Dani's work on dynamical systems on homogeneous spaces

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To Professor S. G. Dani on his 65th birthday

ABSTRACT. We describe some of S. G. Dani's many contributions to the theory and applications of dynamical systems on homogeneous spaces, with emphasis on unipotent flows.

S. G. Dani has written over 100 papers. They explore a variety of topics, including:

- flows on homogeneous spaces
  - unipotent dynamics
  - applications to Number Theory
  - divergent orbits
  - bounded orbits and Schmidt's game
  - topological orbit equivalence
  - Anosov diffeomorphisms
  - entropy and other invariants
- actions of locally compact groups
  - actions of lattices
  - action of $\operatorname{Aut} G$ on the Lie group G
  - stabilizers of points
- convolution semigroups of probability measures on a Lie group
- finitely additive probability measures
- Borel Density Theorem
- history of Indian mathematics

Most of Dani's papers (about 60) are directly related to flows on homogeneous spaces. This survey will briefly discuss several of his important contributions in this field.

NOTATION. Let:

- $G = \operatorname{SL}(n, \mathbb{R})$ (or, for the experts, G may be any connected Lie group),
- $\{g^t\}$ be a one-parameter subgroup of G,
- $\Gamma = \operatorname{SL}(n, \mathbb{Z})$ (or, for the experts, G may be any *lattice* in G), and
- $\varphi_t(x\Gamma) = g^t x\Gamma$ for $t \in \mathbb{R}$ and $x\Gamma \in G/\Gamma$.

Then $\varphi_{s+t} = \varphi_s \circ \varphi_t$, so φ_t is a flow on the homogeneous space G/Γ.

Much is known about flows generated by a general one-parameter subgroup $\{g^t\}$. (For example, the spectrum of $\{g^t\}$, as an operator on $L^2(G/\Gamma)$, is described in [**D77**, Thm. 2.1]. A few of Dani's other results on the general case are mentioned in Section 3 below.) However, the most impressive results apply only to the flows generated by one-parameter subgroups that are "unipotent:"

DEFINITION. A one-parameter subgroup $\{u^t\}$ of $\mathrm{SL}(n, \mathbb{R})$ is said to be *unipotent* if 1 is the only eigenvalue of u^t (for every $t \in \mathbb{R}$). This is equivalent to requiring that

$$\{u^t\} \text{ is conjugate to a subgroup of } \mathbb{U}_n = \begin{bmatrix} 1 & & & & \\ & 1 & & * & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix}.$$

Work of Dani, Margulis, Ratner, and others in the 1970's and 1980's showed that if $\{u^t\}$ is unipotent, then the corresponding flow on G/Γ is surprisingly well behaved. In particular, the closure of every orbit is a very nice C^∞ submanifold of G/Γ, and every invariant probability measure is quite obvious.

Section 1 describes these (and other) fundamental results on unipotent flows, and Section 2 explains some of their important applications in Number Theory. Dani's contributions are of lasting importance in both of these areas.

1. Unipotent flows

Orbit closure (O), equidistribution (E), and measure classification (M). If $\{g^t\}$ is a (nontrivial) group of diagonal matrices, then there exists a $\{g^t\}$-orbit in G/Γ that is very badly behaved — its closure is a fractal (cf. [**St90**, Lem. 2]). Part (**O**) of the following fundamental result tells us that unipotent flows have no such pathology. The other two parts describe additional ways in which the dynamical system is very nicely behaved.

THEOREM 1.1 (Ratner [**R91a**, **R91b**]). *If $\{u^t\}$ is unipotent, then the following hold:*

- (**O**) *The closure of every u^t-orbit on G/Γ is a (finite-volume, homogeneous) C^∞ submanifold.*
- (**E**) *Every u^t-orbit on G/Γ is uniformly distributed in its closure.*
- (**M**) *Every ergodic u^t-invariant probability measure on G/Γ is the natural Lebesgue measure on some (finite-volume, homogeneous) C^∞ submanifold.*

REMARK 1.2. Here is a more precise statement of each part of Theorem 1.1.

(**O**) For each $a \in G/\Gamma$, there is a closed subgroup $L = L_a$ of G, such that $\overline{\{u^t\}a} = La$. Furthermore, the closed submanifold La of G/Γ has finite volume, with respect to an L-invariant volume form on La.

(**E**) Given $a \in G/\Gamma$, let dg be the L_a-invariant volume form on $\overline{\{u^t\}a}$ that is provided by (**O**). (After multiplying dg by a scalar, we may assume the volume of $\overline{\{u^t\}a}$ is 1.) Then, for any continuous function f on G/Γ, with compact support, the average value of f along the orbit of a is equal to the average value of f on the

closure of the orbit. That is, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(u^t a)\, dt = \int_{\overline{\{u^t\}a}} f\, dg.$$

(**M**) Suppose μ is a Borel measure on G/Γ, such that

- $\mu(G/\Gamma) = 1$ (this means μ is a *probability measure*),
- $\mu(u^t A) = \mu(A)$ for every measurable subset A of G/Γ (this means μ is u^t-*invariant*), and
- for every u^t-invariant, measurable subset A of G/Γ, either A has measure 0 or the complement of A has measure 0 (this means μ is *ergodic*).

