Cocycle Superrigidity for Ergodic Actions of Non-semisimple Lie Groups

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Abstract

Suppose L is a semisimple Levi subgroup of a connected Lie group G, X is a Borel G-space with finite invariant measure, and $\alpha\colon X \times G \to \mathrm{GL}_n(\mathbb{R})$ is a Borel cocycle. Assume L has finite center, and that the real rank of every simple factor of L is at least two. We show that if L is ergodic on X, and the restriction of α to $X \times L$ is cohomologous to a homomorphism (modulo a compact group), then, after passing to a finite cover of X, the cocycle α itself is cohomologous to a homomorphism (modulo a compact group).

1 Introduction

Definition 1.1 ([11], Defns. 4.2.1 and 4.2.2, p. 65) Suppose G and H are Lie groups, and X is a Borel G-space. A Borel function $\alpha\colon X \times G \to H$ is a *Borel cocycle* if, for all $g, h \in G$, we have

$$(x, gh)^\alpha = (x, g)^\alpha (xg, h)^\alpha \qquad \text{for a.e. } x \in X.$$

Two cocycles α and β are *cohomologous* if there is a Borel function $\phi\colon X \to H$, such that, for all $g \in G$, we have

$$(x, g)^\alpha = (x^\phi)^{-1} (x, g)^\beta (xg)^\phi \qquad \text{for a.e. } x \in X.$$

Any continuous group homomorphism $\sigma\colon G \to H$ gives rise to a cocycle, defined by $(\cdot, g)^\alpha = g^\sigma$. For actions of semisimple groups, R. J. Zimmer's Cocycle Superrigidity Theorem often shows that (up to cohomology, and modulo a compact group) these are the only examples.

Definition 1.2 (cf. 1.5) Suppose G is a Lie group, X is an ergodic Borel G-space with finite invariant measure, H is a subgroup of $\mathrm{GL}_n(\mathbb{R})$, and $\alpha\colon X \times G \to H$ is a Borel cocycle. We say α is *Zariski dense* if H is contained in the Zariski closure of the range of every cocycle $\beta\colon X \times G \to H$ that is cohomologous to α.

Theorem 1.3 (Zimmer, cf.[11], Theorems. 5.2.5, 7.1.4, 9.1.1)
Suppose G is a connected, semisimple Lie group, X is an ergodic Borel G-space with finite invariant measure, H is a Zariski closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, and $\alpha\colon X \times G \to H$ is a Zariski-dense Borel cocycle. Assume G has finite center, and that the real rank of every simple factor of G is at least two. If H is reductive, then, after replacing G and X by finite covers, there are:

    1. a homomorphism $\sigma\colon G \to H$;

    2. a compact, normal subgroup K of H that centralizes G^σ; and

    3. a cocycle β that is cohomologous to α;

such that, for every $g \in G$, we have $(x,g)^\beta \in g^\sigma K$ for a.e. $x \in X$.

This paper extends Zimmer's result to groups that are not semisimple. Our main theorem reduces the general case to the semisimple case.

Theorem 1.4 *Let G be a connected Lie group, X be a Borel G-space with finite invariant measure and H be a connected Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ that is of finite index in its Zariski closure, and has no nontrivial compact, normal subgroups. Let $\alpha\colon X \times G \to H$ be a Zariski-dense Borel cocycle. Let L be the product of the noncompact, simple factors in a semisimple Levi subgroup of G. Assume that the L-action on X is ergodic. Let $\sigma\colon L \to H$ be a continuous homomorphism and K be a compact subgroup of H that centralizes L^σ. Suppose that $H = (\mathrm{Rad}\,H)L^\sigma K$; and that for every $l \in L$, we have $(x,l)^\alpha \in l^\sigma K$ for o.e. $x \in X$. Then σ extends to a continuous homomorphism defined on all of G and α is cohomologous to the cocycle β, defined by $(\cdot,g)^\beta = g^\sigma$.*

By combining our theorem with Zimmer's, we obtain the following general result.

Definition 1.5 (cf.[[11], Defn. 9.2.2, p. 167]) Suppose L is a connected Lie group, X is an ergodic Borel L-space with finite invariant measure, and $\alpha\colon X \times L \to \mathrm{GL}_n(\mathbb{R})$ is a Borel cocycle. The *Zariski hull* of α is a Zariski closed subgroup J of $\mathrm{GL}_n(\mathbb{R})$, such that α is cohomologous to a cocycle $\beta\colon X \times L \to \mathrm{GL}_n(\mathbb{R})$, such that the range of β is contained in J, but α is not cohomologous to any cocycle whose range is contained in a Zariski-closed, proper subgroup of J. The Zariski hull J always exists, and is unique up to conjugacy.

Corollary 1.6 *Let G, X, H, α and L be as in Theorem 1.4. As in the theorem assume that the L-action on X is ergodic. Suppose further that*

L has finite center, the real rank of every simple factor of L is at least two and that the Zariski hull of the restriction of α to $X \times L$ is reductive. Then, after passing to a finite cover of X, the cocycle α is cohomologous to a homomorphism.

Remark 1.7 The assumption that L is ergodic on X cannot be weakened to the ergodicity of G. To see this, suppose L is not ergodic, and let Y be the space of ergodic components of L on X. The Mautner phenomenon [8, Thm. 1.1] implies that L and $[L, G]$ have the same ergodic components, so the G-action on X factors through to an action of $G/[L, G]$ on Y. Because $G/[L, G]$ is amenable, there may be cocycles of this action that are not related to the algebraic structure of the acting group (cf. [1]). Pulling back to G, these are cocycles on X that have almost nothing to do with G.

On the other hand, it is not known whether the assumption that H is reductive can be omitted from Zimmer's Theorem; perhaps this hypothesis is always satisfied. Zimmer [12, cf. Thm. 1.1] proved this to be the case for cocycles that satisfy an L^1 growth condition. Then, by an argument very much in the spirit of the present paper, he was able to derive a superrigidity theorem for L^1 cocycles of actions of some non-semisimple groups (see [12, Thm. 4.1]).

Zimmer's cocycle superrigidity theorem (1.3) was inspired by the superrigidity theorem for finite-dimensional representations of lattices in semisimple Lie groups, proved by G. A. Margulis [7, Thm. VII.5.9, p. 230]. The present work was suggested by the author's [9], [10, §2, §5] generalization of Margulis' theorem to non-semisimple groups.

After some preliminaries in §2, we prove a restricted version of Theorem 1.4 in §3. The final section of the paper removes the restrictions, and presents a proof of Corollary 1.6.

2   Preliminaries

Standing assumptions 2.1 The notation and hypotheses of Theorem 1.4 are in effect throughout this section.

