$$\tau: \Gamma \backslash G \times \Lambda_1 \backslash H_1 \to \Gamma \backslash G \times \Lambda_2 \backslash H_2 : (\Gamma g, s) \to (\Gamma g, s(g^{-1}\pi_1)\psi \cdot (g\pi_2))$$

is well-defined and that τ is affine for the image of the diagonal embedding of G in $G \times H_1$. Apply the Main Theorem to conclude that τ is affine for $G \times H_1$. Then it is more-or-less obvious (Fubini!) that ψ is affine for H_1. ∎

(11.3) **Theorem.** *Let G and H be connected Lie groups, and Λ be a lattice in H. Assume G is semisimple, and each of its simple factors has real rank at least two. Suppose G acts measurably on $\Lambda \backslash H$ in an arbitrary way that preserves the finite measure. If some unipotent element of G acts by an ergodic translation of $\Lambda \backslash H$, then all elements of G act by translations (a.e.).*

Idea of Proof. Suppose u is a unipotent element of G that acts by an ergodic translation T_u of $\Lambda \backslash H$. (In particular, $\operatorname{rad} H$ is nilpotent.) It will suffice to show that every $c \in C_G(u)^\circ$ acts by a translation, because we can then use this "Affine for the Centralizer" property repeatedly to conclude that every element of g acts by a translation (see the proof of [23, Theorem 4.1]). Now any $c \in C_G(u)$ acts by a transformation ψ_c of $\Lambda\backslash H$ that commutes with T_u. Since T_u has zero entropy (unipotent elements of G must act with entropy 0 or ∞, and no translation has infinite entropy) and H is semisimple, one can prove a version of the Main Theorem which asserts ψ_c is an affine map (a.e.) [23, Main Theorem 1.1]. Thus $c \to \psi_c$ is a (continuous) homomorphism from $C_G(u)$ into the group $\mathrm{Aff}(\Lambda\backslash H)$ of affine transformations of $\Lambda\backslash H$. The identity component of $\mathrm{Aff}(\Lambda\backslash H)$ consists of translations (because the automorphisms of H that stabilize Λ form a *discrete* subgroup of $\mathrm{Aut}\, H$), so each $c \in C_G(u)^\circ$ acts by a translation. ∎

(11.4) **Corollary.** *Let Γ (resp. Λ) be a lattice in a connected semisimple Lie group G (resp. H) with trivial center and no compact factors. Assume the* $\mathbf{R}$*-rank of every simple factor of G is at least 2. Suppose Γ (resp. Λ) admits an injective homomorphism, with dense range, into a connected compact semisimple Lie group K (resp. L), and thus Γ (resp. Λ) acts ergodically (and essentially freely) by translations on any faithful homogeneous space K/A (resp. L/B) of K (resp. L). If the Γ-action on K/A is orbit equivalent to the Λ-action on L/B, then $\Gamma \cong \Lambda$ and, identifying Γ with Λ under this isomorphism, the Γ-actions on K/A and L/B are isomorphic.*

Sketch of Proof. Consider the action of G (resp. H) on the double coset space $\Delta(\Gamma)\backslash K \times G/A$ (resp. $\Delta(\Lambda)\backslash L \times H/B$), where $\Delta(\Gamma)$ (resp. $\Delta(\Lambda)$) is the diagonal embedding of Γ in $K \times G$ (resp. Λ in $L \times H$). (These are the "induced" actions of G and H [25, Def. 4.2.21, p. 75].) Because the Γ-action on K/A is orbit-equivalent to the Λ-action on L/B, one can show the above actions of G and H on these double coset spaces are orbit-equivalent and essentially free (cf. proof of [25, Corollary 5.2.2, p. 96]). Then the Zimmer Super-rigidity Theorem [25, Theorem 5.2.1, p. 95] asserts $G \cong H$ and, after identifying G with H under this isomorphism, the $G-$actions are isomorphic. Thus there is a G-equivariant Borel isomorphism

$$\varphi\colon \Delta(\Gamma)\backslash K \times G/A \to \Delta(\Lambda)\backslash L \times G/B.$$

In addition, there are G-equivariant projections:

$$\begin{aligned}
\pi_1 &\colon \Delta(\Gamma)\backslash K \times G \to \Delta(\Gamma)\backslash K \times G/A,\\
\pi_2 &\colon \Delta(\Lambda)\backslash L \times G \to \Delta(\Lambda)\backslash L \times G/B,\\
\sigma_1 &\colon \Delta(\Gamma)\backslash K \times G/A \to \Gamma\backslash G,\\
\sigma_2 &\colon \Delta(\Lambda)\backslash L \times G/B \to \Lambda\backslash G.
\end{aligned}$$

The Main Theorem implies that the composition

$$\pi_1\varphi\sigma_2\colon \Delta(\Gamma)\backslash K \times G \to \Lambda\backslash G$$

is affine (a.e.). The kernel of the homomorphism associated to the affine map $\pi_1 \varphi \sigma_2$ must be K, which is also the kernel associated to

$$\pi_1 \sigma_1 \colon \Delta(\Gamma) \backslash K \times G \to \Gamma \backslash G.$$

So φ induces a G-isomorphism $\bar{\varphi} \colon \Gamma \backslash G \to \Lambda \backslash G$. Thus $\Gamma \cong \Lambda$. The fiber of σ_1 is the Γ-action on K/A and (identifying Γ with Λ) the fiber of σ_2 is the Γ-action on L/B. Thus these Γ-actions are isomorphic. $\blacksquare$

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