Then there exists a closed subgroup $L = L_\mu$ of G, and some $a = a_\mu \in G$, such that the orbit La is closed, and μ is an L-invariant Lebesgue measure on the submanifold La. (The L-invariant measure on La is unique up to multiplication by a positive scalar, so the probability measure μ is uniquely determined by L.)

DANI'S CONTRIBUTIONS. Dani was a central figure in the activity that paved the way for Ratner's proof of Theorem 1.1. In particular:

(1) Dani's early work on actions of unipotent subgroups was one of the ingredients that inspired Raghunathan to conjecture the truth of (**O**). Raghunathan did not publish the conjecture himself — its first appearance in the literature was in a paper of Dani [**D81**, p. 358].

(2) (**M**) was conjectured by Dani [**D81**, p. 358]. This was an important insight, because the methods developed in the 1980's were able to prove this part of Theorem 1.1 first, and the other two parts are corollaries of it (cf. Section 1 below).

(3) For $G = \mathrm{SL}(2,\mathbb{R})$, (**M**) and (**E**) were proved by Furstenberg [**F73**] when G/Γ is compact, but it was Dani who tackled the noncompact case. First, he [**D78a**] proved (**M**). Then he [**D82**] proved the special case of (**E**) in which $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Subsequent joint work with Smillie [**DS84**] established (**E**) for the remaining lattices in $\mathrm{SL}(2,\mathbb{R})$.

(4) Dani [**D81**, **D86**] proved analogues of (**O**) and (**M**) in which u^t replaced with the larger unipotent subgroup $\mathbb{U}_n$. (See Section 1 below for more discussion of this.)

(5) Dani and Margulis [**DM90a**] proved (**O**) under the assumption that $G = \mathrm{SL}(3,\mathbb{R})$ and u^t is generic.

Furthermore, Dani [**D79**, **D84a**] established a very important special case of (**E**), long before the full theorem was proved. Note that if $a \in G/\Gamma$ and $\epsilon > 0$, then, since (**O**) tells us that $\overline{\{u^t\}a}$ has finite volume, there exists $C > 0$, such that the complement of the ball of radius C has volume less than ϵ. Therefore, letting

- λ be the usual Lebesgue measure on $\mathbb{R}$, and
- $d(x, y)$ be the distance from x to y in G/Γ,

(**E**) implies that

$$(1.3) \qquad \limsup_{T\to\infty} \frac{\lambda\{\, t \in [0, T] \mid d(u^t a, a) > C \,\}}{T} < \epsilon.$$

Dani proved this fundamental inequality:

THEOREM 1.4 (Dani [**D79**, **D84a**]). *For every $a \in G/\Gamma$ and $\epsilon > 0$, there exists $C > 0$, such that* (1.3) *holds.*

This is a strengthening of the following fundamental result:

COROLLARY 1.5 (Margulis [**M71**, **M75**]). *No u^t-orbit diverges to ∞. More precisely, for every $a \in G/\Gamma$, we have $d(u^t a, a) \not\to \infty$ as $t \to +\infty$.*

PROOF. If $d(u^t a, a) \to \infty$, then there exists $T_0 > 0$, such that $d(u^t a, a) > C$ for all $t > T_0$. Therefore

$$\limsup_{T \to \infty} \frac{\lambda\{\, t \in [0, T] \mid d(u^t a, a) > C \,\}}{T} \geq \lim_{T \to \infty} \frac{T - T_0}{T} = 1 > \epsilon. \qquad \square$$

Theorem 1.4 has the following important consequences:

COROLLARY 1.6 (Dani). *Suppose μ is an ergodic u^t-invariant measure on G/Γ. If every compact subset of G/Γ has finite measure, then all of G/Γ has finite measure. (In other words, if μ is locally finite, then μ is finite.)*

COROLLARY 1.7 (Dani). *Suppose μ is an ergodic $\mathrm{SL}(n, \mathbb{Z})$-invariant measure on $\mathbb{R}^n$. If every compact set has finite measure, and every finite set has measure 0, then μ is a scalar multiple of Lebesgue measure.*

The proof of Theorem 1.4 employs an ingenious induction argument of Margulis that was appropriately modified by Dani. The best exposition of this idea is in an appendix of a paper by Dani and Margulis [**DM90b**].

For applications, it is important to have strengthened versions of Theorem 1.4 and (**E**) giving estimates that are *uniform* as the starting point of the orbit varies over a compact set. The first such theorems were proved by Dani and Margulis [**DM91**, **DM93**].

REMARK 1.8. Margulis [**M88**, Rem. 3.12(II)] observed that Dani's Theorem 1.4 provides a short (and rather elementary) proof of the fundamental fact that arithmetic groups are lattices. That is, if $\mathbf{G}$ is an algebraic subgroup of $\mathbf{SL}_n$ that is defined over $\mathbb{Q}$, and $\mathbf{G}$ has no characters that are defined over $\mathbb{Q}$, then the homogeneous space $\mathbf{G}(\mathbb{R})/\mathbf{G}(\mathbb{Z})$ has finite volume.