Notation 2.2 We usually write our maps as superscripts. Thus, if $x \in X$ and $\phi\colon X \to Y$, then x^ϕ denotes the image of x under ϕ.

If g and h belong to H (or to any other group), then g^h denotes the conjugate $h^{-1}gh$.

For any Borel function ϕ whose range lies in H, we use $-\phi$ to denote the function defined by $(\cdot)^{-\phi} = \big((\cdot)^\phi\big)^{-1}$.

Definition 2.3 Let Q be a subset of G, and let $\mathcal{F}\colon X \times G \to H$ be a Borel function. We use $\mathcal{F}|_Q$ to denote the restriction of $\mathcal{F}$ to $X \times Q$.

If Q is countable, or is a Lie subgroup of G, then there is a natural choice of a measure class on Q, and we use $(X \times Q)^{\mathcal{F}}$ to denote the essential range of $\mathcal{F}|_Q$. (Recall that the *essential range* is the unique smallest closed set whose inverse image is conull.)

If, for some $r \in G$, the function $\mathcal{F}|_r$ is essentially constant, then we often omit the reference to X, and simply write $r^{\mathcal{F}}$ for the single point in $(X \times r)^{\mathcal{F}}$.

Definition 2.4 Let Q be a subgroup of G, and let $\mathcal{F}\colon X \times G \to H$ be a Borel function. We say that $\mathcal{F}|_Q$ is a *homomorphism* if there is a homomorphism $\sigma\colon Q \to S$, such that, for all $r \in Q$, we have $(\cdot, r)^{\mathcal{F}} = r^\sigma$ a.e.

The following well-known result is a straightforward consequence of the cocycle identity.

Lemma 2.5 *Let Q be a subgroup of G, and let $\mathcal{F}\colon X \times G \to H$ be a Borel cocycle. Then $\mathcal{F}|_Q$ is a homomorphism if and only if $(\cdot, r)^{\mathcal{F}} = r^{\mathcal{F}}$ is essentially constant, for each $r \in Q$.*

$\square$

Definition 2.6 Let us say that an element $g \in \mathrm{GL}_n(\mathbb{R})$ is *split* if every eigenvalue of g is real and positive. (In other words, the real Jordan decomposition of g [3, Lem. IX.7.1, p. 430] has no elliptic part.)

Let us say that an element of L is *split* if it belongs to a one-parameter subgroup T of L, such that $\mathrm{Ad}_L t$ is split, for all $t \in T$. Note that if l is a split element of L, then l^σ is a split element of H.

Lemma 2.7 *Suppose l is a split element of L, $k \in K$, and T is a connected subgroup of H, such that $l^\sigma k$ normalizes T. Then k normalizes T.*

Proof The normalizer of T is Zariski closed (cf. proof of [11, Thm. 3.2.5, p. 42]), so it contains the elliptic part of each of its elements (cf. proof of [6, Thm. 15.3, p. 99]). Therefore, the desired conclusion follows from the observation that k is the elliptic part of $l^\sigma k$.

□

Theorem 2.8 (Borel Density Theorem [[2], Cor. 2.6]) *Suppose H acts regularly on a variety V, μ is a probability measure on V, and h is a split element of H. If μ is h-invariant, then h fixes the support of μ pointwise.*

■

Corollary 2.9 *Suppose H acts regularly on a variety V. Let $\psi \colon X \to V$ be a Borel function, and let $g \in G$ and $h \in H$, and let K be a compact subgroup of H that centralizes h. Assume $x^{g\psi} \in (x^{\psi h})^K$ for a.e. $x \in X$, and h is split. Then h fixes the essential range of ψ pointwise.*

Proof Let V/K be the space of K-orbits on V. The induced map $\overline{\psi} \colon X \to V/K$ satisfies $x^{g\overline{\psi}} = x^{\overline{\psi}h}$. Therefore, the G-invariant probability measure on X pushes via $\overline{\psi}$ to an h-invariant probability measure $\overline{\mu}$ on V/K. By using $\overline{\mu}$ to integrate together the K-invariant probability measure on each K-orbit, we may lift $\overline{\mu}$ to an h-invariant probability measure μ on V. Note that the support of μ is $(X^\psi)^K$, where X^ψ is the essential range of ψ. On the other hand, the Borel Density Theorem implies that h fixes the support of μ pointwise.

□

Theorem 2.10 (Moore Ergodicity Theorem, cf.[[8], Thm. 1.1]) *If T is a connected subgroup of L, such that, for every simple factor L_i of L, the projection of T into $L_i/Z(L_i)$ has noncompact closure, then T is ergodic on X.*

■

Proposition 2.11 ([4], Thm. XV.3.1, pp. 180–181, and see p. 186)
*If G is a Lie group that has only finitely many connected components, then
G has a maximal compact subgroup K, and every compact subgroup of G is
contained in a conjugate of K.*

□

Proposition 2.12 ([[4], Thm. XIII.1.3, p. 144]) *If a Lie group G is
compact, connected, and solvable, then G is abelian.*

□

Lemma 2.13 *Let $\widetilde{G}$ be a covering group of G, and let $\widetilde{\alpha}\colon X \times \widetilde{G} \to H$ be the
cocycle naturally induced by α. If $\widetilde{\alpha}$ is cohomologous to a homomorphism,
then α is cohomologous to a homomorphism.*

Proof By assumption, there is a Borel function $\phi\colon X \to H$, such that, for
all $g \in \widetilde{G}$, the expression $x^{-\phi}(x,g)^{\widetilde{\alpha}}(xg)^{\phi}$ is essentially independent of x.
Then the same is true with α in place of $\widetilde{\alpha}$, for all $g \in G$, as desired.

□

3   Proof of Theorem 1.4 (The Main Case)

This entire section is devoted to a proof of Theorem 1.4, so the notation
and hypotheses of Thm 1.4 are in effect throughout.

Assumption 3.1 Throughout this section, we assume $\operatorname{Rad} G$ is nilpotent,
and that G has no nontrivial compact semisimple quotients. See 4.1 for an
explanation of how to obtain the full theorem from this special case.

Notation 3.2 Let $R = \operatorname{Rad} G$, so $G = RL$. By passing to a covering group
of G, we may assume R is simply connected (see 2.13).
    By assumption (and perhaps replacing K with a larger compact group),
we may write $H = S \mathbin{\mathrm{o}} (MK)$, where

  (1) S is a connected, split, solvable subgroup;

  (2) $M = L^{\sigma}$ is connected and semisimple, with no compact factors;

  (3) K is a compact subgroup that centralizes M;

  (4) $(X \times L)^{\alpha} \subset MK$; and

  (5) $M \cap K = Z(M)$.

, Now α induces a cocycle $\overline{\alpha}\colon X \times G \to H/(SK) \cong M/(M \cap K)$. Note that $\overline{\alpha}|_L = \overline{\sigma}$ is a homomorphism.

