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Actions of Semisimple Lie Groups on Circle Bundles

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Abstract. Suppose G is a connected, simple, real Lie group with $\mathbb{R}$-rank $(G) \geqslant 2$, M is an ergodic G-space with invariant probability measure μ, and α: $G \times M \to \text{Homeo}(\mathbb{T})$ is a Borel cocycle. We use an argument of É. Ghys to show that there is a G-invariant probability measure ν on the skew product $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ. Furthermore, if $\alpha(G \times M) \subset \text{Diff}^1(\mathbb{T})$, then ν can be taken to be equivalent to $\mu \times \lambda$, where λ is Lebesgue measure on $\mathbb{T}$; therefore, α is cohomologous to a cocycle with values in the isometry group of $\mathbb{T}$.

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1. Introduction

É. Ghys [Gh] recently proved that irreducible lattices in most semisimple Lie groups of higher real rank do not have any interesting differentiable actions on the circle $\mathbb{T}$.

DEFINITION 1.1. A lattice Γ in a connected, semisimple, real Lie group G is *irreducible* if $N\Gamma$ is dense in G, for every closed, connected, noncompact, normal subgroup N of G.

NOTATION 1.2. We use $\text{Diff}^1(\mathbb{T})$ to denote the group of C^1 diffeomorphisms of $\mathbb{T}$, and $\text{Diff}^1_+(\mathbb{T})$ to denote the subgroup of orientation preserving diffeomorphisms.

THEOREM 1.3 (Ghys [Gh, Thm. 1.1]). *Let Γ be an irreducible lattice in a connected, semisimple, real Lie group G, such that*

(1) $\mathbb{R}$-rank $G \geqslant 2$*; and*
(2) *there is no continuous homomorphism from G onto* $\text{PSL}(2, \mathbb{R})$.

Then every homomorphism from Γ to $\text{Diff}^1(\mathbb{T})$ has finite image

Remark 1.4. Under the additional assumption that $H^2(\Gamma; \mathbb{R}) = 0$ (and in many other cases), the conclusion of the theorem was also proved by M. Burger and N. Monod [BM1, BM2, BM3], as a consequence of vanishing theorems for bounded cohomology. (The results of Burger and Monod also apply to the setting where $\mathbb{R}$ is replaced by other local fields; for example, Γ could be an S-arithmetic group (cf. 6.10 and 6.11).) For a more restricted class of lattices in real semisimple Lie groups, B. Farb and P. Shalen [FS] proved finiteness of the image of homomorphisms into the group $\mathrm{Diff}^{\omega}(M)$ of real analytic diffeomorphisms of some higher-dimensional manifolds.

In this paper, we extend Ghys' Theorem to the context of semisimple Lie group actions on circle bundles, or, more generally, $\mathrm{Diff}^1(\mathbb{T})$-valued Borel cocycles for ergodic actions of G. We first recall:

DEFINITION 1.5 ([Zi, Defns 4.2.1 and 4.2.2, p. 65, and top of p. 75]). Suppose M is a Borel G-space with quasi-invariant measure μ, and H is a topological group (such that the Borel structure on H is countably generated).

- A Borel function $\alpha\colon G \times M \to H$ is a *Borel cocycle* if, for all $g, h \in G$, we have $\alpha(gh, m) = \alpha(g, hm)\,\alpha(h, m)$ for a.e. $m \in M$.
- Two Borel cocycles $\alpha, \beta\colon G \times M \to H$ are *cohomologous* if there is a Borel function $\phi\colon M \to H$, such that, for each $g \in G$, we have $\beta(g, m) = \phi(gm)^{-1}\,\alpha(g, m)\phi(m)$, for a.e. $m \in M$.
- A Borel cocycle $\alpha\colon G \times M \to H$ is *strict* if, for all $g, h \in G$, we have $\alpha(gh, m) = \alpha(g, hm)\,\alpha(h, m)$ for *every* $m \in M$. For every Borel cocycle $\alpha\colon G \times M \to H$, there is a strict Borel cocycle $\alpha'\colon G \times M \to H$, such that, for every $g \in G$, we have $\alpha'(g, m) = \alpha(g, m)$ for a.e. $m \in M$ [Zi, Thm. B.9, p. 200].
- If $\alpha\colon G \times M \to H$ is a strict Borel cocycle and S is a Borel H-space, the *skew-product* action $M \times_{\alpha} S$ is the Borel action of G on $M \times S$ defined by $g \cdot (m, s) = \big(gm, \alpha(g, m)s\big)$.

Recall that any smooth action on a circle bundle defines a $\mathrm{Diff}^1(\mathbb{T})$-valued cocycle on the base, and that the action on the bundle is measurably conjugate to the skew product action defined by this cocycle. Conversely, the skew product defined by any $\mathrm{Diff}^1(\mathbb{T})$-valued cocycle can be viewed as an action on a measurable circle bundle over the base.

For $M = G/\Gamma$, cohomology classes of Borel cocycles $\alpha\colon G \times M \to \mathrm{Diff}^1(\mathbb{T})$ are in bijective correspondence with conjugacy classes of homomorphisms $\hat{\alpha}\colon \Gamma \to \mathrm{Diff}^1(\mathbb{T})$ [Zi, Prop. 4.2.13, p. 70]. Then the conclusion of Ghys' Theorem asserts that α is cohomologous to a Borel cocycle whose image is a finite subgroup of $\mathrm{Diff}^1(\mathbb{T})$. However, the following example shows that this conclusion is not valid for Borel cocycles for more general G-spaces; not even for Borel cocycles that arise from a C^{∞}, volume-preserving action of G on a principal $\mathbb{T}$-bundle over a compact manifold.

EXAMPLE 1.6. Let

- H be a connected, semisimple Lie group;
- Γ be a torsion-free, cocompact lattice in H;
- T be a subgroup of H that is isomorphic to $\mathbb{T}$;
- G be a closed subgroup of H that centralizes T and acts ergodically on H/Γ (see 2.11); and
- $M = T\backslash H/\Gamma$.

Because Γ is torsion free and cocompact, we know that M is a compact manifold. Because G centralizes T, the action of G by translation on H/Γ factors through to an action on M; we see that H/Γ is a principal $\mathbb{T}$-bundle over M, and G acts on H/Γ by bundle automorphisms. Thus, there is a Borel cocycle α: $G\times M \to \mathbb{T}$, such that the action of G on H/Γ is isomorphic to the skew product $M \times_\alpha \mathbb{T}$. By assumption, the action of G on H/Γ is ergodic, so, if β is any cocycle cohomologous to α, then $M \times_\beta \mathbb{T}$ must be ergodic. Therefore, the image of β cannot be contained in any finite group of transformations of $\mathbb{T}$.

These examples show that there can be nontrivial cocycles into Isom($\mathbb{T}$), the isometry group of $\mathbb{T}$. Our extension of Ghys' Theorem shows that if G has Kazhdan's property (T) (see 2.14), then every cocycle into $\mathrm{Diff}^1(\mathbb{T})$ for a much more general G-action is cohomologous to one into Isom($\mathbb{T}$). (However, as far as we know, the homeomorphisms in the image of the map implementing the cohomology may not be differentiable, but only absolutely continuous.) In more geometric terms, this asserts that for G-actions on very general circle bundles, there is a measurable choice of metric on each fiber that is preserved by the action. I.e., the action on the bundle is an 'isometric extension' of the base.

DEFINITION 1.7. Let G be a connected, semisimple Lie group, and let M be an ergodic G-space with quasi-invariant measure. We say that M is *irreducible* if every closed, connected, noncompact, normal subgroup of G is ergodic on M.

NOTATION 1.8. $\mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$ denotes the group of all homeomorphisms ϕ of $\mathbb{T}$, such that $\phi_*\lambda$ has the same null sets as λ, where λ is the Lebesgue measure on $\mathbb{T}$.

THEOREM 5.4′. *Let*

- *G be a connected, real, semisimple Lie group, such that*
  - *G has Kazhdan's property (T), and*
  - *$\mathbb{R}$-rank $G \geqslant 2$;*
- *M be an irreducible ergodic G-space with finite invariant measure μ; and*
- *α: $G \times M \to \mathrm{Diff}^1(\mathbb{T})$ be a Borel cocycle.*

Then, as a cocycle into $\mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$, *$\alpha$ is cohomologous to a cocycle with values in* Isom($\mathbb{T}$).

Furthermore, if $\alpha(g, m)$ is orientation preserving, for almost every $(g, m) \in G \times M$, then, as a cocycle into $\mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$, *$\alpha$ is cohomologous to a cocycle with values in the rotation group* $\mathrm{Rot}(\mathbb{T})$.