Linearization. Dani and Margulis [**DM93**, §3] developed an important method that is called "*Linearization*," because it replaces the action of u^t on G/Γ with the much simpler action of u^t by linear transformations on a vector space. It has become a crucial tool in applications of unipotent flows in Number Theory and related areas. To see the main idea (which has its roots in earlier work of Dani-Smillie [**DS84**] and Shah [**Sh91**]), consider the following proof that (**E**) is a consequence of (**O**) and (**M**):

IDEA OF PROOF OF (**E**) FROM (**O**) AND (**M**). The start of the proof is straightforward. Fix $a \in G/\Gamma$. From (**O**), we know there is a closed subgroup L of G, such that $\overline{\{u^t\}a} = La$. Furthermore, there is an L-invariant probability measure μ on La.

Assume, for simplicity, that G/Γ is compact, and let $\mathrm{Meas}(G/\Gamma)$ be the set of probability measures on G/Γ. (The Riesz Representation Theorem tells us that $\mathrm{Meas}(G/\Gamma)$ can be identified with the set of positive linear functionals on $C(G/\Gamma)$ that have norm 1, so it has a natural weak* topology.) For each $T > 0$, define $M_T \in \mathrm{Meas}(G/\Gamma)$ by

$$M_T(f) = \frac{1}{T} \int_0^T f(u^t a) \, dt.$$

We wish to show $M_T \to \mu$ as $T \to \infty$.

Since $\mathrm{Meas}(G/\Gamma)$ is a closed, convex subset of the unit ball in $C(G/\Gamma)^*$, the Banach-Alaoglu Theorem tells us that $\mathrm{Meas}(G/\Gamma)$ is compact (in the weak* topology). Therefore, in order to show $M_T \to \mu$, it suffices to show that μ is the only accumulation point of $\{M_T\}$.

Thus, given an accumulation point μ_∞ of $\{M_T\}$, we wish to show $\mu_\infty = \mu$. It is not difficult to see that μ_∞ is u^t-invariant (since M_T is nearly invariant when T is large). Assume, for simplicity, that μ_∞ is ergodic. Then (**M**) tells us there is a closed subgroup L_∞ of G, and some $a_\infty \in G$, such that the orbit $L_\infty a_\infty$ is closed, μ_∞ is L_∞-invariant, and μ_∞ is supported on $L_\infty a_\infty$.

Since $M_T \to \mu_\infty$, we know $L_\infty a_\infty \subseteq \overline{\{u^t\}a} = La$. To complete the proof, it suffices to show the opposite inclusion $La \subseteq L_\infty a_\infty$. (This suffices because the two inclusions imply $L = L_\infty$, which means that μ_∞ must be the (unique) L-invariant probability measure on La, which is μ.)

We will now use *Linearization* to show $u^t a \in L_\infty a_\infty$ for all t. (Since $\overline{\{u^t\}a} = La$, and $L_\infty a_\infty$ is closed, this implies the desired inclusion $La \subseteq L_\infty a_\infty$.) Roughly speaking, one can show that the subgroup L_∞ is Zariski closed (since the ergodicity of μ_∞ implies that the unipotent elements generate a cocompact subgroup). Therefore, Chevalley's Theorem (from the theory of Algebraic Groups) tells us there exist:

- a finite-dimensional real vector space V,
- a homomorphism $\rho \colon G \to \mathrm{SL}(V)$, and
- $\vec{v} \in V$,

such that $L_\infty = \mathrm{Stab}_G(\vec{v})$. The Euclidean distance formula tells us that $d(\vec{x}, \vec{v})^2$ is a polynomial of degree 2 (as a function of $\vec{x}$). Also, since $\rho(u^t)$ is unipotent, elementary Lie theory shows that each matrix entry of $\rho(u^t)$ is a polynomial function of t. Therefore, if we choose $g \in G$, such that $a = g\Gamma$, then

$$d_g(t) := d\big(\rho(u^t g)\vec{v}, \vec{v}\big)$$

is a polynomial function of t. Furthermore, the degree of this polynomial is bounded, independent of g.

Since $M_T \to \mu_\infty$, and μ_∞ is supported on $L_\infty a_\infty$, we know that if T is large, then for most $t \in [0, T]$, the point $u^t a$ is close to $L_\infty a_\infty$. Assuming, for simplicity, that $a_\infty = e$, so $\rho(L_\infty a_\infty)\vec{v} = \rho(L_\infty)\vec{v} = \vec{v}$, this implies there exists $\gamma \in \Gamma$, such that $d_{g\gamma}(t)$ is very small. However, a polynomial of bounded degree that is very small on a large fraction of an interval must be small on the entire interval. We conclude that $d_{g\gamma}(t)$ is small for all $t \in \mathbb{R}^+$ (and that γ is independent of t). Since constants are the only bounded polynomials, this implies that the distance from $u^t a$ to $L_\infty a_\infty$ is a constant (independent of t). Then, from the first sentence of this paragraph, we see that this distance must be 0, which means $u^t a \in L_\infty a_\infty$ for all t, as desired. $\square$

REMARK 1.9. The above argument can be modified to derive both (**O**) and (**E**) from (**M**), without needing to assume (**O**). Therefore, as was mentioned on page 133, (**M**) implies both (**O**) and (**E**).

