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COCOMPACT SUBGROUPS OF SEMISIMPLE LIE GROUPS

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Abstract. Lattices and parabolic subgroups are the obvious examples of cocompact subgroups of a connected, semisimple Lie group with finite center. We use an argument of C. C. Moore to show that every cocompact subgroup is, roughly speaking, a combination of these.

We study a cocompact subgroup H of a connected, semisimple Lie group G with finite center. The case where H is discrete is very important and has been widely studied [6, 9], though it is not yet fully understood. Apparently, C. C. Moore [5] made the first (and, up to now, the only) assault on the case where H is not discrete: he treated the case where the identity component of H is nilpotent. In this paper, we modify Moore's argument to eliminate the assumption of nilpotence. Given G, we will explicitly describe what subgroups can arise as the identity component of a cocompact subgroup H. This essentially reduces the problem of finding all the cocompact subgroups to the problem of finding *discrete* cocompact subgroups.

The identity component of H plays a key role in the proof of Moore's theorem [5]. Perhaps the only major difference between Moore's proof and ours is that we have selected a different subgroup—the unipotent radical of H—to play this key role.

For the statement of the main theorem, it will be convenient to construct a refinement of the Langlands decomposition of a parabolic subgroup.

Notation. For any Lie group X, we let X° be the identity component of X.

Definition 1.1. Suppose P is a parabolic subgroup of a connected, semisimple Lie group G with finite center. Recall that P has a Langlands decomposition $P = MAN$ [8, p. 81]. Let L be the product of all the noncompact simple factors of M°, and let E be the maximal compact factor of M°. Then $P^\circ = LEAN$; we call this the *refined Langlands decomposition* of P°.

Main Theorem 1.2. Let G be a connected, semisimple Lie group with finite center, let P be any parabolic subgroup of G, and let $P^\circ = LEAN$ be the refined Langlands decomposition of P°. For any connected, normal subgroup X of L, and any connected, closed subgroup Y of EA, there is a closed, cocompact subgroup H of G such that (a) H is contained in P, and (b) $H^\circ = XYN$.

Conversely, given any closed, cocompact subgroup of H of G, there is a parabolic subgroup P and corresponding subgroups X and Y satisfying (a) and (b).

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The proof of the main theorem will employ a number of results from the theory of real-algebraic groups. Humphreys' book [4] is a good introduction to the theory of algebraic groups; the books of Raghunathan [6] and Zimmer [9] contain useful information specifically on algebraic groups over $\mathbf{R}$.

Definition 1.3. A *real-algebraic group* is a Lie group that is a subgroup of finite index in the real points of an (affine) algebraic group defined over $\mathbf{R}$. A *real-algebraic subgroup* of a real-algebraic group G is a Lie subgroup that is of finite index in a Zariski-closed subgroup of G.

Definition. Let G be a real-algebraic group. A subgroup U is *unipotent* if every element of U is a unipotent element of G. Recall that a connected, unipotent subgroup of G is necessarily Zariski closed [6, §P.2.2, p. 9].

If H is any subgroup of G, we let unip H be the unique largest connected, unipotent, normal subgroup of H (the *unipotent radical* of H).

A semisimple element of G is *hyperbolic* if each of its eigenvalues is real and positive; it is *elliptic* if each of its eigenvalues lies on the unit circle in the complex plane.

Theorem 1.4. ("The Borel Density Theorem," S. G. Dani [3, Corollary 2.6]). Suppose H is a Zariski-closed subgroup of a real-algebraic group G, and let ν be a finite measure on G/H. Set

$$G_\nu = \{g \in G \mid \text{the g-action on } G/H \text{ preserves } \nu\} \text{ and}$$
$$N_\nu = \{g \in G \mid gs = s \text{ for all } s \in \operatorname{supp} \nu\}.$$

Then G_ν and N_ν are Zariski-closed subgroups of G, and N_ν is a cocompact, normal subgroup of G_ν.

Corollary 1.5. Suppose H is a real-algebraic subgroup of a real-algebraic group G, and that G/H has a finite G-invariant volume. Then H contains a cocompact, normal, real-algebraic subgroup of G. In particular, H contains every unipotent element and every hyperbolic element of G.

Corollary 1.6. Suppose H is a closed subgroup of a real-algebraic group G, and that G/H has a finite G-invariant volume. Then every unipotent element of G normalizes H°.