Proposition 3.3 $(X \times R)^{\overline{\alpha}} = e.$

Proof Let P be a minimal parabolic subgroup of G. From [11, Step 1 of proof of Thm. 5.2.5, p. 103], we know there is an (almost) Zariski closed, proper subgroup L of M and a Borel function $\phi\colon X \times P\backslash G \to L\backslash M$ such that, for all $g \in G$, we have

$$(xg, cg)^{\phi} = (x, c)^{\phi}(x, g)^{\overline{\alpha}} \qquad \text{for a.e. } (x,c) \in X \times P\backslash G.$$

In particular, for $l \in L$, we have $(xl, cl)^{\phi} = (x,c)^{\phi}l^{\overline{\alpha}}$. Then, by Fubini's Theorem, we see that, for a.e. $c \in P\backslash G$, $l \in P^c \cap L$, and $x \in X$, we have $(xl, c)^{\phi} = (x,c)^{\phi}l^{\overline{\alpha}}$. So 2.9 implies that $l^{\overline{\alpha}}$ fixes $(X \times c)^{\phi}$ pointwise (assuming that l is split), which implies $(xl, c)^{\phi} = (x,c)^{\phi}$. So, from the ergodicity of $P^c \cap L$ (see 2.10), we conclude that $(\cdot, c)^{\phi} = c^{\phi}$ is essentially constant. Therefore, for $c \in P\backslash G$ and $r \in \mathrm{Rad}\,G$, because $cr = c$, we have

$$(x,c)^{\phi} = (xr, cr)^{\phi} = (x,c)^{\phi}(x,r)^{\overline{\alpha}}.$$

Therefore, $(x,r)^{\overline{\alpha}}$ fixes $(X \times P\backslash G)^{\phi}$ pointwise, so $(x,r)^{\overline{\alpha}}$ is trivial (cf. [11, proof of Lem. 5.2.8, p. 102]). $\qquad\blacksquare$

Definition 3.4 For $r \in R$ and $l \in L$, we have

$$(x, rl)^{\overline{\alpha}} = (x,r)^{\overline{\alpha}}(xr, l)^{\overline{\alpha}} = l^{\overline{\alpha}},$$

so we see that $\overline{\alpha}$ is a homomorphism. By replacing G with a finite cover, we may assume that $\overline{\alpha}$ lifts to a homomorphism $\mathcal{M}\colon G \to M$ (see 2.13). Because $H = S \mathbin{\mathfrak{o}} (MK)$, there are well-defined Borel functions $\mathcal{S}\colon X \times G \to S$ and $\mathcal{K}\colon X \times G \to K$, such that

$$(x,g)^{\alpha} = (x,g)^{\mathcal{S}} g^{\mathcal{M}}(x,g)^{\mathcal{K}} \qquad \text{for a.e. } x \in X.$$

Note that, for $u \in L$ and $r \in R$, we have

$$(x,u)^{\alpha} = u^{\mathcal{M}}(x,u)^{\mathcal{K}} \text{ for a.e. } x \in X; \text{ and}$$
$$(x,r)^{\alpha} = (x,u)^{\mathcal{S}}(x,u)^{\mathcal{K}} \text{ for a.e. } x \in X.$$

Note also that $G^{\mathcal{M}}$ centralizes $(X \times G)^{\mathcal{K}}$, because M centralizes K.

Because $\mathcal{S}$ may not be a cocycle, Lemma 2.5 may not apply to $\mathcal{S}$. However, the following lemma is a suitable replacement.

Definition 3.5 Let Q be a subgroup of R. A function $\sigma \colon Q \to S$ is a *crossed homomorphism* if there is a homomorphism $\kappa \colon R \to K'$, such that $(rs)^\sigma = r^\sigma s^{\sigma \tau^\kappa}$ for all $r, s \in Q$, where $K' = N_K(\langle Q^\sigma \rangle)/C_K(\langle Q^\sigma \rangle)$.

We say that $S|_Q$ is a *crossed homomorphism* if there is a crossed homomorphism $\sigma \colon Q \to S$ such that, for all $r \in Q$, we have $(\cdot, r)^S = r^\sigma$ a.e..

Lemma 3.6 *Let Q be a subgroup of R.*

1. *$S|_Q$ is a crossed homomorphism if and only if $(\cdot, r)^S$ is essentially constant, for each $r \in Q$.*

2. *$S|_Q$ is a homomorphism if and only if $S|_Q$ is a crossed homomorphism and $(X \times Q)^\kappa$ centralizes Q^S.*

Proof (1) We need only prove the nontrivial direction, so assume $(\cdot, r)^S = r^S$ is essentially constant, for each $r \in Q$. For convenience, let $N = N_K(\langle Q^S \rangle)$ and $C = C_K(\langle Q^S \rangle)$.

For $r, s \in Q$, we have $(x, rs)^\alpha = (x, r)^\alpha (xr, s)^\alpha$, so $(rs)^S = r^S s^{S(x,r)^{-\kappa}}$. This implies that $s^{S(x,r)^{-\kappa}} \in \langle Q^S \rangle$, so we see that $(X \times Q)^\kappa \subset N$. This also implies that $(\cdot, r)^\kappa$ is essentially constant, modulo C. Therefore, the induced cocycle $\overline{\mathcal{K}} \colon X \times N/C$ is a homomorphism, as desired.

(2) Because $(rs)^S = r^S s^{S(x,r)^{-\kappa}}$, we see that $(rs)^S = r^S s^S$ if and only if $(x, r)^\kappa$ centralizes s^S. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 3.7 (of proof) *Suppose Q is a subgroup of R. Let*

$$N = N_K(\langle Q^S \rangle) \quad and \quad C = C_K(\langle Q^S \rangle).$$

If $S|_Q$ is a crossed homomorphism, then the cocycle $\overline{\mathcal{K}} \colon X \times Q \to N/C$, induced by $\mathcal{K}$, is a homomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Notation 3.8 Let U be a maximal connected unipotent subgroup of L, let U^- be a maximal connected unipotent subgroup that is opposite to U, and let A be a maximal split torus of L that normalizes both U and U^-. Thus, $N_L(U) \cap N_L(U^-)$ is reductive, and contains A in its center.