It is an open question whether Ghys' Theorem 1.3 remains valid if $\mathrm{Diff}^1(\mathbb{T})$ is replaced with the homeomorphism group $\mathrm{Homeo}(\mathbb{T})$. (Witte [Wi] showed that the answer is affirmative if Γ is an arithmetic lattice of $\mathbb{Q}$-rank at least two.) However, Ghys (and, in most cases, also Burger and Monod) made the following major step toward an affirmative answer.

THEOREM 1.9 ([Gh, Thm. 3.1]). *Let Γ be an irreducible lattice in a connected, semisimple, real Lie group G, such that*

(1) $\mathbb{R}\text{-rank}\, G \geqslant 2$; *and*
(2) *there is no continuous homomorphism from G onto* $\mathrm{PSL}(2, \mathbb{R})$.

Then every continuous action of Γ on $\mathbb{T}$ has an invariant probability measure.
In fact, every continuous action of Γ on $\mathbb{T}$ has a finite orbit.

Ghys obtained Theorem 1.3 by combining Theorem 0.2 with the Thurston Stability Theorem 5.1. (He also proved that if G does have a continuous homomorphism onto $\mathrm{PSL}(2, \mathbb{R})$, then any action of Γ on $\mathbb{T}$ either preserves a probability measure or is semi-conjugate to a finite cover of the restriction of a G-action (cf. 6.13).)

THEOREM 5.1′ (Thurston [Th]). *Suppose Γ is a finitely generated group, such that $\Gamma/[\Gamma, \Gamma]$ is finite. If $\sigma\colon \Gamma \to \mathrm{Diff}^1_+(\mathbb{T})$ is any homomorphism, such that $\sigma(\Gamma)$ has a fixed point, then $\sigma(\Gamma)$ is trivial.*

The following theorem is the natural generalization of Theorem 1.9 to the setting of ergodic.

G-actions. Although Ghys did not state this result, it can be proved by translating his proof in a straightforward way from the setting of homomorphisms of lattices to the setting of Borel cocycles for ergodic G-actions. In Section 4, we provide a proof that is based on Ghys' ideas, but is much shorter than a direct translation.

THEOREM 1.10. *Let*

- *G be a connected, semisimple, real Lie group, such that*
  - $\mathbb{R}\text{-rank}\, G \geqslant 2$, *and*
  - *there is no continuous homomorphism from G onto* $\mathrm{PSL}(2, \mathbb{R})$;
- *M be an irreducible ergodic G-space with invariant probability measure μ; and*
- *$\alpha\colon G \times M \to \mathrm{Homeo}(\mathbb{T})$ be a strict Borel cocycle.*

Then there is a G-invariant probability measure ν on $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ.

We obtain Theorem 5.4 by combining Theorem 1.10 with the following generalization of Theorem 5.1.

DEFINITION 1.11. Let α: $G \times M \to H$ be a Borel cocycle, and let Y be an H-space. A function f: $M \to Y$ is *α-equivariant* if, for each $g \in G$, we have $f(gm) = \alpha(g, m)f(m)$ for almost every $m \bullet M$.

THEOREM 5.3′. *Let*

- *G be a connected Lie group with Kazhdan's property (T);*
- *M be an ergodic G-space with finite invariant measure μ;*
- *α: $G \times M \to \mathrm{Diff}^1(\mathbb{T})$ be a Borel cocycle; and*
- *f: $M \to \mathbb{T}$ be an α-equivariant measurable map. (In bundle theoretic terms, f is a measurable G-invariant section.)*

Then, as a cocycle into $\mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$, α is cohomologous to the trivial cocycle.

Theorems 5.4 and 1.10 can be generalized to allow G to be a S-algebraic group (see 6.5), and there are also analogues for Γ-actions, where Γ is a lattice in G (see 6.3). Thus, as was already mentioned in Remark 2, Ghys' Theorem 1.3 can be generalized to allow Γ to be an S-arithmetic group (see 6.11).

The paper is organized as follows. Section 2 establishes notation, and recalls various results from measure theory, Lie theory, ergodic theory, and Kazhdan's property (T). Section 3 constructs a pair of subgroups that play a crucial role in the proof of Theorem 1.10, which is presented in Section 4. Section 5 proves Theorems 5.3 and 5.4, our results on differentiable actions. Section 6 extends our main results to slightly different settings.

2. Preliminaries

2A. PROBABILITY MEASURES

NOTATION 2.1. We use I to denote the unit interval $[0, 1]$, and $\mathbb{T}$ to denote the unit circle. For $\Omega = \mathbb{T}$ or I:

- λ denotes the Lebesgue measure on Ω; and
- $\mathrm{Prob}(\Omega)$ denotes the space of probability measures on Ω, with the weak* topology.

DEFINITION 2.2. Measures μ_1 and μ_2 on a Borel space X are *equivalent* (or in the same *measure class*) if they have the same null sets.

LEMMA 2.3. *Let* α: $G \times M \to \mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$ *be a Borel cocycle. There is a* G*-invariant probability measure on* $M \times_\alpha \mathbb{T}$ *that is equivalent to* $\mu \times \lambda$ *if and only if* α *is equivalent to a cocycle with values in* $\mathrm{Isom}(\mathbb{T})$.

Proof. ($\Leftarrow$) By assumption, there is a Borel cocycle β: $G \times M \to \mathrm{Isom}(\mathbb{T})$, and a Borel function ϕ: $M \to \mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$, such that, for each $g \in G$, we have

$$\alpha(g, m) = \phi(gm)^{-1}\beta(g, m)\phi(m)$$

for a.e. $m \in M$. Let

$$\nu = \int_M \left(m \times \phi(m)_*^{-1}\lambda\right) d\mu(m) \in \mathrm{Prob}(M \times \mathbb{T}).$$

Because $\phi(m) \in \mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$, we know that $\phi(m)_*\lambda$ is equivalent to λ, for every $m \in M$, so ν is equivalent to $\mu \times \lambda$. Because λ is invariant under $\mathrm{Isom}(\mathbb{T})$, it is easy to see that ν is invariant under the action of G on $M \times_\alpha \mathbb{T}$.

($\Rightarrow$) Let ν be a G-invariant probability measure on $M \times_\alpha \mathbb{T}$ that is equivalent to $\mu \times \lambda$. We may write

$$\nu = \int_M (m \times \nu_m)\, d\mu(m),$$

where ν_m is a probability measure on $\mathbb{T}$. Because ν is equivalent to $\mu \times \lambda$, we know that ν_m is equivalent to λ, for a.e. $m \in M$. Thus, for a.e. $m \in M$, there exists $\phi(m) \in \mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$, such that $\nu_m = \phi(m)_*^{-1}\lambda$. Now define β: $G \times M \to \mathrm{Homeo}^{\mathrm{Leb}}(\mathbb{T})$ by $\beta(g, m) = \phi(gm)\alpha(g, m)\phi(m)^{-1}$. Then $\mu \times \lambda$ is a G-invariant measure on $M \times_\beta \mathbb{T}$, so we see, for each $g \in G$, that $\beta(g, m)$ preserves λ, and hence is in $\mathrm{Isom}(\mathbb{T})$, for a.e. $m \in M$. ∎

2B. LIE THEORY [WA, Chap. 1]

Let G be a connected, semisimple, real Lie group.

NOTATION 2.4. We use lower-case gothic letters $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{p}$, $\mathfrak{q}$, etc. for the Lie algebras of Lie groups G, H, P, Q, etc.

DEFINITION 2.5. A subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is a *maximal split toral subalgebra* of $\mathfrak{g}$ if

(1) $\mathfrak{a}$ is Abelian;
(2) $\mathrm{ad}_{\mathfrak{g}}\, a$ is diagonalizable over $\mathbb{R}$, for every $a \in \mathfrak{a}$; and
(3) $\mathfrak{a}$ is maximal, with respect to (1) and (2).

A *maximal split torus* of G is a closed, connected subgroup A of G, such that the Lie algebra $\mathfrak{a}$ of A is a maximal split toral subalgebra of G.

DEFINITION 2.6. Let A be a maximal split torus of G.

- For each linear functional α: $\mathfrak{a} \to \mathbb{R}$, we let

$$\mathfrak{g}_\alpha = \{ v \in \mathfrak{g} \mid (\mathrm{ad}_{\mathfrak{g}} a)(v) = \alpha(a)v \text{ for all } a \in \mathfrak{a} \}.$$

- A linear functional α: $\mathfrak{a} \to \mathbb{R}$ is a *real root* of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$.
- The *relative Weyl group* of G is $N_G(A)/C_G(A)$.

DEFINITION 2.7. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is *parabolic* if $\mathfrak{p} \otimes \mathbb{C}$ contains a maximal solvable subalgebra of $\mathfrak{g} \otimes \mathbb{C}$.