Proof. Let $\overline{H}$ be the Zariski closure of H in G. The G-invariant probability measure on G/H pushes to a G-invariant probability measure on $G/\overline{H}$, so Corollary 1.5 implies that $\overline{H}$ contains every unipotent element of G. Because $\overline{H} \subset N_G(H^\circ)$, this implies that every element of G normalizes H°. ∎

Lemma 1.7. (cf. [7, Theorem 3.8.3(ii), p. 206]). Let G be a connected, real-algebraic group. Then $[G, \operatorname{rad} G] \subset \operatorname{unip} G$.

Lemma 1.8. [5, Lemma 1]. Let $\overline{H}$ be a locally compact group, with a closed, unimodular, cocompact subgroup H. Then $\overline{H}$ is unimodular, and $\overline{H}/H$ has a finite $\overline{H}$-invariant measure.

Lemma 1.9. If H is a Lie group, and $[H, \operatorname{rad} H] = e$, then H is unimodular.

Proof. Every automorphism of $H/\operatorname{rad} H$ (indeed, of any semisimple Lie group) is volume-preserving. So any inner automorphism of H is volume-preserving on $H/\operatorname{rad} H$, and (by assumption) trivial—hence volume-preserving—on $\operatorname{rad} H$. Thus any inner automorphism of H is volume-preserving on H. ∎

Lemma 1.10. Let H be a Lie subgroup of a real-algebraic group G. If $\operatorname{unip} H = e$, then H is unimodular.

Proof. We may assume that H is Zariski dense in G, and, by replacing H with a subgroup of finite index, that G is connected. It is clear that (i) $[G, \operatorname{rad} H]$ is connected. Because G normalizes $\operatorname{rad} H$ (since H is Zariski dense in G), we have $\operatorname{rad} H \subset \operatorname{rad} G$. Then, because $[G, \operatorname{rad} G] \subset \operatorname{unip} G$ (see Lemma 1.7), it is immediate that (ii) $[G, \operatorname{rad} H] \subset \operatorname{unip} G$. Because H normalizes both G and $\operatorname{rad} H$, it is clear that (iii) H normalizes $[G, \operatorname{rad} H]$. Because G normalizes $\operatorname{rad} H$, it must be true that (iv) $[G, \operatorname{rad} H] \subset \operatorname{rad} H$. Putting (i), (ii), (iii), and (iv) together, we see that $[G, \operatorname{rad} H]$ is a connected, unipotent, normal subgroup of H. By hypothesis, then $[G, \operatorname{rad} H] = e$. This implies $[H, \operatorname{rad} H] = e$. So Lemma 1.9 asserts that H is unimodular. ∎

Lemma 1.11. (a trivial exercise in group theory). Let H be a subgroup of a semidirect product $A \ltimes N$, and assume that H contains N. Then $H = (H \cap A) \ltimes N$.

Proof of the main theorem (cf. C. C. Moore [5]). ($\Rightarrow$) Because L/X is a connected, semisimple Lie group, a fundamental theorem of A. Borel [1] (or see [6, Theorem 14.1, p. 215]) asserts that L/X has a discrete cocompact subgroup Γ_1; let H_1 be the inverse image of Γ_1 in L. Thus H_1 is a cocompact subgroup of L, and $(H_1)^\circ = X$.

Now A is a simply connected abelian group, and $YE \cap A$ is a connected, closed subgroup. Let V be a subgroup of A complementary to $YE \cap A$ in A; let Γ_2 be a lattice in V, and let $H_2 = \Gamma_2 \cdot Y$. Thus H_2 is a cocompact subgroup of AE, and $(H_2)^\circ = Y$.

Let $H = H_1 \cdot H_2 \cdot N$. Then $H^\circ = (H_1)^\circ \cdot (H_2)^\circ \cdot N = X \cdot Y \cdot N$. Because the map $L \times EA \times N \to LEAN : (l, a, n) \mapsto lan$ is a finite cover, hence proper, we know that H is closed. By construction, $H \subset P$, and H is cocompact in $LEAN = P^\circ$. Because P° is cocompact in G, we conclude that H is cocompact in G.

($\Leftarrow$) Because G is semisimple with finite center, it finitely covers a real-algebraic group [9, Proposition 3.1.6, p. 35]. In studying the identity component of H, there is no loss in simply ignoring this finite cover and assuming that G itself is a real-algebraic group. Thus, we may let $\overline{H}$ be the Zariski closure of H in G, and let U be the unipotent radical of H.