Lemma 3.9 *Suppose $r \in R$, and u is a split element of L. Let $w = [u, r] \in R$, and assume $(\cdot, w)^S = w^S$ is essentially constant, and that $(X \times u)^\kappa$ and $(X \times w)^\kappa$ centralize w^S. Then $(xu, r)^S = (x, r)^{S(x,u)^\kappa}$ for a.e. $x \in X$.*

Proof For a.e. $x \in X$, because $r = u^{-1}ruw$, we have

$$
\begin{aligned}
(xu, r)^\alpha &= (x, u)^{-\alpha}(x, r)^\alpha(xr, u)^\alpha(xru, w)^\alpha \\
&= (x, u)^{-\mathcal{K}}(x, r)^{\alpha u^\mathcal{M}} w^\mathcal{S}(xr, u)^\mathcal{K}(xru, w)^\mathcal{K} \\
&\in \left((x, r)^{\alpha u^\mathcal{M}} w^\mathcal{S}\right)^{(x, u)^\mathcal{K}} C_K(w^\mathcal{S}).
\end{aligned}
$$

So the Borel Density Theorem (see 2.9) implies that $(x, r)^{\alpha u^\mathcal{M}} w^\mathcal{S} = (x, r)^\alpha$. Thus, we have

$$
(xu, r)^\alpha \in (x, r)^{\alpha(x, u)^\mathcal{K}} C_K(w^\mathcal{S}),
$$

which implies $(xu, r)^\mathcal{S} = (x, r)^{\mathcal{S}(x, u)^\mathcal{K}}$.

$\square$

Corollary 3.10 *Suppose $r \in R$, u is a split element of L, and W is a subgroup of R. Assume $\mathcal{S}|_W$ is a homomorphism, $(X \times u)^\mathcal{K}$ centralizes $W^\mathcal{S}$, and that $w = [u, r] \in W$. If $(\cdot, r)^\mathcal{S} = r^\mathcal{S}$ is essentially constant, then $(X \times u)^\mathcal{K}$ centralizes $r^\mathcal{S}$.*

$\square$

Corollary 3.11 *Suppose u is a unipotent element of L, and W is a subgroup of R that is normalized by u. If $\mathcal{S}|_W$ is a homomorphism, then $(X \times u)^\mathcal{K}$ centralizes $W^\mathcal{S}$.*

Proof Because u is unipotent, we may assume by induction on $\dim W$ that $(X \times u)^\mathcal{K}$ centralizes $([u, W][W, W])^\mathcal{S}$. Then 3.10 implies that $(X \times u)^\mathcal{K}$ centralizes $W^\mathcal{S}$.

$\square$

Definition 3.12 Given a Borel function $\phi \colon X \to K$, let $\alpha^\phi \colon X \times G \to H$ be the cocycle cohomologous to α defined by

$$
(x, g)^{\alpha^\phi} = x^{-\phi}(x, g)^\alpha(xg)^\phi.
$$

Also define a Borel function $\mathcal{S}^\phi \colon X \times G \to \mathcal{S}$ by $(x, g)^{\mathcal{S}^\phi} = (x, g)^{\mathcal{S}x^\phi}$. Note that $(x, g)^{\alpha^\phi} \in (x, g)^{\mathcal{S}^\phi} g^\mathcal{M} K$, for all $(x, g) \in X \times G$.

Corollary 3.13 *Suppose Q and W are Lie subgroups of R, and u is a split element of L, such that $[u, Q] \subset W$. Assume $\mathcal{S}|_W$ is a homomorphism, and that $(X \times u)^\mathcal{K}$ centralizes $W^\mathcal{S}$. Let K_u be the closure of $\langle (X \times u)^\mathcal{K} \rangle$. If u is ergodic on X, then there is a Borel function $\phi \colon X \to K_u$, such that $\mathcal{S}^\phi|_Q$ is a crossed homomorphism.*

Proof Let $\mathrm{Func}(Q,S)$ be the the space of Borel functions from Q to S, where two functions are identified if they agree almost anywhere. (The topology of convergence in measure defines a countably generated Borel structure on this space [11, pp. 49–50].) The function $\mathcal{S}|_Q\colon X \times Q \to S$ determines a Borel function $\mathcal{F}\colon X \to \mathrm{Func}(Q,S)$. From 3.9, we see that, for a.e. $x \in X$, we have $(xu)^{\mathcal{F}} = x^{\mathcal{F}(x,u)^{\mathcal{K}}} \in x^{\mathcal{F}K_u}$, where K_u acts on $\mathrm{Func}(Q,S)$ via conjugation on the range space S. Thus, the ergodicity of u implies that there is a Borel function $\phi\colon X \to K_u$, and some $\sigma \in \mathrm{Func}(Q,S)$, such that $x^{\mathcal{F}x^{\phi}} = \sigma$ for a.e. $x \in X$. That is, for a.e. $x \in X$ and a.e. $r \in Q$, we have $(x,r)^{\mathcal{S}x^{\phi}} = r^{\sigma}$. In other words, $(\cdot,r)^{\mathcal{S}^{\phi}} = r^{\sigma}$ is essentially constant, for a.e. $r \in Q$. Then, because Q has no proper subgroups of full measure, we conclude from the cocycle identity (applied to α^{ϕ}) that we have $(\cdot,r)^{\mathcal{S}^{\phi}} = r^{\sigma}$, for *all* $r \in Q$.

□

The following is the special case where W is trivial.

Corollary 3.14 *Let u be a split element of L that is ergodic on X, and let K_u be the closure of $\langle (X \times u)^{\mathcal{K}} \rangle$. Then there is a Borel function $\phi\colon X \to K_u$, such that $\mathcal{S}^{\phi}|_{C_R(u)}$ is a crossed homomorphism.*

□

Most of the work in this section is devoted to showing that we may assume $\mathcal{S}|_R$ is a homomorphism. The following proposition represents our first real progress toward this goal. Most of the rest is achieved by an inductive argument based on the unipotence of U and the solvability of R.

Proposition 3.15 *We may assume $\mathcal{S}|_{C_R(L)}$ is a homomorphism.*

Proof For convenience, let $Q = C_R(L)$, let V be the Zariski closure of $\langle Q^{\mathcal{S}} \rangle$, $N = N_K(V)$ and $C = C_K(V)$. From 3.14, we see that, by replacing α with a cohomologous cocycle α^{ϕ}, we may assume $\mathcal{S}|_{C_R(L)}$ is a crossed homomorphism. Then 3.7 implies that the induced cocycle $\overline{\mathcal{K}}\colon X \times Q \to N/C$ is a homomorphism. Therefore, the induced cocycle $\overline{\alpha}\colon X \times Q \to VN/C$ is a homomorphism. Then, because Q is nilpotent (see 3.1), the Zariski closure of $Q^{\overline{\alpha}}$ in VN/C is of the form $W \times T/C$, where $W \subset V$ is split and T is a compact torus (cf. [6, Prop. 19.2, p. 122]). Because maximal compact subgroups are conjugate (see 2.11), there is some $v \in V$ with $T \subset N^v$. Because v normalizes C (indeed, it centralizes C), and 3.10 implies that $(X \times L)^{\mathcal{K}} \subset C$, we know that K^v contains $(X \times L)^{\mathcal{K}}$, so there is no harm in replacing K with K^v. Thus, we may assume $T \subset K$.