REMARK 1.10. The above proof assumes that G/Γ is compact. To eliminate this hypothesis, one passes to the one-point compactification, replacing $\mathrm{Meas}(G/\Gamma)$ with $\mathrm{Meas}\big(G/\Gamma \cup \{\infty\}\big)$. The bulk of the argument can remain unchanged, because

Dani's Theorem 1.4 tells us that if μ_∞ is any accumulation point of $\{M_T\}$, then $\mu_\infty\big(\{\infty\}\big) = 0$, so $\mu_\infty \in \mathrm{Meas}(G/\Gamma)$.

Actions of horospherical subgroups and actions of lattices. As has already been mentioned on page 133, Dani proved the analogues of (**M**) and (**O**) in which the one-dimensional unipotent subgroup $\{u^t\}$ is replaced by a unipotent subgroup that is maximal:

THEOREM 1.11 (Dani [**D81**, **D86**]). *Let*

- $G = \mathrm{SL}(n, \mathbb{R})$ (*or, more generally, let G be a connected, reductive, linear Lie group*),
- $U = \mathbb{U}_n$ (*or, more generally, let U be any maximal unipotent subgroup of G*), *and*
- Γ *be a discrete subgroup of G, such that G/Γ has finite volume* (*in other words, let Γ be a lattice in G*).

Then:

(**O**′) *The closure of every U-orbit on G/Γ is a* (*finite-volume, homogeneous*) C^∞ *submanifold.*

(**M**′) *Every ergodic U-invariant probability measure on G/Γ is the natural Lebesgue measure on some* (*finite-volume, homogeneous*) *submanifold.*

REMARK 1.12. Although the statement of Theorem 1.11 requires the unipotent subgroup U to be maximal, Dani actually proved (**O**′) under the weaker assumption that U is "horospherical:"

For each $g \in G$, the corresponding *horospherical subgroup* is
$$U_g = \{\, u \in G \mid g^{-k} u g^k \to e \text{ as } k \to +\infty \,\}.$$

Since 1 is the only eigenvalue of the identity matrix, and similar matrices have the same eigenvalues, it is easy to see that every element of U_g is unipotent. Conversely, the maximal unipotent subgroup $\mathbb{U}_n$ is horospherical. (Namely, we have $\mathbb{U}_n = U_g$ if $g = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any diagonal matrix with $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$.)

Dani's first published paper was joint work with Mrs. Dani [**DD73**], while they were students at the Tata Institute of Fundamental Research. It proved a p-adic version of the following interesting consequence of (**O**′):

COROLLARY 1.13 (Greenberg [**G63**]). *Let $G = \mathrm{SL}(n, \mathbb{R})$, and let Γ be a discrete subgroup of G, such that G/Γ is compact. Then the Γ-orbit of every nonzero vector is dense in $\mathbb{R}^n$.*

PROOF. 1.11(**O**′) tells us that, for each $g \in G$, there is a (unique) connected subgroup L_g of G, such that $\overline{\mathbb{U}_n g \Gamma} = L_g \Gamma$. Since $\mathbb{U}_n$ is horospherical (see Remark 1.12) and G/Γ is compact, one can show that $L_g = G$. This means $\mathbb{U}_n g \Gamma$ is dense in G (for all $g \in G$).

So $\Gamma g \mathbb{U}_n$ is dense in G. Since $\mathbb{U}_n$ fixes the vector $\vec{e_n} = (0, \ldots, 0, 1)$, and $G\vec{e_n} = \mathbb{R}^n \smallsetminus \{0\}$, this implies that $\Gamma g \vec{e_n}$ is dense in $\mathbb{R}^n$. This is the desired conclusion, since $g\vec{e_n}$ is an arbitrary nonzero vector in $\mathbb{R}^n$.   □

Corollary 1.13 provides information about an action of the "lattice" Γ. This particular result is a consequence of the theory of unipotent dynamics, but other theorems of Dani about lattice actions do not come from this theory. As an example of this, we mention Dani's topological analogue of a famous measure-theoretic result of Margulis [**M78**, Thm. 1.14.2]:

THEOREM 1.14 (Dani [**D84b**]). *Let*

- $G = \mathrm{SL}(n, \mathbb{R})$ *with* $n \geq 3$,
- $\Gamma = \mathrm{SL}(n, \mathbb{Z})$,
- B *be the "Borel" subgroup of all upper-triangular matrices,*
- X *be a compact, Hausdorff space on which* Γ *acts by homeomorphisms, and*
- $\phi\colon G/B \to X$ *be a continuous, surjective,* Γ-*equivariant map.*

Then B *is contained in a closed subgroup* P *of* G, *such that* X *is* Γ-*equivariantly homeomorphic to* G/P.