By definition, U is a connected, unipotent subgroup of the semisimple group G, so a theorem of A. Borel and J. Tits [2, Proposition 3.1] asserts that there is a parabolic subgroup P of G such that P contains $N_G(U)$, and the unipotent radical of P contains U.

Because $\overline{H} \subset N_G(U)$, then we have $\overline{H} \subset P$. Let $P^\circ = LEAN$ be the refined Langlands decomposition of P°; then N is the unipotent radical of P, so we have $U \subset N$.

Step 1. *We have $A \subset \overline{H}$ and $N \subset$ unip $\overline{H}$.* Now $\overline{H}$ is cocompact in G (because H is cocompact), and $\overline{H}AN$ is closed (because it is a real-algebraic subgroup), so we conclude that $\overline{H} \cap AN$ is cocompact in AN. Because AN is a real-algebraic group with no elliptic elements, and $\overline{H} \cap AN$ is a real-algebraic subgroup, this implies that $\overline{H} \cap AN = AN$ (for example, this follows from Corollary 1.5). We conclude that $AN \subset \overline{H}$. This implies that $A \subset \overline{H}$ and, because N is a connected, unipotent, normal subgroup of P, that $N \subset$ unip $\overline{H}$.

Step 2. *We have $U = N$.* It follows from Lemma 1.10 that H/U is unimodular. Then, because H/U is cocompact in $\overline{H}/U$, Lemma 1.8 implies that $\overline{H}/U$ is unimodular.

Step 1 asserts that $A \subset \overline{H}$, so it follows from the unimodularity of $\overline{H}/U$ that the action of A by conjugation on $\overline{H}/U$ is volume-preserving. From the structure of the parabolic subgroup P, we know that A centralizes P/N (because $P = MAN$ and A centralizes M in the Langlands decomposition); because $\overline{H} \subset P$, this implies that A centralizes $\overline{H}/N$. By combining the conclusions of the preceding two sentences, we conclude that the action of A by conjugation on N/U must be volume-preserving. From the structure of P (namely, because N is a subgroup corresponding only to *positive* roots of A), we see that this implies $U = N$ as desired.

Step 3. *We have $H = H_1 \ltimes N$, where H_1 is a closed subgroup of LEA such that $(H_1)^\circ$ is normalized by L.* Let $H_1 = H \cap LEA$. Because $N \subset H$ (by Step 2), and $H \subset P = LEA \ltimes N$, Lemma 1.11 asserts that $H = H_1 \ltimes N$. So we need only show that $(H_1)^\circ$ is normalized by L.

Because $H_1 \cong H/N$ $(= H/U)$, Lemma 1.10 shows that H_1 is unimodular. Because H_1 is cocompact in LEA, then Lemma 1.8 implies that LEA/H_1 has a finite LEA-invariant volume. So S. G. Dani's version of the Borel Density Theorem (see Corollary 1.6) implies that $(H_1)^\circ$ is normalized by L.

Step 4. *We have $(H_1)^\circ = XY$, where X is a connected, closed, normal subgroup of L, and Y is a connected, closed subgroup of EA.* Let $\mathcal{H}$ be the Lie algebra of the subgroup H_1. Because L normalizes $(H_1)^\circ$, we see that $\mathcal{H}$ is L-invariant. Because every finite-dimensional representation of L is completely reducible, this means that $\mathcal{H}$ has an L-invariant decomposition $\mathcal{H} = \mathcal{X} \oplus \mathcal{Y}$ where $[L, \mathcal{X}] = \mathcal{X}$ and $[L, \mathcal{Y}] = 0$. Thus $\mathcal{X}$ is the Lie algebra of a connected subgroup X of $[L, LEA] = L$, and $\mathcal{Y}$ is the Lie algebra of a connected subgroup Y of $C_{LEA}(L)^\circ = EA$. Because $\mathcal{X}$ and $\mathcal{Y}$ are L-invariant, we know that X and Y are normalized by L. Since $X = (H_1 \cap L)^\circ$ and $Y = (H_1 \cap EA)^\circ$, we see that each of X and Y is the identity component of the intersection of two closed subgroups. Hence X and Y are closed. ∎

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