Then, for any $r \in Q$, $(x, w)^S$ and $(x, w)^{\mathcal{K}}$ are the projections of $(x, w)^\alpha$ into W and T, respectively. Because T centralizes W, we conclude that $(X \times Q)^{\mathcal{K}}$ centralizes W^S, as desired.

□

Lemma 3.16 *Let Q be a one-parameter subgroup of R that is normalized, but not centralized, by A. If $S|_Q$ is a crossed homomorphism, then $S|_Q$ is a homomorphism.*

Proof For convenience, let V be the Zariski closure of $\langle Q^S \rangle$, $N = N_K(V)$ and $C = C_K(V)$. From 3.7, we know that the induced cocycle $\overline{\mathcal{K}} \colon X \times Q \to N/C$ is a homomorphism. We wish to show that $\overline{\mathcal{K}}$ is trivial.

Because A does not centralize Q, there is some $a \in A$ with $r^a = r^2$ for all $r \in Q$. Because $r^{aS} = r^{S(x,a)^{\mathcal{K}} a^{\mathcal{M}}}$, we see that $(x, a)^{\mathcal{K}} \in N$ (see 2.7), and that $(\cdot, a)^{\mathcal{K}}$ is constant, modulo C. Thus, $a^{\overline{\mathcal{K}}} = (\cdot, a)^{\mathcal{K}} C$ is a well-defined element of N/C, so $\overline{\mathcal{K}}$ extends to a homomorphism defined on QA. Thus, we have $r^{\overline{\mathcal{K}} a^{\overline{\mathcal{K}}}} = r^{a\overline{\mathcal{K}}} = (r^2)^{\overline{\mathcal{K}}} = (r^{\overline{\mathcal{K}}})^2$, for all $r \in Q$. If $Q^{\overline{\mathcal{K}}}$ is nontrivial, this implies that 2 is an eigenvalue of $\mathrm{Ad}_{N/C} a^{\overline{\mathcal{K}}}$. But eigenvalues in a compact group all have absolute value 1 - contradiction.

□

Lemma 3.17 (cf. 3.10) *Given $r, s \in R$, let $w = [s, r] \in R$. If $(\cdot, r)^S = r^S$, $(\cdot, s)^S = s^S$, and $(\cdot, w)^S = w^S$ are essentially constant, and $(X \times \langle s, w \rangle)^{\mathcal{K}}$ centralizes $\langle s, w \rangle^S$, then $(X \times s)^{\mathcal{K}}$ centralizes r^S.*

Proof Same as 3.10 (and 3.9), with s and s^S in place of u and $u^{\mathcal{M}}$.

□

Corollary 3.18 *Let P and Q be subgroups of R, such that $S|_P$ is a homomorphism, $S|_Q$ is a crossed homomorphism, and $[P, Q] \subset P$. Then $S|_{PQ}$ is a crossed homomorphism.*

Proof Because $[P, Q] \subset P$, we see from 3.17 that $(X \times P)^{\mathcal{K}}$ centralizes Q^S. Thus, for any $p \in P$ and $q \in Q$, we have

$$(\cdot, pq)^S = p^S q^{S(\cdot, p)^{-\mathcal{K}}} = p^S q^S$$

is essentially constant, as desired.

□

Corollary 3.19 *Let P and Q be subgroups of R, such that $\mathcal{S}|_P$ and $\mathcal{S}|_Q$ are homomorphisms, and $[P,Q] \subset P$. Then $\mathcal{S}|_{PQ}$ is a homomorphism.*

Proof From the preceding corollary, we know that $\mathcal{S}|_{PQ}$ is a crossed homomorphism.

Because $\mathcal{S}|_P$ is a homomorphism, we know that $(X \times P)^{\mathcal{K}}$ centralizes $P^{\mathcal{S}}$. So, from 3.17 (and the fact that $[Q,P] \subset P$), we see that $(X \times P)^{\mathcal{K}}$ centralizes $Q^{\mathcal{S}}$.

Because P is nilpotent (see 3.1), we may assume, by induction on $\dim P$, that $\mathcal{S}|_{Q[Q,P]}$ is a homomorphism, so $(X \times Q)^{\mathcal{K}}$ centralizes $[Q,P]^{\mathcal{S}}$. Therefore, from 3.17, we see that $(X \times Q)^{\mathcal{K}}$ centralizes $(X \times P)^{\mathcal{S}}$.

By combining the conclusions of the preceding two paragraphs, we conclude that $(X \times QP)^{\mathcal{K}}$ centralizes $(X \times QP)^{\mathcal{S}}$, as desired.

$\square$

Notation 3.20 Fix a normal subgroup Q of G, contained in R, such that $\mathcal{S}|_{[Q,Q]}$ is a homomorphism.

Proposition 3.21 *We may assume $\mathcal{S}|_{C_Q(U)}$ is a homomorphism.*

Proof Let $\phi\colon X \to K_u$ be as in 3.14. From 3.15 and 3.19, we know that $\mathcal{S}|_{C_R(L)[Q,Q]}$ is a homomorphism. Thus, from 3.11, we know that K_u centralizes $C_R(L)[Q,Q]$, so $\mathcal{S}^{\phi}|_{C_R(L)[Q,Q]} = \mathcal{S}|_{C_R(L)[Q,Q]}$ is a homomorphism. Thus, there is no harm in replacing α with α^{ϕ}, in which case, from the choice of ϕ, we see that $\mathcal{S}|_{C_Q(U)}$ is a crossed homomorphism. In addition, 3.16 (plus the fact that $\mathcal{S}|_{C_Q(L)}$ is a homomorphism) implies that $C_Q(U)$ is generated by subgroups T (one of which is $[Q,Q]$), such that $\mathcal{S}|_T$ is a homomorphism. Therefore, 3.19 implies that $\mathcal{S}|_Q$ is a homomorphism.

$\square$

For each L-module V, we now define an AU-submodule V^+ and an AU^--submodule V^-. The specific definition does not matter; what we need are the properties described in the proposition that follows.

Definition 3.22 Let Φ be the system of $\mathbb{R}$-roots of L, let Δ be the base for Φ determined by U, and let $\langle\Delta\rangle$ be the $\mathbb{Z}$-span of Δ. (Note that $\Phi \subset \langle\Delta\rangle$.) Define

$$\langle\Delta\rangle^+ = \left\{ \sum_{\alpha\in\Delta} k_\alpha\alpha \in \langle\Delta\rangle \;\middle|\; \exists\alpha, k_\alpha > 0 \right\},$$

and $\langle\Delta\rangle^- = -\langle\Delta\rangle^+$.