REMARK 1.15. To make the statement more elementary, we assume $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ in Theorem 1.14, but Dani actually proved the natural generalization in which G is allowed to be any semisimple Lie group of real rank greater than one, and Γ is an irreducible lattice in G.

2. Applications in Number Theory

Unipotent dynamics (which was discussed in Section 1) became well known through its use in the solution of the "Oppenheim Conjecture." Before explaining this, let us look at a similar problem whose solution is much more elementary.

EXAMPLE 2.1. Suppose

$$L(x_1, x_2, \ldots, x_n) = \sum_i a_i x_i = a_1 x_1 + \cdots + a_n x_n$$

is a homogeneous polynomial of degree 1, with real coefficients. (Assume, to avoid degeneracies, that L is not identically zero.) For any $b \in \mathbb{R}$, it is easy to find a solution of the equation $L(\vec{x}) = b$.

However, a number theorist may wish to require the coordinates of $\vec{x}$ to be integers (i.e., $\vec{x} \in \mathbb{Z}^n$). Since $\mathbb{Z}^n$ is countable, most choices of b will yield an equation that does not have an integral solution, so it is natural to ask for an approximate solution: for every $\epsilon > 0$,

does there exist $\vec{x} \in \mathbb{Z}^n$, such that $|L(\vec{x}) - b| < \epsilon$?

Obviously, to say an approximate solution exists for every $b \in \mathbb{R}$ (and every $\epsilon > 0$) is the same as saying that $L(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

It is obvious that if the coefficients of L are integers, then L takes integer values at any point whose coordinates are integers. This means $L(\mathbb{Z}^n) \subseteq \mathbb{Z}$, so $L(\mathbb{Z}^n)$ is not dense in $\mathbb{R}$. More generally, if L is a scalar multiple of a polynomial with integer coefficients, then $L(\mathbb{Z}^n)$ is not dense in $\mathbb{R}$. It is less obvious (but not difficult to prove) that the converse is true:

If $L(\vec{x})$ *is a a homogeneous polynomial of degree* 1, *with real coefficients, and* L *is not a scalar multiple of a polynomial with integer coefficients, then* $L(\mathbb{Z}^n)$ *is dense in* $\mathbb{R}$.

Everything in the above discussion is trivial, but the problem becomes extremely difficult if polynomials of degree 1 are replaced by polynomials of degree 2. In this setting, a conjecture made by A. Oppenheim [**O29**] in 1929 was not proved until almost 60 years later. The statement of the result needs to account for the following counterexamples:

EXAMPLES 2.2. Let $Q(x_1, x_2, \ldots, x_n) = \sum_{i,j} a_{i,j} x_i x_j$ be a homogeneous polynomial of degree 2, with real coefficients. (In other words, Q is a "real quadratic form.")

(1) We say Q is *positive-definite* if $Q(\mathbb{R}^n) \subseteq \mathbb{R}^{\geq 0}$. (Similarly, Q is *negative-definite* if $Q(\mathbb{R}^n) \subseteq \mathbb{R}^{\leq 0}$.) Obviously, this implies that $Q(\mathbb{R}^n)$ is not dense in $\mathbb{R}$, so it is obvious that the smaller set $Q(\mathbb{Z}^n)$ is not dense in $\mathbb{R}$. For example, if we let $Q(x_1, x_2, \ldots, x_n) = a_1 x_1^2 + \cdots a_n x_n^2$, with each a_i positive, then Q is positive-definite, so $Q(\mathbb{Z}^n)$ is not dense in $\mathbb{R}$.

(2) Let $Q(x_1, x_2) = x_1^2 - \alpha^2 x_2^2$. It is not difficult to see that if α is badly approximable, then 0 is *not* an accumulation point of $Q(\mathbb{Z}^2)$. Obviously, then $Q(\mathbb{Z}^2)$ is not dense in $\mathbb{R}$. (Recall that, to say $\alpha \in \mathbb{R}$ is *badly approximable* means there exists $\epsilon > 0$, such that $|\alpha - (p/q)| > \epsilon/q^2$ for all $p, q \in \mathbb{Z}$. It is well known that quadratic irrationals, such as $1 + \sqrt{2}$, are always badly approximable.) This means that the "obvious" converse can fail when $n = 2$.

(3) Given a 2-variable counterexample $x_1^2 - \alpha^2 x_2^2$, it is easy to construct counterexamples in any number of variables, such as
$$Q(x_1, x_2, \ldots, x_n) = (x_1 + \cdots + x_{n-1})^2 - \alpha^2 x_n^2.$$
Note that this quadratic form has n variables, but a linear change of coordinates can transform it into a form with less than n variables. This means it is *degenerate*.

The following theorem shows that all quadratic counterexamples are of the above types.

THEOREM 2.3 (Margulis [**M87**]). *Suppose:*

- $Q(x_1, x_2, \ldots, x_n)$ *is a homogeneous polynomial of degree* 2, *with real coefficients,*
- Q *is not a scalar multiple of a polynomial with integer coefficients,*
- Q *is neither positive-definite nor negative-definite,*
- $n \geq 3$, *and*
- Q *is not degenerate.*

Then $Q(\mathbb{Z}^n)$ *is dense in* $\mathbb{R}$.