A subgroup P of G is *parabolic* if

- $\mathfrak{p}$ is parabolic and
- $P = N_G(\mathfrak{p})$.

Remark 2.8 ([Wa, Thm. 1.2.4.8, p. 75]). If P is any parabolic subgroup of G, then P contains a maximal split torus A of G. We have $C_G(A) \subset P$ and, for any real root α of $\mathfrak{g}$, we have either $\mathfrak{g}_\alpha \subset \mathfrak{p}$ or $\mathfrak{g}_{-\alpha} \subset \mathfrak{p}$.

Remark 2.9. A proper subgroup P of $\mathrm{SL}(2, \mathbb{R})$ is parabolic if and only if P is conjugate to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

LEMMA 2.10. *Let P be a parabolic subgroup of G, and let L be a closed, connected subgroup of G that is locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$. *If $\mathfrak{p} \cap \mathfrak{l}$ is a parabolic subalgebra of $\mathfrak{l}$, then $P \cap L$ is a parabolic subgroup of L.*

Proof. Because $\mathfrak{p} \cap \mathfrak{l}$ is a parabolic subalgebra of $\mathfrak{l}$, there is a parabolic subgroup Q of L, such that $Q^\circ = (P \cap L)^\circ$. We wish to show that $Q \subset P$. By definition, P is the normalizer of $\mathfrak{p}$, so it suffices to show that every subalgebra of $\mathfrak{g}$ normalized by Q° is also normalized by Q.

Because $\mathrm{SL}(2, \mathbb{R})$ is simply connected as an algebraic group, the adjoint representation of L on $\mathfrak{g}$ must factor through either $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{R})$. Then, because parabolic subgroups of $\mathrm{SL}(2, \mathbb{R})$ are Zariski connected, we conclude that every subalgebra of $\mathfrak{g}$ normalized by Q° is also normalized by Q, as desired. ∎

2C. ERGODIC ACTIONS

THEOREM 2.11 ('Moore Ergodicity Theorem', cf. [Zi, Thm. 2.2.15, p. 21]). *Let*

- *G be a connected, semisimple, real Lie group;*
- *M be an irreducible, ergodic G-space with finite invariant measure; and*
- *H be a closed subgroup of G, such that $\mathrm{Ad}_G H$ is not precompact.*

Then

(1) *the action of H on M is ergodic; and*
(2) *the diagonal action of G on $(G/H) \times M$ is ergodic.*

COROLLARY 2.12. *Let M be an irreducible, ergodic G-space with finite invariant measure, and let P be a minimal parabolic subgroup of G. Then the diagonal action of G on $(G/P) \times (G/P) \times M$ is ergodic.*

DEFINITION 2.13. An action of G on a space X is *triply transitive* if G is transitive on the set of ordered triples of distinct points of X.

We note that if G acts triply transitively on X, then X has no nontrivial, proper G-equivariant quotients. (In particular, every G-equivariant quotient of X is triply transitive.) Namely, if there are two distinct points in the same fiber of a quotient map, then, by double transitivity, G can move them to two points in different fibers. This is impossible if the quotient map is G-equivariant.

2D. KAZHDAN'S PROPERTY (T)

DEFINITION 2.14 (Kazhdan, cf. [Ma, Prop. III.2.8(A), p. 116]). A locally compact group G has *Kazhdan's property* (T) if, for every unitary representation ρ of G on a Hilbert space V, there is a compact subset C of G, such that, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if $v \in V$ is any vector with the property that

$$\|\rho(g)v - v\| \leqslant \delta\|v\| \text{ for every } g \in C,$$

then there is a $\rho(G)$-invariant vector $w \in V$, such that $\|w\| = \|v\|$ and $\|w - v\| \leqslant \varepsilon\|v\|$.

The following well-known theorem describes exactly which connected, semisimple, real Lie groups have Kazhdan's property (T). We note, in particular, that $\mathrm{SL}(2, \mathbb{R})$ does not have Kazhdan's property (T).

THEOREM 2.15 (Kazhdan, Kostant, Serre, Wang). *Let G be a connected semisimple real Lie group.*

(1) *Assume G is simple. Then G has Kazhdan's property (T) if and only if either*
- *$\mathbb{R}$-rank$(G) \geqslant 2$ or*
- *G is compact, or*
- *G is locally isomorphic to either $\mathrm{Sp}(1, n)$ or the real-rank one form of F_4.*

(2) *G has Kazhdan's property (T) if and only if each simple factor of G has Kazhdan's property (T).*

(3) *G has Kazhdan's property (T) if and only if $G/Z(G)$ has Kazhdan's property (T).*

Proof. For (1), see [Ma, Thm. III.5.6(c), p. 131]. For (2$\Leftarrow$), see [Ma, Cor. III.2.10, p. 117]. For (2$\Rightarrow$ and 3$\Rightarrow$), see [Ma, Lem. III.2.4, p. 115]. For (3$\Leftarrow$), see [HV, Thm. 2.12, p. 28]. ∎

In the proof of our generalization of the Thurston Stability Theorem 5.3, the following lemma is used to construct vectors v as in Definition 2.14.

LEMMA 2.16 (cf. [Zi, 2nd par. of pf. of Thm. 9.1.1, p. 163]). *Let* α: $G \times M \to \mathrm{Diff}^1(I)$ *be a Borel cocycle. For each* $g \in G$, *assume that for almost every* $m \in M$, *we have* $\alpha(g, m)(0) = 0$ *and* $\alpha(g, m)'(0) = 1$. *Then, for every compact subset* C *of* G, *and every* $\varepsilon > 0$, *there is a nontrivial interval* I' *containing* 0, *such that, for every* $g \in C$, *we have*

$$\mu\{ m \in M \mid \forall s \in I', \left|\alpha(g, m)'(s) - 1\right| < \varepsilon\} > 1 - \varepsilon.$$

3. A Crucial Lemma

Ghys' proof of Theorem 1.3 is based on the existence certain subgroups P and L of G, such that $P \subset L$, and the action of L on L/P is triply transitive. (Then this is contrasted with the fact that the group of orientation-preserving homeomorphisms of $\mathbb{T}$ is not triply transitive on $\mathbb{T}$.) Ghys describes P and L quite explicitly, in geometric terms, but this depends on a case-by-case study that uses the classification of semisimple Lie groups. By giving a uniform construction, the following lemma allows us to avoid case-by-case analysis (or, at least, to condense it into this one lemma).

LEMMA 3.1. *Let*

- H *be a connected, noncompact, almost simple, real Lie group;*
- P *be a minimal parabolic subgroup of* H; *and*
- A *be a maximal split torus of* H *contained in* P.

If H *is not locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$, *then there is a connected Lie subgroup* L *of* H, *such that:*

(1) $L \not\subset P$;
(2) $\mathfrak{a} \cap [\mathfrak{l}, \mathfrak{l}]$ *is nontrivial;*
(3) $C_{\mathfrak{a}}(\mathfrak{l})$ *has codimension one in* $\mathfrak{a}$; *and*
(4) $LN_P(L)$ *is triply transitive on* $LN_P(L)/N_P(L)$.

Proof. Let us begin by making our goal more specific.

CLAIM. *It suffices to find a connected, closed subgroup* L *of* H, *a real root* α *of* H, *and an element* g *of* H, *such that:*

(a) L *is locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$;
(b) $\mathfrak{l} = \langle \mathfrak{h}_\alpha \cap \mathfrak{l}, \mathfrak{h}_{-\alpha} \cap \mathfrak{l} \rangle$;
(c) $g \bullet C_H(A)$; *and*
(d) g *normalizes* L, *and acts on* L *by an outer automorphism.*

Proof of Claim. (1) Because P is minimal parabolic, we know that $P/\mathrm{Rad}P$ is compact, so P does not contain L (or any other noncompact, semisimple subgroup).

(2) By definition, $[\mathfrak{l}, \mathfrak{l}]$ contains the nontrivial subalgebra $[\mathfrak{h}_\alpha \cap \mathfrak{l}, \mathfrak{h}_{-\alpha} \cap \mathfrak{l}]$ of $\mathfrak{a}$.

(3) Because $\ker(\alpha)$ centralizes $\mathfrak{l}$, we know that $C_{\mathfrak{a}}(\mathfrak{l})$ has codimension one in $\mathfrak{a}$.