Definition 3.23 Let V be a finite-dimensional L-module. For each linear functional λ on A, let V_λ be the corresponding weight space. In particular, $V_0 = C_V(A)$.

(1) If $V = V_0$, let $V^+ = V^- = V$.

(2) If V is irreducible, and $V_0 = 0$, let $V^+ = V^- = V$.

(3) If V is irreducible, and $V \neq V_0 \neq 0$, then every weight of V belongs to $\langle \Delta \rangle$; let $V^+ = \sum_{\lambda \in \langle \Phi \rangle^+} V_\lambda$ and $V^- = \sum_{\lambda \in \langle \Phi \rangle^-} V_\lambda$.

(4) In general, define $V^+ = \sum W^+$ and $V^- = \sum W^-$, where each sum is over all irreducible submodules W of V.

Proposition 3.24 *Let V be a finite-dimensional L-module. Then:*

1. $V = V^- + V_0 + V^+$;

2. $V^+ \cap V_0 = V^- \cap V_0 = C_V(L)$;

3. V^+ *is an AU-submodule of V;*

4. V^- *is an AU^- submodule; and*

5. $V_0 + V^+$ *is the smallest U-submodule of V that contains both V_0 and $V^+ \cap V^-$.*

Proof Only (5) is perhaps not clear from the definition. We may assume V is irreducible. Assume, furthermore, that $V \neq V_0 \neq 0$, for otherwise we have $V^+ \cap V^- = V$, so the desired conclusion is obvious. Fix some $\lambda \in \langle \Delta \rangle^+$. If $\lambda \notin \langle \Delta \rangle^-$, then, in the unique representation $\lambda = \sum k_\alpha \alpha$ of λ as a linear combination of the elements of Δ, it must be the case that every k_α is nonnegative. Because $\|\lambda\| > 0$, this implies that there is some $\beta \in \Delta$ whose inner product with λ is strictly positive. Let $\mathcal{U}$, $\mathcal{A}$, and $\mathcal{L}$ be the Lie algebras of U, A, and L, respectively. By induction on $\sum k_\alpha$, we may assume that $V_{\lambda - \beta}$ is in the $\mathcal{U}$-submodule of V generated by V_0 and $V^+ \cap V^-$, so it suffices to show that $[\mathcal{L}_\beta, V_{\lambda - \beta}] = V_\lambda$.

Choose $u \in \mathcal{L}_\beta$, $h \in \mathcal{A}$, and $v \in \mathcal{L}_{-\beta}$, such that the linear span $\langle u, h, v \rangle$ is a subalgebra of $\mathcal{L}$ isomorphic to $sl_2(\mathbb{R})$, and such that h^μ is equal to the inner product of β with μ, for every weight μ (see [3, Eqn. (7) of §IX.1, p. 407]). Then, if we restrict V to a representation of $\langle u, h, v \rangle$, we see that vectors in the space V_λ have strictly positive weight, so the structure theory of $sl_2(\mathbb{R})$-modules implies that $[u, V_{\lambda - \beta}] = V_\lambda$.

$\square$

Definition 3.25 Let $\mathcal{Q}$ be the Lie algebra of Q. Clearly, $\mathcal{Q}^+[\mathcal{Q},\mathcal{Q}]$ and $\mathcal{Q}^-[\mathcal{Q},\mathcal{Q}]$ are Lie subalgebras of $\mathcal{Q}$; let Q^+ and Q^- be the corresponding connected Lie subgroups of Q.

In addition, we let

$$Q_0 = C_Q(A)[Q,Q], \; Q_0^+ = Q_0Q^+, \;\; \text{and} \;\; Q_0^- = Q_0Q^-.$$

Let $Q^\pm = Q^+ \cap Q^-$, and $Q_0^\pm = Q_0Q^\pm$.

Proposition 3.26 *We may assume* $\mathcal{S}|_{Q^+}$ *is a homomorphism, and* $\mathcal{S}|_{Q_0^+}$ *is a crossed homomorphism.*

Proof (Q^+) For a proof by induction, it suffices to show that if T is a one-parameter subgroup of Q that is normalized by A, and N is a subgroup of Q^+, normalized by TU, such that $[T,U] \subset N$ and $\mathcal{S}|_N$ is a homomorphism, then we may assume $\mathcal{S}|_{TN}$ is a homomorphism. From 3.13, we see that there is a Borel function $\phi \subset X \to K_u$, such that $\mathcal{S}^\phi|_T$ is a crossed homomorphism. From 3.11, we see that K_u centralizes $N^{\mathcal{S}}$, so there is no harm in replacing α with α^ϕ, so we may assume $\mathcal{S}|_T$ is a crossed homomorphism.

If $T \subset C_Q(A)$, then, by definition of Q^+ (and 3.24(2)), we must have $T \subset C_Q(L)[Q,Q]$, so $\mathcal{S}|_T$ is a homomorphism (see 3.15 and 3.19). On the other hand, if $T \not\subset C_Q(A)$, then 3.16 applies, so we again conclude that $\mathcal{S}|_T$ is a homomorphism. Thus, from 3.19, we conclude that $\mathcal{S}|_{TN}$ is a homomorphism, as desired.

(Q_0^+) From 3.13 (and 3.11), we see that we may assume $\mathcal{S}|_{Q_0}$ is a crossed homomorphism. So 3.18 implies that $\mathcal{S}_{Q_0^+}$ is a crossed homomorphism. $\qquad\square$

Similarly, by considering the opposite unipotent subgroup U^-, we obtain:

Proposition 3.27 *There is a Borel function* $\phi\colon X \to K$, *such that the restriction* $\mathcal{S}^\phi|_{Q^-C_R(L)}$ *is a homomorphism, and* $\mathcal{S}^\phi|_{Q_0^-C_R(L)}$ *is a crossed homomorphism.* $\qquad\square$

Proposition 3.28 *We may assume* $\mathcal{S}|_{Q^-}$ *is a homomorphism, and* $\mathcal{S}|_Q$ *is a crossed homomorphism.*

Proof Because $Q = Q^- Q_0^+$ and $[Q,Q] \subset Q^-$, we see from 3.18 that the second conclusion follows from the first. Choose a Borel function $\phi \colon X \to K$, as in Prop. 3.27. It suffices to show that we may assume $X^\phi = e$.

Step 1. We may assume X^ϕ centralizes $(Q_0^\pm)^S$. For any $r \in Q_0^\pm$ and a.e. $x \in X$, we have $r^{S^\phi} = r^{S x^\phi}$, so there is some $k \in K$, such that, for a.e. $x \in X$, we have $x^\phi \in C_K\big((Q_0^\pm)^S\big)k$. We may replace x^ϕ with the function $x \mapsto x^\phi k^{-1}$.