IDEA OF PROOF. We will explain how the result can be obtained as a corollary of Ratner's Theorem (1.1). (The result was originally proved directly, because Ratner's theorem was not yet available in 1987. See [**D01**] for an elementary exposition of a direct proof of this type.)

Assume, for simplicity, that $n = 3$. Given $b \in \mathbb{R}$, we wish to show there exists $\vec{m} \in \mathbb{Z}^3$, such that $Q(\vec{m}) \approx b$. Since Q is neither positive-definite nor negative-definite, we have $Q(\mathbb{R}^3) = \mathbb{R}$, so there exists $\vec{v} \in \mathbb{R}^3$, such that $Q(\vec{v}) = b$.

Let $G = \mathrm{SL}(3, \mathbb{R})$, $\Gamma = \mathrm{SL}(3, \mathbb{Z})$, and
$$H = \mathrm{SO}_3(Q) = \{\, h \in \mathrm{SL}(3, \mathbb{R}) \mid Q(h\vec{x}) = Q(\vec{x}) \text{ for all } \vec{x} \in \mathbb{R}^3 \,\}.$$
Note that H is a subgroup of G that is generated (up to finite index) by unipotent one-parameter subgroups. Therefore, Ratner's Theorem (1.1) implies that the closure $\overline{H\Gamma}$ of $H\Gamma$ is a very nice submanifold of G. More precisely, there is a closed subgroup L of G, such that $\overline{H\Gamma} = L\Gamma$, and $H \subseteq L$. However, it can be shown that H is a maximal subgroup of G, and, since $Q(\vec{x})$ does not have integer coefficients

(up to a scalar multiple), that $\overline{H\Gamma} \neq H\Gamma$ (so $L \neq H$). This implies $L = G$. In other words, $H\Gamma$ is dense in G.

For convenience, let $\vec{e_1} = (1,0,0) \in \mathbb{R}^3$. Since $G = \mathrm{SL}(3,\mathbb{R})$ is transitive on the nonzero vectors in $\mathbb{R}^3$, we know there exists $g \in G$, such that $g\vec{e_1} = \vec{v}$. Then the conclusion of the preceding paragraph implies there exist $h \in H$ and $\gamma \in \Gamma = \mathrm{SL}(3,\mathbb{Z})$, such that

$$(*) \qquad\qquad\qquad h\gamma\vec{e_1} \approx \vec{v}.$$

Let $\vec{m} = \gamma\vec{e_1} \in \mathbb{Z}^3$. Then

$$\begin{aligned}
Q(\vec{m}) &= Q(\gamma\vec{e_1}) && \text{(definition of } \vec{m}) \\
&= Q(h\gamma\vec{e_1}) && \text{(definition of } H) \\
&\approx Q(\vec{v}) && \text{(polynomial } Q \text{ is continuous, and } (*)) \\
&= b && \text{(definition of } \vec{v}). \qquad \square
\end{aligned}$$

DANI'S CONTRIBUTIONS. Joint work of Dani and Margulis made the following important improvements to this theorem:

[**DM89**] *approximation by primitive vectors:* $Q(\mathcal{P})$ is dense in $\mathbb{R}$, where

$$\mathcal{P} = \{\, (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n \mid \gcd(m_1, m_2, \ldots, m_n) = 1 \,\}.$$

[**DM90a**] *simultaneous approximation:* Given two quadratic forms Q_1 and Q_2 (satisfying appropriate conditions), the set $\big\{\, \big(Q_1(\vec{m}), Q_2(\vec{m})\big) \mid \vec{m} \in \mathcal{P} \,\big\}$ is dense in $\mathbb{R}^2$.

[**DM93**] *quantitative estimates:* For any nonempty open interval $I \subset \mathbb{R}$, Theorem 2.3 shows there exists at least one $\vec{m} \in \mathbb{Z}^n$ with $Q(\vec{m}) \in I$. In fact, there exist *many* such $\vec{m}$ (of bounded norm). Namely, if λ is the Lebesgue measure on $\mathbb{R}^n$, then

$$\liminf_{C \to \infty} \frac{\#\left\{ \vec{m} \in \mathbb{Z}^n \;\middle|\; \begin{array}{c} Q(\vec{m}) \in I, \\ \|\vec{m}\| < C \end{array} \right\}}{\lambda\left(\left\{ \vec{v} \in \mathbb{R}^n \;\middle|\; \begin{array}{c} Q(\vec{v}) \in I, \\ \|\vec{v}\| < C \end{array} \right\} \right)} > 0.$$

Furthermore, the estimate is uniform when Q varies over any compact set of quadratic forms that all satisfy the hypotheses of Theorem 2.3.

The paper [**DM93**] has been especially influential, because it introduced *Linearization* (which was discussed in Section 1).

Dani has continued his contributions to this field of research in recent years. For example, the corollary of the following theorem is an approximation theorem for linear forms.