(4) Because P is parabolic and contains A, we know that $\mathfrak{p}$ contains either $\mathfrak{h}_\alpha$ or $\mathfrak{h}_{-\alpha}$ (see 2.8). Thus, $\mathfrak{p} \cap \mathfrak{l}$ is a parabolic subalgebra of $\mathfrak{l}$ (see 2.9). Then Lemma 2.10 implies that $P \cap L$ is parabolic in L. Thus, we may identify $L/(P \cap L)$ with $\mathbb{R}P^1 \approx \mathbb{T}$, so there is an L-invariant circular order on $L/(P \cap L)$, and L has only two orbits on the ordered triples of distinct points in $L/(P \cap L)$: the positively oriented triples and the negatively oriented triples. Modulo inner automorphisms, there is only one outer automorphism of L, so it is easy to verify that any outer automorphism of L that fixes $P \cap L$ must take each positively oriented triple to a negatively oriented triple. Thus, because $g \blacklozenge C_G(A) \subset P$ (see 2.8), we see from (d) that all ordered triples of distinct points in $L/(P \cap L)$ are in the same $\big(L N_P(L)\big)$-orbit.

This completes the proof of the claim.

We now consider two cases, based on the real rank of H.

Case 1. Assume $\mathbb{R}$-rank $H = 1$. From the classification of simple Lie groups of real rank one (cf. [He, Table X.V, p. 518]) (and the fact that H is not locally isomorphic to $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}(1, 2)$), we know that H must contain a subgroup locally isomorphic to either $\mathrm{SO}(1, 3)$ (if H is locally isomorphic to $\mathrm{SO}(1, n)$) or $\mathrm{SU}(1, 2)$ (if H is locally isomorphic to $\mathrm{SU}(1, n)$, $\mathrm{Sp}(1, n)$, or the rank one form of F_4). Then the proof is completed by explicitly constructing L and g for $\mathrm{SO}(1, 3)$ and $\mathrm{SU}(1, 2)$.

Subcase 1.1. Assume H is locally isomorphic to $\mathrm{SO}(1, 3)$. We may assume that $H = \mathrm{SL}(2, \mathbb{C})$, that A consists of diagonal matrices, and that P is the group of upper triangular matrices. The matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ acts by an outer automorphism of $\mathrm{SL}(2, \mathbb{R})$.

Subcase 1.2. Assume $H = \mathrm{SU}(1, 2)$. We use the Hermitian form $\langle x|y\rangle = x_1\overline{y_3} + x_2\overline{y_2} + x_3\overline{y_1}$. We may assume that A consists of diagonal matrices, and that P is the group of upper triangular matrices in H. Let

$$\mathfrak{l} = \left\{ \begin{pmatrix} a & t & 0 \\ s & 1 & -t \\ 0 & -s & -a \end{pmatrix} \;\middle|\; a, s, t \in \mathbb{R} \right\} \text{ and } g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Case 2. Assume $\mathbb{R}$-rank $H > 1$. It is well known (see, for example, [Ma, Prop. I.1.6.2, p. 46]) that H contains a closed, connected subgroup that is locally isomorphic to either $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{Sp}(4, \mathbb{R})$. Therefore, by passing to a subgroup, and then passing to a locally isomorphic group, we may assume that H is either $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{Sp}(4, \mathbb{R})$.

Subcase 2.1. Assume $H = \mathrm{SL}(3, \mathbb{R})$. We may assume that A consists of diagonal matrices, and that P is the group of upper triangular matrices. Let

$$L = \begin{pmatrix} \mathrm{SL}(2, \mathbb{R}) & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ acts by an outer automorphism of L, and centralizes A.

Subcase 2.2. Assume $H = \mathrm{Sp}(4, \mathbb{R})$. We use the symplectic form defined by

$$\langle (x_1, x_2) | (y_1, y_2) \rangle = x_1 \cdot y_2 - x_2 \cdot y_1,$$

for $x_i, y_i \in \mathbb{R}^2$, and we may assume that A consists of diagonal matrices. Let

$$L = \left\{ \begin{pmatrix} R & 0 \\ 0 & \theta(R) \end{pmatrix} \middle| R \in \mathrm{SL}(2, \mathbb{R}) \right\},$$

where θ is the Cartan involution (transpose-inverse). The matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

acts by an outer automorphism of L, and centralizes A. ∎

Remark 3.2. By using more theory, one can give a more conceptual proof of Lemma 0.1, without using the classification of real simple Lie algebras.

Case 1. Assume $\mathbb{R}\text{-rank}\, H = 1$. Write $P^\circ = CAU$, where C is a compact, connected subgroup of $C_H(A)$ and U is the unipotent radical of P. Let α be the simple real root of H, and assume without loss of generality that $\mathfrak{h}_\alpha \subset \mathfrak{u}$. Because the compact, connected group C acts nontrivially on $\mathfrak{h}_\alpha$, there is some $g \in C$ and $u \in \mathfrak{h}_\alpha$, such that $\mathrm{Ad}g(u) = -u$. From the Jacobson–Morosov Theorem, we know that u is contained in a subalgebra $\mathfrak{l}$ that is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Since $\mathbb{R}\text{-rank}\, H = 1$, we know that $N_{\mathfrak{h}}(\langle u \rangle) \subset \mathfrak{p}$, so $\mathfrak{p} \cap \mathfrak{l}$ contains a maximal split torus of $\mathfrak{l}$. Thus, because all maximal split toral subalgebras of $\mathfrak{p}$ are conjugate, there is some $v \in U$, such that $(\mathrm{Ad}v)(\mathfrak{a})$ is a maximal split toral subalgebra of $\mathfrak{l}$ that normalizes $\langle u \rangle$. Then $\mathfrak{a}$ normalizes $(\mathrm{Ad}v^{-1})\langle u \rangle$, so $(\mathrm{Ad}v^{-1})u \in \mathfrak{h}_\alpha$. Because u is also in $\mathfrak{h}_\alpha$, and $[\mathfrak{u}, \mathfrak{u}] \cap \mathfrak{h}_\alpha = 0$, we conclude that $(\mathrm{Ad}v^{-1})u = u$. Thus, replacing $\mathfrak{l}$ by $(\mathrm{Ad}v^{-1})\mathfrak{l}$, we may assume that $\mathfrak{a} \subset \mathfrak{l}$.

Then g normalizes the parabolic subalgebra $\mathfrak{a} + \langle u \rangle$ of $\mathfrak{l}$, so it must normalize $\mathfrak{l}$. Also, we know that g acts on $\mathfrak{l}$ by an outer automorphism, because g

conjugates u to $-u$, whereas no nontrivial unipotent element is conjugate to its inverse in $\mathrm{SL}(2, \mathbb{R})$.

Case 2. Assume $\mathbb{R}$-rank $H > 1$. For simplicity, let us assume that H is $\mathbb{R}$-split. Choose two roots α and β, such that the β-string through α has odd length, let $\mathfrak{l} = \langle \mathfrak{h}_\alpha, \mathfrak{h}_{-\alpha} \rangle$, let L_β be the connected Lie subgroup of H corresponding to the subalgebra $\langle \mathfrak{h}_\beta, \mathfrak{h}_{-\beta} \rangle$, and let V be the L_β-submodule of $\mathfrak{h}$ generated by $\mathfrak{h}_\alpha$. Then, identifying L_β with $\mathrm{SL}(2, \mathbb{R})$, the highest weight of V is odd, so $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as -1 on the highest weight space $\mathfrak{h}_\alpha$. ∎

4. Proof of Theorem 1.10

The reader is encouraged to read Ghys' beautiful proof [Gh, Section 4] for the case of lattices in $\mathrm{SL}(3, \mathbb{R})$ before looking at the general case considered here. Many of the ideas of this section can be found in [Gh], but we have reorganized them, and changed some of the emphasis. Ghys' proof is presented in geometric terms, but we have reformulated the argument in group-theoretic terms.

NOTATION 4.1.

- G, M, and α are always assumed to be as described in the statement of Theorem 1.10. (In particular, G has no factors locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$.)
- P is a minimal parabolic subgroup of G.
- For any natural number k, $\mathbb{T}_k$ denotes the collection of all k-element subsets of $\mathbb{T}$.

LEMMA 4.2. *We may assume* α: $G \times M \to \mathrm{Homeo}_+(\mathbb{T})$.

Proof. Let sgn: $\mathrm{Homeo}(\mathbb{T}) \to \{\pm 1\}$ be the homomorphism with kernel $\mathrm{Homeo}_+(\mathbb{T})$, and let $\epsilon = \mathrm{sgn} \circ \alpha$, so ϵ: $G \times M \to \{\pm 1\}$ is a Borel cocycle.

Let $M^+ = M \times_\epsilon \{\pm 1\}$. Because M^+ is a two-point extension of M and M is irreducible, it is clear that each closed, connected, noncompact, normal subgroup of G has no more than two ergodic components on M^+. We may assume that G is ergodic on M^+, for, otherwise, ϵ is equivalent to the trivial cocycle, so α is equivalent to a cocycle into $\mathrm{Homeo}_+(\mathbb{T})$, as desired. Then G must act ergodically on the space of ergodic components of any normal subgroup. Because G, being connected, has no nontrivial action on any finite set, we conclude that M^+ is irreducible.