Step 2. X^ϕ centralizes $(Q_0^\pm)^{Sk}$, for every $k \in (X \times Q_0^\pm)^{\mathcal{K}}$. For $r, s \in Q_0^\pm$, and a.e. $x \in X$, we have $(rs)^S = r^S s^{S(x,r)^{-\mathcal{K}}}$. Because X^ϕ centralizes $(rs)^S$ and r^S (see Step 1), this implies that X^ϕ centralizes $s^{S(x,r)^{-\mathcal{K}}}$, as desired.

Step 3. For every $r \in Q_0^+$ and $k \in \langle (X \times Q_0^+)^{\mathcal{K}} \rangle$, there is some $k' \in \langle X \times Q_0 \rangle^{\mathcal{K}}$ with $r^k = r^{k'}$. First note that, modulo $C_K\big((Q_0^+)^S\big)$, the cocycle $\mathcal{K}|_{Q_0^+}$ is a homomorphism (see 3.7). Because the image of a homomorphism is always a subgroup, this implies that, for any $k \in \langle (X \times Q_0^+)^{\mathcal{K}} \rangle$, there is some $r \in Q_0^+$ with $(\cdot, r)^{\mathcal{K}} \in C_K\big((Q_0^+)^S\big)k$ a.e..

Therefore, it will suffice to show that $(X \times Q^+)^{\mathcal{K}}$ centralizes $(Q_0^+)^S$. Because $S|_{Q^+}$ is a homomorphism (so $(X \times Q^+)^{\mathcal{K}}$ centralizes $(Q^+)^S$), and $[Q_0^+, Q_0^+] \subset Q^+$, Lem. 3.17 provides this conclusion.

Step 4. $(Q_0^+)^S$ is centralized by X^ϕ. For $r \in Q_0^\pm$ and $u \in U$, we have

$$(xu, r^u)^\alpha = (x,u)^{-\alpha}(x,r)^\alpha (xr, u)^\alpha \in (x,r)^{\alpha(x,u)^{\mathcal{K}} u^{\mathcal{M}}} K,$$

so $r^{uS} = r^{S(x,u)^{\mathcal{K}} u^{\mathcal{M}}} = r^{Su^{\mathcal{M}}}$, because $(x,u)^{\mathcal{K}}$ centralizes $(Q_0^\pm)^S$ (see 3.10). Therefore, for $k \in \langle (X \times Q_0^\pm)^{\mathcal{K}} \rangle$, we have $r^{uSk} = r^{Su^{\mathcal{M}} k} = r^{Sku^{\mathcal{M}}}$. Since X^ϕ centralizes $u^{\mathcal{M}}$ and r^{Sk} (for the latter, see Steps 2 and 3), this implies that X^ϕ centralizes r^{uSk}. Thus, because

$$(Q_0^+)^S = \langle r^u \mid r \in Q_0^\pm, u \in U \rangle^S \qquad \text{(see 3.24(5))}$$
$$\subset \langle r^{uSk} \mid r \in Q_0^\pm, u \in U, k \in \langle (X \times Q_0^\pm)^{\mathcal{K}} \rangle \rangle,$$

we have the desired conclusion.

Step 5. We may assume $X^\phi = e$. From Step 4, we see that replacing α with the equivalent cocycle α^ϕ will not change $S|_{Q_0^+}$, so we may assume $x^\phi = e$.

$\square$

Proposition 3.29 $S|_Q$ *is a homomorphism.*

Proof For $g \in G$, let $(x,g)^{\mathcal{P}} = (x,g)^{\mathcal{S}}(x,g)^{\mathcal{M}}$. For $r \in Q$ and $u \in U$, because $(\cdot,r)^{\mathcal{K}}$ centralizes $u^{\mathcal{M}}$, we have $(\cdot,ru)^{\mathcal{P}} = r^{\mathcal{S}}u^{\mathcal{M}}$ is essentially constant. Thus, $\mathcal{P}|_{QU}$ is a crossed homomorphism, so the cocycle $\mathcal{K}|_{QU}$ is a homomorphism, modulo $C_K((QU)^{\mathcal{S}})$ (see 3.7). Then, because QU is solvable, the fact that connected, compact, solvable groups are abelian (see 2.12), implies that $[QU,QU]^{\mathcal{K}} \subset C_K((QU)^{\mathcal{S}})$, so $\mathcal{P}|_{[QU,QU]}$ is a homomorphism. In particular, $\mathcal{S}|_{Q_0 \cap [U,Q]}$ is a homomorphism.

Now Q_0 is generated by $Q_0 \cap [U,Q]$, $C_Q(L)$, and $[Q,Q]$. The restriction of $\mathcal{S}$ to each of these subgroups is a homomorphism, so we conclude from 3.19 that $\mathcal{S}|_{Q_0}$ is a homomorphism. Then, because Q is generated by Q_0, Q^-, and Q^+, we conclude from 3.19 that $\mathcal{S}|_Q$ is a homomorphism.

$\square$

This proposition provides the induction step in a proof of the following important corollary:

Corollary 3.30 *We may assume $\mathcal{S}|_R$ is a homomorphism.*

$\square$

Proposition 3.31 *We have $C_K(R^{\mathcal{S}}) = e$.*

Proof Because $C_K(S)$ is centralized (hence normalized) by M and S, and is normalized by K, we see that $C_K(S)$ is a normal subgroup of H. Being also compact, it must be trivial. Thus, letting S' be the Zariski closure of $\langle R^{\mathcal{S}} \rangle$, it suffices to show $S' \vartriangleleft H$, for then $S' = S$. Furthermore, because $G = LR$, we need only show that $(X \times L)^{\alpha}$ and $(X \times R)^{\alpha}$ each normalize S'.

For $u \in L$ and $r \in R$, we have $r^{\mathcal{S}(x,u)^{\alpha}} = r^{u\mathcal{S}} \in R^{\mathcal{S}}$, so we see that $(x,u)^{\alpha}$ normalizes S'.

Because $\mathcal{S}|_R$ is a homomorphism, we know that $(X \times R)^{\mathcal{K}}$ centralizes S'. It is obvious from the definition of S' that $R^{\mathcal{S}}$ normalizes S'.

$\square$

Corollary 3.32 *α is a homomorphism.*

Proof Because $\mathcal{M}$ and $\mathcal{S}|_R$ are homomorphisms, it only remains to show that $\mathcal{K}$ is a homomorphism.

For $u \in L$ and $r \in R$, we have $r^{u\mathcal{S}} = r^{\mathcal{S}(x,u)^{\alpha}}$, so the proposition implies that $(\cdot,u)^{\alpha}$ is essentially constant.