THEOREM 2.4 (Dani-Nogueira [**DN09**]). *Let Γ^+ be the set of matrices in $\mathrm{SL}(n,\mathbb{Z})$ that do not have any negative entries. If $\vec{a} \in \mathbb{R}^n$, such that a_1/a_2 is an irrational number that is negative, and $n \geq 3$, then $\Gamma^+\vec{a}$ is dense in $\mathbb{R}^n$.*

COROLLARY 2.5 (Dani-Nogueira). *Let $L(\vec{x}) = \sum_{i=1}^{n} a_i x_i$, and assume a_1/a_2 is an irrational number that is negative. Then $L(\mathcal{P}_k)$ is dense in $\mathbb{R}$, for every k, where*

$$\mathcal{P}_k = \left\{ (m_1, \ldots, m_n) \in \mathbb{Z}^n \;\middle|\; \begin{array}{c} \gcd(m_1, \ldots, m_n) = 1, \\ \text{and } m_i > k, \; \forall i \end{array} \right\}.$$

Dani [**D08**] also proved a simultaneous approximation theorem for certain pairs consisting of a linear form and a quadratic form.

3. Dynamics of non-unipotent homogeneous flows

This section presents a few fundamental questions that Dani worked on, but remain open.

Kolmogorov automorphisms. It is believed that almost all translations on homogeneous spaces are isomorphic to Bernoulli shifts:

DEFINITION 3.1. If $p\colon X \to \mathbb{R}^+$ is a probability distribution on a finite set $X = \{x_1, \ldots, x_n\}$, then there is a natural product measure $p^{\mathbb{Z}}$ on the infinite product $X^{\mathbb{Z}} = \{f\colon \mathbb{Z} \to X\}$, and the associated *Bernoulli shift* is the measurable map $B_{X,p}\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$, defined by $B_{X,p}(f)(k) = f(k-1)$.

CONJECTURE 3.2. *Suppose*
- $G = \mathrm{SL}(n, \mathbb{R})$ (*or, more generally, let G be a connected, semisimple, linear Lie group with no compact factors*),
- $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ (*or, more generally, let Γ be an irreducible lattice in G*),
- $g \in G$, *and*
- $T_g\colon G/\Gamma \to G/\Gamma$ *be defined by* $T_g(x\Gamma) = gx\Gamma$.

If there is an eigenvalue λ of g, such that $|\lambda| \neq 1$ (in other words, if the entropy of T_g is nonzero), then T_g is measurably isomorphic to a Bernoulli shift.

DANI'S CONTRIBUTIONS. Two results of Dani [**D76**] are the inspiration for this conjecture:
(1) The conjecture is true if the matrix g is diagonalizable over $\mathbb{C}$.
(2) In the general case, T_g has no zero-entropy quotients (if $\exists \lambda, |\lambda| \neq 1$).
A transformation with no zero-entropy quotients is called a "*Kolmogorov automorphism.*" Examples of non-Bernoulli Kolmogorov automorphisms are rare, so (2) is good evidence that the conjecture is true for all g, not just those that are diagonalizable.

Anosov diffeomorphisms. Dani has worked on the longstanding conjecture that Anosov diffeomorphisms can only be constructed on infranilmanifolds:

DEFINITION 3.3. A diffeomorphism f of a compact, connected manifold M is *Anosov* if, at every point $x \in M$, the tangent space T_xM has a splitting $T_xM = \mathcal{E}^+ \oplus \mathcal{E}^-$, such that
- for $v \in \mathcal{E}^+$, $D(f^k)(v) \to 0$ exponentially fast as $k \to -\infty$, and
- for $v \in \mathcal{E}^-$, $D(f^k)(v) \to 0$ exponentially fast as $k \to +\infty$.

CONJECTURE 3.4 (from the 1960's). *If there is an Anosov diffeomorphism f on M, then some finite cover of M is a nilmanifold. (This means the cover is a homogeneous space G/Γ, where G is a nilpotent Lie group, and Γ is a discrete subgroup of G.) Furthermore, lifting f to the finite cover yields an affine map on the nilmanifold.*

DANI'S CONTRIBUTIONS.
[**D78b**] Dani constructed many nilmanifolds that have Anosov diffeomorphisms. Much more recently, joint work with M. Mainkar [**DMa05**] constructed examples of every sufficiently large dimension.

[**D80**] Dani proved Conjecture 3.4 under the assumption that M is a double-coset space $K\backslash G/H$ (and certain additional technical conditions are satisfied).

Divergent trajectories. We know, from Corollary 1.5, that if u^t is unipotent, then no u^t-orbit diverges to ∞. On the other hand, orbits of a diagonal subgroup can diverge to ∞.

EXAMPLE 3.5. Let

$$G = \mathrm{SL}(2, \mathbb{R}), \quad a^t = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix}, \quad \Gamma = \mathrm{SL}(2, \mathbb{Z}), \quad \text{and} \quad \vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $a^t \vec{e_1} \to \vec{0}$ as $t \to \infty$, so $a^t \Gamma \to \infty$ in G/Γ.