Define α^+: $G \times M^+ \to \mathrm{Homeo}(\mathbb{T})$ by $\alpha^+\big(g, (m, \pm 1)\big) = \alpha(g, m)$. If there is a G-invariant probability measure ν^+ on $M^+ \times_{\alpha^+} \mathbb{T}$, such that ν^+ projects to μ^+ on M^+, then simply let ν be the projection of ν^+ to $M \times_\alpha \mathbb{T}$.

Now let f be any orientation-reversing homeomorphism of $\mathbb{T}$, and define σ: $M^+ \to \mathrm{Homeo}(\mathbb{T})$ by

$$\sigma(m, \epsilon) = \begin{cases} \mathrm{Id} & \text{if } \epsilon = 1 \\ f & \text{if } \epsilon = -1 \end{cases}.$$

For any $m^+ \in M^+$, we have

$$\sigma(gm^+)\alpha^+(g, m^+)\sigma(m^+)^{-1} \in \mathrm{Homeo}_+(\mathbb{T}),$$

so we see that α^+ is cohomologous (via σ) to a cocycle with values in $\mathrm{Homeo}_+(\mathbb{T})$. □

Henceforth, we assume $\alpha(G \times M) \subset \mathrm{Homeo}_+(\mathbb{T})$.

It suffices to show that there is an α-equivariant Borel map ψ: $M \to \mathrm{Prob}(\mathbb{T})$, for then we may set $\nu = \int_M (m \times \psi(m))\, d\mu(m)$. The action of G on $(G/P) \times M$ is amenable (because P is amenable) [Zi, 4.1.7bis, 4.3.2, 4.3.4], and the space of measurable functions from $(G/P) \times M$ to $\mathrm{Prob}(\mathbb{T})$ is an affine G-space over $(G/P) \times M$ [Zi, Defn 4.3.1]. Thus, from the definition of an amenable action [Zi, Defn 4.3.1], we know that there is an α-equivariant Borel map Ψ: $(G/P) \times M \to \mathrm{Prob}(\mathbb{T})$ (cf. [Zi, pf. of Step 1 of Thm. 5.2.5, bot. of p. 103]). The following theorem completes the proof of Theorem 1.10.

THEOREM 4.3. *Suppose* Ψ: $(G/P) \times M \to \mathrm{Prob}(\mathbb{T})$ *is an* α*-equivariant Borel map. For each* $m \in M$, *define* Ψ_m: $G/P \to \mathrm{Prob}(\mathbb{T})$ *by* $\Psi_m(x) = \Psi(x, m)$. *Then* Ψ_m *is essentially constant, for a.e.* $m \in M$.

Proof. If almost every $\Psi(x, m)$ is atomless, the desired conclusion is given by Theorem 4.4 below. If there is some k, such that almost every $\Psi(x, m)$ consists of k atoms of equal weight, the desired conclusion is given by Corollary 4.6 below. Because G is ergodic on $(G/P) \times M$ (see 2.11), it is not difficult to reduce the problem to these two cases.

Namely, any $\nu \in \mathrm{Prob}(\mathbb{T})$ has a unique decomposition of the form $\nu = \nu_0 + \nu_1$, where ν_0 has no atoms, and ν_1 consists entirely of atoms. (Either of the terms in the decomposition may be 0.) Thus, we may write $\Psi = \Psi_0 + \Psi_1$, where $\Psi_i(x, m) = [\Psi(x, m)]_i$. Because the decomposition $\nu = \nu_0 + \nu_1$ is $\mathrm{Homeo}(\mathbb{T})$-equivariant (and unique), we see that Ψ_0 and Ψ_1 are α-equivariant. Then, because G is ergodic on $(G/P) \times M$, we see, for $i = 0, 1$, that either $\Psi_i = 0$ for a.e. (x, m) or $\Psi_i \neq 0$ for a.e. (x, m). Thus, either $\Psi_i = 0$ a.e. (in which case $\Psi = \Psi_{1-i}$), or, after renormalizing, Ψ_i defines an α-equivariant Borel map into $\mathrm{Prob}(\mathbb{T})$. Then, because the sum of α-equivariant functions is α-equivariant, there is no harm in assuming that either $\Psi = \Psi_0$ or $\Psi = \Psi_1$.

If $\Psi = \Psi_0$, then Theorem 4.4 shows that Ψ_m is essentially constant.

Thus, we henceforth assume that $\Psi = \Psi_1$. For any $\nu \in \mathrm{Prob}(\mathbb{T})$ that consists entirely of atoms, and any rational number $q \in (0, 1)$, let $\nu^{>q} \subset \mathbb{T}$ be the set of atoms of weight $> q$. Because this definition is $\mathrm{Homeo}(\mathbb{T})$-equivariant, and G is ergodic on $(G/P) \times M$, we see that the cardinality of $\Psi^{>q}$ is constant a.e., so $\Psi^{>q}$ is an α-equivariant Borel map into $\mathbb{T}_k$, for some k. Then Corollary 4.6 asserts that $\Psi_m^{>q}$ is essentially constant. Because this is true for all rational q, we conclude that Ψ_m itself is essentially constant, as desired. □

THEOREM 4.4. *Suppose* Ψ: $(G/P) \times M \to \mathrm{Prob}(\mathbb{T})$ *is an* α*-equivariant Borel map. For each* $m \bullet M$, *define* Ψ_m: $G/P \to \mathrm{Prob}(\mathbb{T})$ *by* $\Psi_m(x) = \Psi(x, m)$.

If $\Psi(x, m)$ *is atomless, for almost every* $(x, m) \in (G/P) \times M$, *then* Ψ_m *is essentially constant, for a.e.* $m \bullet M$.

Proof. Let $\mathrm{Prob}_0(\mathbb{T})$ be the set of atomless probability measures on $\mathbb{T}$. Define

$$\Psi^2\colon (G/P)^2 \times M \to \mathrm{Prob}_0(\mathbb{T})^2$$

by

$$\Psi^2(x_1, x_2, m) = \big(\Psi(x_1, m), \Psi(x_2, m)\big)$$

and

$$D\colon \mathrm{Prob}_0(\mathbb{T})^2 \to [0, 1]$$

by

$$D(\mu_1, \mu_2) = \sup_J \big|\mu_1(J) - \mu_2(J)\big|,$$

where J ranges over all subintervals of $\mathbb{T}$. It suffices to show that the composite function $D \circ \Psi^2$: $(G/P)^2 \times M \to [0, 1]$ is 0 a.e.

Step 1. D is continuous. Given $\mu_1, \mu_2 \in \mathrm{Prob}_0(\mathbb{T})$. Because μ_1 and μ_2 are atomless, there is a mesh $t_0, t_1, \ldots, t_n = t_0$ of points in $\mathbb{T}$, such that $\mu_k([t_i, t_{i+1}]) < \varepsilon/40$, for each i and for $k = 1, 2$. Also, for each $i, j \in \{0, \ldots, n\}$, there are continuous functions f_{ij}^+, f_{ij}^-: $\mathbb{T} \to [0, 1]$, such that $\mathrm{supp} f_{ij}^- \subset (t_i, t_j)$, $f_{ij}^+\big([t_i, t_j]\big) = 1$, and $\mu_k(f_{ij}^+ - f_{ij}^-) < \varepsilon/40$ for $k = 1, 2$. If ν_k is a measure so close to μ_k that $|\nu_k(f_{ij}^\epsilon) - \mu_k(f_{ij}^\epsilon)| < \varepsilon/40$ for all $i, j \in \{0, \ldots, n\}$ and $\epsilon \in \{+, -\}$, then

$$\Big|\nu_k\big([t_i, t_j]\big) - \mu_k\big([t_i, t_j]\big)\Big| < \frac{\varepsilon}{20}$$

for all i and j. Therefore $\big|\nu_k(J) - \mu_k(J)\big| < \varepsilon/2$ for every interval J, so $|D(\nu_1, \nu_2) - D(\mu_1, \mu_2)| \leqslant \varepsilon$. This proves the continuity of D.

Step 2. $D \circ \Psi^2$ is essentially constant. Because Ψ^2 is α-equivariant and D is $\mathrm{Homeo}(\mathbb{T})$-invariant, we know that $D \circ \Psi^2$ is essentially G-invariant. The Moore Ergodicity Theorem 2.12 implies that G is ergodic on $(G/P)^2 \times M$, so we conclude that $D \circ \Psi^2$ is essentially constant.