Because $\mathcal{S}|_R$ is a homomorphism, we know that $(X \times R)^{\mathcal{K}}$ centralizes $R^{\mathcal{S}}$. Hence, the proposition implies that $(X \times R)^{\mathcal{K}} = e$.

$\square$

4   The Remaining Proofs

In this section, we describe how to finish the proof of Theorem 1.4 (see 4.1) and we prove Corollary 1.6 (see 4.2).

4.1   Proof of the remaining case of Theorem 1.4

The notation and hypotheses of Theorem 1.4 are in effect throughout this subsection, but, unlike in §3, we do not assume that $\operatorname{Rad} G$ is nilpotent, nor that G has no nontrivial, compact, semisimple quotients (see 3.1). Let

  (1) $N = \operatorname{nil} G$; and

  (2) R be the product of $\operatorname{Rad} G$ with the maximal compact factor of a Levi subgroup of G.

Proposition 4.1 *We may assume* $\mathcal{S}|_R$ *is a crossed homomorphism.*

Proof From the main proof (§3), applied to the group LN, we see that we may assume $\mathcal{S}|_N$ is a homomorphism. Let K_u be the closure of $\langle (X \times U)^{\mathcal{K}} \rangle$. Then 3.14 implies there is a Borel function $\phi \colon X \to K_u$, such that $\mathcal{S}^\phi|_{C_R(U)}$ is a crossed homomorphism. Because K_u centralizes $N^{\mathcal{S}}$ (see 3.10), we have $\mathcal{S}^\phi|_N = \mathcal{S}|_N$, so there is no harm in replacing α with the cohomologous cocycle α^ϕ. Thus, we may assume $\mathcal{S}|_{C_R(U)}$ is a crossed homomorphism. Then, because $R = N\,C_R(U)$ and $[N, R] \subset N$, Lemma 3.18 implies that $\mathcal{S}|_R$ is a crossed homomorphism.

<div align="right">□</div>

Proposition 4.2 (cf. 3.31) *We have* $C_K(R^{\mathcal{S}}) = e$.

Proof Let S' be the Zariski closure of $\langle R^{\mathcal{S}} \rangle$. As in the proof of 3.31, we wish to show S' is normalized by $(X \times L)^\alpha$ and $(X \times R)^\alpha$.

    For $u \in L$ and $r \in R$, we have $r^{u\mathcal{S}} = r^{\mathcal{S}(x,u)^\alpha}$, so we see that $(x, u)^\alpha$ normalizes S'.

    For any $r, s \in R$, we have $(rs)^{\mathcal{S}} = r^{\mathcal{S}} s^{\mathcal{S}(x,r)^{-\mathcal{K}}}$, so $s^{\mathcal{S}(x,r)^{-\mathcal{K}}} \in \langle R^{\mathcal{S}} \rangle$. This implies that $(X \times R)^{\mathcal{K}}$ normalizes S'. It is obvious from the definition of S' that $R^{\mathcal{S}}$ also normalizes S'.

<div align="right">□</div>

Corollary 4.3 (cf. 3.32) α *is a homomorphism.*

Proof Because $G = LR$, it suffices to show that $\alpha|_L$ and $\alpha|_R$ are homomorphisms.

For $u \in L$ and $r \in R$, we have $r^{u\mathcal{S}} = r^{\mathcal{S}(x,u)^\alpha}$, so the proposition implies that $(\cdot, u)^\alpha$ is essentially constant. Therefore, $\alpha|_L$ is a homomorphism.

For any $r, s \in R$, we have $(rs)^\mathcal{S} = r^\mathcal{S} s^{\mathcal{S}(x,r)^{-\mathcal{K}}}$, so the proposition implies that $(\cdot, r)^\mathcal{K}$ is essentially constant. Therefore, $\mathcal{K}|_R$ is a homomorphism. Because $\mathcal{S}|_R$ is a crossed homomorphism, this implies that $\alpha|_R$ is a homomorphism.

$\square$

4.2   Proof of Corollary 1.6

The notation and hypotheses of Corollary 1.6 are in effect throughout this subsection. We wish to verify the hypotheses of Thm. 1.4.

By passing to an ergodic component of $X \times_\alpha H/H^\circ$, which is a finite cover of X, we may assume H is connected (cf. [11, Prop. 9.2.6, p. 168]). Then we may write $H = S \mathbin{\mathfrak{o}} (MK)$, where

(1) S is a connected, split, solvable subgroup;

(2) M is connected and semisimple, with no compact factors;

(3) K is a compact subgroup that centralizes M; and

(4) $M \cap K = Z(M)$.

By assumption, the Zariski hull of $\alpha|_{X \times L}$ is reductive. Then, because L has the Kazhdan Property (see [11, Thm. 7.1.4, p. 130]), we see that the center of this Zariski hull is compact (see [11, Thm. 9.1.1, p. 162]). Thus, the Zariski hull is contained in (a conjugate of) MK, so, by replacing α with an equivalent cocycle, we may assume $(X \times L)^\alpha \subset MK$.

Now α induces a cocycle $\overline{\alpha} \colon X \times G \to H/(SK) \cong M/(M \cap K)$.

Although the statement of Zimmer's Theorem (1.3) assumes that G is semisimple, the proof shows that this hypothesis is not necessary if the Zariski hull of the cocycle is assumed to be semisimple. Thus, we have the following:

Theorem 4.4 ([[11], cf. proof of Theorem 5.2.5, p. 98]) $\overline{\alpha}$ *is cohomologous to a homomorphism.*

$\square$

So there is no harm in assuming that $\bar{\alpha}$ itself is a homomorphism. By replacing G with a finite cover (see 2.13), we may assume that $\overline{\alpha}$ lifts to a homomorphism $\sigma \colon G \to M$.

Note that, because $(X \times L)^\alpha \subset MK$, the definition of σ implies that, for every $l \in L$, we have $(x,l)^\alpha \in l^\sigma K$ for a.e. $x \in X$.

Let J be the product of $\operatorname{Rad} G$ with the maximal compact factor of a Levi subgroup of G; then J is a connected, normal subgroup of G. Therefore, because $G^{\overline{\alpha}}$ is Zariski dense in $M/(M \cap K)$, we see that $J^{\overline{\alpha}}$ is normal in $M/(M \cap K)$. But $M/(M \cap K)$ has no nontrivial, normal subgroups that are solvable or compact, so we conclude that $J^{\overline{\alpha}}$ is trivial. Because $G = LJ$, this implies that L^σ is Zariski dense in M. Because connected, semisimple subgroups have finite index in their Zariski closure [5, Thm. VIII.3.2, p. 112], this implies $L^\sigma = M$, so $H = SL^\sigma K$.

Thus, the hypotheses of Theorem 1.4 are verified, so we conclude that α is cohomologous to a homomorphism. This completes the proof of Corollary 1.6.

$\square$

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