The divergent orbits of the diagonal matrices in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ are well known, and quite easy to describe. Dani vastly generalized this, by proving that all divergent orbits are obvious in a much wider setting. Here is a special case of his result:

THEOREM 3.6 (Dani [**D85**]). *Let*

- $G = \mathrm{SO}(1,n)$ *(or, more generally, let G be a connected, almost simple algebraic $\mathbb{Q}$-subgroup of $\mathrm{SL}_{n+1}(\mathbb{R})$, with $\mathrm{rank}_{\mathbb{Q}} G = 1$),*
- $\Gamma = G \cap \mathrm{SL}_{n+1}(\mathbb{Z})$,
- $\{a^t\}$ *be a one-parameter subgroup of G that is diagonalizable over $\mathbb{R}$, and*
- $g \in G$.

Then $a^t g \Gamma$ diverges to ∞ in G/Γ if and only if there exist

- *a continuous homomorphism $\rho\colon G \to \mathrm{SL}(\ell, \mathbb{R})$, for some ℓ, such that $\rho(\Gamma) \subseteq \mathrm{SL}(\ell, \mathbb{Z})$, and*
- *a nonzero vector $\vec{v} \in \mathbb{Z}^\ell$,*

such that $\rho(a^t g)\, \vec{v} \to \vec{0}$ as $t \to \infty$.

Surprisingly, he was also able to exhibit many cases in which there are divergent orbits that do not come from the construction in Theorem 3.6:

THEOREM 3.7 (Dani [**D85**]). *Let*

- $G = \mathrm{SL}(n, \mathbb{R})$, *with $n \geq 3$ (or, more generally, let G be the $\mathbb{R}$-points of a connected, almost simple algebraic $\mathbb{Q}$-group with $\mathrm{rank}_{\mathbb{Q}} G = \mathrm{rank}_{\mathbb{R}} G \geq 2$),*
- $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ *(or, in general, let G be the $\mathbb{Z}$-points of G), and*
- $\{a^t\}$ *be a (nontrivial) one-parameter subgroup of G that is diagonalizable over $\mathbb{R}$.*

Then there are a^t-orbits in G/Γ that diverge to ∞, but do not correspond to a continuous homomorphism $\rho\colon G \to \mathrm{SL}(\ell, \mathbb{R})$, as in Theorem 3.6.

It remains an open problem to determine the set of divergent orbits in G/Γ in cases where Theorem 3.6 does not apply.

Bounded orbits and Schmidt's game. Dani also pioneered the study of orbits at the opposite extreme from the divergent trajectories discussed in Section 3: an orbit in G/Γ is *bounded* if its closure is compact.

It is well known that if $\{a^t\}$ is any nontrivial one-parameter group of diagonal matrices in $\mathrm{SL}(n,\mathbb{R})$, then almost every a^t-orbit is dense in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$. Since bounded orbits cannot be dense, this implies that the set $\mathcal{B}$ of bounded orbits has measure 0, which obviously means that $\mathcal{B}$ is very small. However, Dani showed that, from the viewpoint of Hausdorff dimension, this set can be as large as possible:

THEOREM 3.8 (Dani [**D89**, Thm. 4.2]). *There is a nontrivial one-parameter group $\{a^t\}$ of diagonal matrices in $\mathrm{SL}(n, \mathbb{R})$, such that the Hausdorff dimension of the set of bounded orbits in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ is equal to $n^2 - 1$, the dimension of $\mathrm{SL}(n, \mathbb{R})$.*

The proof uses a method, now called "Schmidt's game," which had been introduced into the theory of Diophantine Approximation by W. M. Schmidt [**Sc66**]. We have two players, A and B. In the setting of Theorem 3.8, Player B starts the game, by choosing any closed ball B_0 in the space $X = G/\Gamma$, and the game proceeds inductively. Given ball B_{i-1}, Player A chooses a closed ball A_i that is contained in B_{i-1}, and whose radius is exactly one-half of the radius of B_{i-1}. Then Player B chooses a ball B_i that is contained in A_i, and whose radius is exactly one-half of the radius of A_i. We say $\mathcal{B}$ is a *winning set* if Player A has a strategy to guarantee that the unique point in $\bigcap_{i=0}^{\infty} B_i$ belongs to $\mathcal{B}$. (More precisely, $\mathcal{B}$ is a "$(1/2, 1/2)$-winning set," since we have specified that the radius of each ball is one-half of the radius of the preceding ball.)

Results like Theorem 3.8 can be proved by showing that the set of bounded orbits is a winning set. It is obvious that a winning set must be dense in G/Γ. (Otherwise, Player B could win by choosing B_0 to be disjoint from $\mathcal{B}$.) A more careful argument shows that the Hausdorff dimension of a winning set must be equal to the dimension of G/Γ.

Dani's introduction of this method into the study of dynamical systems was very influential; Schmidt's game has now been used to prove that numerous other sets of exceptional orbits are large. For example, it can be used to study the set of orbits whose closure does not contain a certain point in the space.

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