Step 3. We have $D \circ \Psi^2 = 0$ a.e. From Lusin's Theorem, we know that Ψ is continuous on some compact subset C of positive measure in G/P. Therefore, $D \circ \Psi^2$ is continuous on $C \times C$. By replacing C with a smaller compact set, we may assume that every conull subset of C is dense. Then, because $D \circ \Psi^2$ is essentially constant, we conclude that $D \circ \Psi^2$ is constant on $C \times C$. Obviously, $D \circ \Psi^2$ is 0 on the diagonal $\{(c, c)\}$, so we conclude that $D \circ \Psi^2$ is 0 a.e. ∎

THEOREM 4.5. *Suppose* Ψ: $(G/P) \times M \to \mathbb{T}_k$ *is an* α*-equivariant Borel map. For each* $m \in M$, *define* Ψ'_m: $G \to \mathbb{T}_k$ *by* $\Psi'_m(g) = \Psi(gP, m)$. *Suppose* L *is a closed, connected subgroup of* G, *such that*

(1) $C_P(L)$ *is not compact; and*
(2) $LN_P(L)$ *acts triply transitively on* $LN_P(L)/N_P(L)$.

Then Ψ'_m *is essentially right* L*-invariant, for a.e.* $m \in M$. *(That is, for each* $l \in L$, *we have* $\Psi'_m(gl) = \Psi'_m(g)$ *for a.e.* $g \in G$.*)*

Proof. Because Ψ'_m is right P-invariant, we may assume that $L \not\subset P$. The inclusion $N_P(L) \hookrightarrow LN_P(L)$ induces a G-equivariant smooth submersion

$$\pi\colon\ G/N_P(L) \to G/\big(LN_P(L)\big).$$

Define

$$\overline{X} = \Big\{ (x_1, x_2, x_3) \in \big(G/N_P(L)\big)^3 \ \Big|\ \pi(x_1) = \pi(x_2) = \pi(x_3) \Big\}$$

and

$$X = \big\{ (x_1, x_2, x_3) \in \overline{X} \mid x_1, x_2, x_3 \text{ distinct} \big\}.$$

Then $\overline{X}$ is a closed submanifold of $\big(G/N_P(L)\big)^3$, and X is conull open subset of $\overline{X}$ (with respect to any smooth measure on $\overline{X}$). For $i = 1, 2, 3$, let π_i: $\overline{X} \to G/P$ be the G-equivariant map defined by $\pi_i(x_1, x_2, x_3) = x_i P$.

Because G is transitive on $G/N_P(L)$, any G-orbit on X contains a point (x_1, x_2, x_3), such that $x_1 = N_P(L)$. Then x_1, x_2, x_3 are three points in $LN_P(L)/N_P(L)$. Thus, assumption (2) implies that G is transitive on X. In particular, this implies that the class χ of Lebesgue measure is the unique σ-finite G-invariant measure class on X. For $i = 1, 2, 3$, the projection $(\pi_i)_*\chi$ must be the G-invariant measure class on G/P. Thus, we have an essentially well-defined Borel map Ψ^3: $\overline{X} \times M \to (\mathbb{T}_k)^3$ given by

$$\Psi^3(x, m) = \Big(\Psi\big(\pi_1(x), m\big), \Psi\big(\pi_2(x), m\big), \Psi\big(\pi_3(x), m\big)\Big).$$

Note that Ψ^3 is α-equivariant.

The stabilizer of a triple of points in $LN_P(L)/N_P(L)$ obviously contains $C_P(L)$, which, by (1), is not compact. Thus, we conclude from the Moore Ergodicity Theorem 2.11 that G is ergodic on $\overline{X} \times M$. This implies that there is a single Homeo($\mathbb{T}$)-orbit O on $(\mathbb{T}_k)^3$, such that $\Psi^3(x, m) \in O$ for a.e. (x, m). For any permutation σ of $\{1, 2, 3\}$, and any $(x_1, x_2, x_3) \in \overline{X}$, we know that $(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ also belongs to $\overline{X}$. Therefore, Lemma 4.7 implies that $O = \{(A, A, A) | A \in \mathbb{T}_k\}$.

The map $G \times L^2 \to \overline{X}$ given by

$$(g, l, l') \mapsto \big(gN_P(L), glN_P(L), gl'N_P(L)\big)$$

is a submersion, so it preserves the class of Lebesgue measure. Thus, from the conclusion of the preceding paragraph, we see that, for almost every $m \in M$, $g \in G$, and $l, l' \in L$, we have $\Psi'_m(g) = \Psi'_m(gl) = \Psi'_m(gl')$. From Fubini's Theorem (and ignoring l'), we conclude, for a.e. $m \in M$, that Ψ'_m is essentially right L-invariant. ∎

COROLLARY 4.6. *Suppose* Ψ: $(G/P) \times M \to \mathbb{T}_k$ *is an* α*-equivariant Borel map. For each* $m \in M$, *define* Ψ_m: $G/P \to \mathbb{T}_k$ *by* $\Psi_m(x) = \Psi(x, m)$.

Then Ψ_m *is essentially constant, for a.e.* $m \in M$.

Proof. Let $P = MAN$ be the Langlands decomposition of P [Wa, p. 81]. (Because the parabolic subgroup P is minimal, we know that A is a maximal split torus of G.)

It suffices to show, for each simple factor H of G, that there are subgroups $L_1, L_2, \ldots, L_n$ of H, such that

(a) each subgroup L_i satisfies the hypotheses of Theorem 4.5, and
(b) $\{[L_i, L_i] \cap A\}$ generates $A \cap H$.

To see that this suffices, let J be the subgroup generated by $\{P \cap H\} \cup \{L_1, L_2, \ldots, L_n\}$. Then Theorem 4.5 implies that Ψ'_m is essentially right J-invariant. Because $J \supset P \cap H$, we know that J is parabolic in H; let $J = M_J A_J N_J$ be the Langlands decomposition of J, with $A_J \subset A \cap H$. Then $[J^\circ, J^\circ] \subset M_J N_J$, so $[J^\circ, J^\circ] \cap A_J = e$. On the other hand, we have $(A \cap H)^\circ \subset [J^\circ, J^\circ]$ (see (b)). We conclude that A_J is trivial, so

$$J \supset M_J A_J = C_H(A_J) = C_H(e) = H.$$

Therefore Ψ'_m is essentially right H-invariant. Because this is true for every simple factor H, we conclude that Ψ'_m is essentially right G-invariant, so Ψ'_m is essentially constant, for a.e. $m \in M$.

If $\mathbb{R}\text{-rank}\, H = 1$, then Lemma 3.1 provides an appropriate subgroup L satisfying (a) and (b). (Because L is centralized by all the simple factors other than H, the requirement that $C_P(L)$ be noncompact is automatically satisfied.)

We may now assume that $\mathbb{R}\text{-rank}\, H > 1$. Let L be as in Lemma 3.1, and let W be the relative Weyl group of $\mathfrak{H}$ (with respect to $\mathfrak{a} \cap \mathfrak{H}$). Because $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{a}$ is nontrivial and W acts irreducibly on $\mathfrak{a}$, we know that $\{w([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{a}) \mid w \in W\}$ spans $\mathfrak{a}$, so $\{w(L) \mid w \in W\}$ satisfies (b). Because $C_{\mathfrak{a}}(w(\mathfrak{l}))$ has codimension one in $\mathfrak{a}$ (and, being a subspace of $\mathfrak{a}$, is contained in $\mathfrak{p}$), we know that $C_P(L)$ is noncompact. Thus, we see that each $w(L)$ satisfies the hypotheses of Theorem 4.5. □

The following result was used in the proof of Theorem 4.5. For completeness, we include the proof. We also remark that, as explained by Ghys [Gh, Step 3 of §4, bot. of p. 210], the group $\text{Homeo}_+(\mathbb{T})$ has only finitely many orbits on $(\mathbb{T}_k)^3$.

LEMMA 4.7 (Ghys). *Let* O *be an orbit of* $\text{Homeo}_+(\mathbb{T})$ *on* $(\mathbb{T}_k)^3$, *and assume there is an element* (A_1, A_2, A_3) *of* O, *such that*

$$\left(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}\right) \in O,$$

for every permutation σ of $\{1, 2, 3\}$. Then

$$O = \{ (A, A, A) \mid A \in \mathbb{T}_k \}.$$

Proof [Gh, bot. of p. 211]. Let $B = A_1 \cup A_2 \cup A_3$, and let $H = \{h \in \text{Homeo}_+(\mathbb{T}) \mid h(B) = B\}$. For each permutation σ of $\{1, 2, 3\}$, there is an orientation-preserving homeomorphism h_σ (not unique) of $\mathbb{T}$, such that $h_\sigma(A_i) = A_{\sigma(i)}$. Then $h_\sigma \in H$, and the restriction of H to B is a cyclic group, so the commutator of any two of these homeomorphisms acts trivially on B. Because the permutation $\sigma = (1, 2, 3)$ is a commutator in the symmetric group S_3, we conclude that $h_{(1,2,3)}$ acts trivially on B. Because $h_{(1,2,3)}(A_1) = A_2$ and $h_{(1,2,3)}(A_2) = A_3$, this implies $A_1 = A_2 = A_3$.

5. The Reeb–Thurston Stability Theorem

Ghys' proof of Theorem 1.3 relies on the following one-dimensional case of the Reeb–Thurston Stability Theorem [Th]. (See [RS] and [Sc] for elegant proofs.)

THEOREM 5.1 (Thurston [Th]). *If Γ is a finitely generated group, such that $\Gamma/[\Gamma, \Gamma]$ is finite, then there is no nontrivial homomorphism $\Gamma \to \text{Diff}^1_+(I)$.*

For the proof of Theorem 5.4, we provide the following generalization in the setting of Borel cocycles. Applying this result to G/Γ recovers Thurston's theorem in the special case where Γ is a lattice in G, and G has Kazhdan's property (T) (see 2.14).

THEOREM 5.2. *Let*

- *G be a locally compact group with Kazhdan's property (T);*
- *M be a Borel G-space with invariant probability measure μ; and*
- *α: $G \times M \to \text{Diff}^1_+(I)$ be a Borel cocycle.*

Then there is a G-invariant probability measure ν on $M \times_\alpha I$, such that ν is equivalent to $\mu \times \lambda$ (and ν projects to μ on M).

Therefore, as a cocycle into $\text{Homeo}^{\text{Leb}}_+(I)$, α is cohomologous to the trivial cocycle.

Before proving Theorem 5.2, let us explain how it implies Theorem 5.4.

COROLLARY 5.3. *Let*

- *G be a locally compact group with Kazhdan's property (T);*
- *M be a Borel G-space with finite invariant measure μ;*

- α: $G \times M \to \mathrm{Diff}^1_+(\mathbb{T})$ *be a Borel cocycle; and*
- f: $M \to \mathbb{T}$ *be an* α*-equivariant measurable map.*

Then there is a G-invariant probability measure ν *on* $M \times_\alpha \mathbb{T}$*, such that* ν *is equivalent to* $\mu \times \lambda$.

Therefore, as a cocycle into $\mathrm{Homeo}^{\mathrm{Leb}}{}_+(\mathbb{T})$, α *is cohomologous to the trivial cocycle.*

Proof. Cutting $\mathbb{T}$ open at the point $f(m)$ yields an interval I_m, so we may define a cocycle $\hat{\alpha}$: $G \times M \to \mathrm{Diff}^1_+(I)$. Then Theorem 5.2 applies. ∎

THEOREM 5.4. *Let*

- *G be a connected, semisimple, real Lie group, such that*
  - G has Kazhdan's property (T), and
  - $\mathbb{R}$-rank $G \geqslant 2$;
- *M be an irreducible ergodic G-space with finite invariant measure* μ*; and*
- α: $G \times M \to \mathrm{Diff}^1_+(\mathbb{T})$ *be a Borel cocycle.*

Then there is a G-invariant probability measure ν *on* $M \times_\alpha \mathbb{T}$*, such that* ν *is equivalent to* $\mu \times \lambda$.

Therefore, as a cocycle into ${}_+(\mathbb{T})$, α *is cohomologous to a cocycle with values in the rotation group* $\mathrm{Rot}(\mathbb{T})$.

Proof. Because G has Kazhdan's property (T), we know that G has no factors locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ (see 2.15). Therefore, Theorem 1.10 implies that there is a G-invariant probability measure σ on $M \times_\alpha \mathbb{T}$, such that σ projects to μ on M.

Define a cocycle β: $G \times (M \times_\alpha \mathbb{T}) \to \mathrm{Diff}^1_+(\mathbb{T})$ by $\beta(g, m, s) = \alpha(g, m)$. The map f: $M \times_\alpha \mathbb{T} \to \mathbb{T}$ defined by $f(m, s) = s$ is β-equivariant, so we know, from Corollary 5.3, that there is a G-invariant probability measure $\hat{\nu}$ on $(M \times_\alpha \mathbb{T}) \times_\beta \mathbb{T}$, such that $\hat{\nu}$ is equivalent to $\sigma \times \lambda$.

Let ν be the image of $\hat{\nu}$ under the projection $(m, s, t) \mapsto (m, t)$. Because $\hat{\nu}$ is equivalent to $\sigma \times \lambda$, and σ projects to μ on M, we see that ν is equivalent to $\mu \times \lambda$, as desired. ∎

To motivate the proof of Theorem 5.2, let us sketch the analogous proof of Theorem 5.1, under the assumption that Γ has Kazhdan's property (T). (It is well known that, because Γ is discrete, Kazhdan's property (T) implies both that Γ is finitely generated [Zi, Thm. 7.1.5, p. 131] and that $\Gamma/[\Gamma, \Gamma]$ is finite [Zi, Cor. 7.1.7, p. 131].)

Proof of Theorem 5.1 when Γ *has Kazhdan's property* (T). It suffices to show that the set of fixed points of Γ is dense in I. Suppose not. Then, replacing I by the closure of a component of the complement of the fixed-point set, we may assume that there are no fixed points in the interior of I.

We have a unitary representation ρ of Γ on $L^2(I)$ given by

$$\big(\rho(\gamma)f\big)(t) = [\gamma'(t)]^{1/2} f(\gamma^{-1}t).$$

Let $\varepsilon = 1/2$, and let $C \subset \Gamma$ and $\delta > 0$ be as in Definition 2.14. Because $\Gamma/[\Gamma, \Gamma]$ is finite, the homomorphism $\Gamma \to \mathbb{R}^+$: $\gamma \mapsto \gamma'(0)$ must be trivial. Thus, $\gamma'(0) = 1$, for every $\gamma \in \Gamma$. Therefore, there is a nontrivial interval I' containing 0, such that $|\gamma'(t) - 1| < \delta^2/4$, for every $\gamma \in C$ and every $t \in I'$. Let χ be the characteristic function of I'. Then $\|\rho(\gamma)f - f\| < \delta\|f\|$, for every $\gamma \in C$, so we conclude from the choice of C and δ that there is some nonzero $\rho(\Gamma)$-invariant function ϕ in $L^2(I)$. Then $|\phi|^2 d\lambda$ is a Γ-invariant measure on I, so every point in the support of this measure is fixed by Γ. This contradicts the assumption that Γ has no fixed points in the interior of I.

Our proof of Theorem 5.2 is a fairly straightforward translation of this argument to the setting of cocycles for Borel actions, except that it is not convenient to use a topological argument in this setting. Therefore, instead of obtaining a contradiction by finding a fixed point that does not belong to the closure of the fixed-point set, we find a set of fixed points whose measure is greater than the measure of the set of fixed points.

Proof of Theorem 5.2. By passing to ergodic components, we may assume that G is ergodic on M.

The map $G \times M \to \mathbb{R}^+$ defined by $(g, m) \mapsto \alpha(g, m)'(0)$ is a cocycle. Because G has Kazhdan's property (T), and $\mathbb{R}^+$ is amenable and has no compact subgroups, the cocycle must be cohomologous to the trivial cocycle [Zi, Thm. 9.1.1, p. 162], so, by replacing α with an equivalent cocycle, we may assume, for each $g \in G$, that $\alpha(g, m)'(0) = 1$ for a.e. $m \in M$.

Because μ is G-invariant, we have

$$\int_{M \times_\alpha I} \psi \, d(\mu \times \lambda) = \int_{M \times_\alpha I} \psi\big(g(m, s)\big)\, \alpha(g, m)'(s)\, d(\mu \times \lambda)(m, s)$$

for any $g \in G$ and $\psi \in L^1(M \times_\alpha I)$. Therefore, a unitary representation ρ of G on $L^2(M \times_\alpha I)$ is given by

$$\big(\rho(g)\phi\big)(m, s) = \phi\big(g^{-1}(m, s)\big)\, \big(\alpha(g^{-1}, m)'(s)\big)^{1/2}$$

for $g \in G$, $\phi \in L^2(M \times_\alpha I)$, and $(m, s) \in M \times_\alpha I$.

Fix a compact subset C of G, as in Definition 2.14.

- Fix some $\varepsilon_1 \in (0, 1)$, and let $\delta_1 = \delta(\varepsilon_1) > 0$ be the corresponding δ-value given by Definition 2.14.
- Choose $\varepsilon_2 > 0$ small enough that $9\varepsilon_2 < \delta_1^2$, and let $\delta_2 = \delta(\varepsilon_2) > 0$ be the corresponding δ-value given by Definition 2.14.
- Choose $\varepsilon_3 > 0$ small enough that $13\varepsilon_3 < \delta_2^2$.

We may assume that $1 > \varepsilon_1 > \delta_1 > \varepsilon_2 > \delta_2 > \varepsilon_3 > 0$.

Lemma 2.16 tells us that there is a nontrivial interval I' containing 0, such that, for every $g \in C$, we have

$$\mu\big(\{m \in M \mid \forall s \in 2I', |\alpha(g^{-1}, m)'(s) - 1| < \varepsilon_3\}\big) > 1 - \varepsilon_3. \tag{5.5}$$

Let $\mathcal{F}$ be the space of all $\rho(G)$-invariant functions in $L^2(M \times_\alpha I)$, and choose $\phi \in \mathcal{F}$, such that $(\mu \times \lambda)\big(\phi^{-1}(0)\big)$ is minimal. The minimum exists, because any convex combination of (the absolute values of) countably many $\rho(G)$-invariant functions is $\rho(G)$-invariant. Furthermore,

$$\text{for every } \psi \in \mathcal{F}, \text{ we have } \psi = 0 \text{ a.e. on } \phi^{-1}(0). \tag{5.6}$$

Because ϕ is $\rho(G)$-invariant, we know that $\nu = |\phi|^2 \cdot (\mu \times \lambda)$ is a G-invariant measure on $M \times_\alpha I$. (*A priori*, ϕ could be identically 0, so this measure could be trivial.) To complete the proof, we will show that this measure is equivalent to $\mu \times \lambda$; that is, we will show that $(\mu \times \lambda)\big(\phi^{-1}(0)\big) = 0$. (Then ν projects to μ on M. Indeed, because G is ergodic on (M, μ), we know that any G-invariant probability measure on $M \times_\alpha I$ that is equivalent to $\mu \times \lambda$ must project to μ on M.)

NOTATION 5.7. For each $m \in M$, let λ_m be the Lebesgue measure on the interval $m \times I$. Thus, $\lambda_m(E) = \lambda_m\big(E \cap (m \times I)\big)$ for every Borel subset E of I, and we have $\mu \times \lambda = \int_M \lambda_m d\mu(m)$.

Assume that $(\mu \times \lambda)\big(\phi^{-1}(0)\big) \neq 0$. (This will lead to a contradiction.) Because μ is ergodic and $\phi^{-1}(0)$ is G-invariant, we must have $\lambda_m\big(\phi^{-1}(0)\big) \neq 0$, for a.e. $m \in M$. By discarding a set of measure 0, we may assume $\lambda_m\big(\phi^{-1}(0)\big) \neq 0$ for every $m \in M$. Then we may define $f\colon M \to I$ by

$$f(m) = \max\left\{ t \in I \middle| \lambda_m\Big(\phi^{-1}(0) \cap \big(m \times [0, t]\big)\Big) = 0 \right\}.$$

Replacing $M \times_\alpha I$ with the invariant subset

$$\big\{ (m, s) \in M \times_\alpha I \mid f(m) \leqslant s \leqslant 1 \big\},$$

we may assume that

$$f \text{ is identically } 0. \tag{5.8}$$

Step 1. There is some $\delta_0 > 0$, such that

$$\mu\left\{ m \in M \middle| \int_{m \times I'} |\phi|^2 \, d\lambda_m < \delta_0 \right\} < \varepsilon_2.$$

(To avoid confusion, we emphasize that the integral is over $m \times I'$, not the entire interval $m \times I$.) Let χ be the characteristic function of $M \times I'$. We will show, for every $g \in C$, that $\|\rho(g)\chi - \chi\| < \delta_2 \|\chi\|$ (see Claim 1.1 below). From the definition

of δ_2, this implies that there is some $\psi \bullet \mathcal{F}$, such that $\|\psi\| = \|\chi\|$ and $\|\chi - \psi\| \leqslant \varepsilon_2 \|\chi\|$. Therefore,

$$\mu\{m \in M \mid \psi = 0 \text{ a.e. on } m \times I'\} \leqslant \varepsilon_2^2 < \varepsilon_2.$$

From (5.5), we conclude that the same inequality is true with ϕ in the place of ψ. In other words, we have

$$\mu\left\{ m \in M \middle| \int_{m \times I'} |\phi|^2 \, d\lambda_m = 0 \right\} < \varepsilon_2.$$

Thus, the desired conclusion is obtained by taking δ_0 sufficiently small.

Claim 1.1. *For each* $g \bullet C$, *we have* $\|\rho(g)\chi - \chi\| < \delta_2 \|\chi\|$. Let

$$E = \{m \in M \mid \forall s \in 2I', |\alpha(g^{-1}, m)'(s) - 1| < \varepsilon_3\}.$$

For $m \in E$, we have:

$$\left|(\rho(g)\chi)(m, s) - \chi(m, s)\right| \leqslant \begin{cases} \varepsilon_3 & \text{if } (1 + \varepsilon_3)s \in I' \\ 2 & \text{if } (1 + \varepsilon_3)s \notin I' \text{ and } s/(1 + \varepsilon_3) \bullet I' \\ 0 & \text{if } s/(1 + \varepsilon_3) \notin I' \end{cases}$$

Therefore,

$$\begin{aligned} &\int_{E \times I} \left|(\rho(g)\chi) - \chi\right|^2 d(\mu \times \lambda) \\ &\quad \leqslant \varepsilon_3^2 \lambda(I') + 2^2 \left((1 + \varepsilon_3) - \frac{1}{1 + \varepsilon_3}\right) \lambda(I') + 0 \leqslant 9\varepsilon_3 \lambda(I'). \end{aligned} \tag{5.9}$$

If F is any subset of M with $\mu(F) < \varepsilon_3$, then $\int_F \times I \chi^2 d(\mu \times \lambda) < \varepsilon_3 \lambda(I')$, because $|\chi| \leqslant 1$. Then we must also have $\int_F \times I \left(\rho(g)\chi\right)^2 d(\mu \times \lambda) < \varepsilon_3 \lambda(I')$, because ρ is unitary, $g^{-1}(F \times I) = (g^{-1}F) \times I$, and μ is G-invariant. In particular, letting $F = M \setminus E$ and using the triangle inequality, we obtain

$$\int_{(M \setminus E) \times I} \left|(\rho(g)\chi) - \chi\right|^2 d(\mu \times \lambda) < 4\varepsilon_3 \lambda(I'). \tag{5.10}$$

Combining (5.9) and (5.9) yields

$$\|\rho(g)\chi - \chi\|^2 \leqslant 9\varepsilon_3 \lambda(I') + 4\varepsilon_3 \lambda(I') = 13\varepsilon_3 \|\chi\|^2 < \left(\delta_2 \|\chi\|\right)^2.$$

This completes the proof of the claim.

Step 2. We obtain a contradiction. Let χ' be the characteristic function of the G-invariant set

$$X = \phi^{-1}(0) \cap \left\{ (m, s) \in M \times_\alpha I \;\middle|\; \int_0^s |\phi(m, t)|^2 \, d\lambda_m(t) < \delta_0 \right\}.$$

We have $\lambda_m(X) \neq 0$ for a.e. $m \in M$ (see 5.7), so we may define a unit vector $\omega \in L^2(M \times_\alpha I)$ by

$$\omega(m, s) = \frac{\chi'(m, s)}{\lambda_m(X)^{1/2}}.$$

We will show, for every $g \in C$, that $\|\rho(g)\omega - \omega\| < \delta_1 \|\omega\|$ (see Claim 2.2 below). Then the definition of δ_1 implies that ω is not orthogonal to $\mathcal{F}$. Thus, there is some $\psi \in \mathcal{F}$, such that ψ is not essentially 0 on X. From (5.5), we conclude that ϕ is not essentially 0 on X. Because $X \subset \phi^{-1}(0)$, this is a contradiction.

Claim. 2.2. *For every $g \in C$, we have $\|\rho(g)\omega - \omega\| < \delta_1 \|\omega\|$. Let*

$$E = \left\{ m \in M \;\middle|\; \forall s \in 2I', \, |\alpha(g^{-1}, m)'(s) - 1| < \varepsilon_3 \right\} \bigcap \left\{ m \in M \;\middle|\; \int_{m \times I'} |\phi|^2 \, d\lambda_m \geqslant \delta_0 \right\}.$$

By comparing the rightmost terms in the definitions of X and E, we see that $X \cap (E \times I) \subset E \times I'$. Thus, from the left term in the definition of E, we see that $|\alpha(g^{-1}, m)'(s) - 1| < \varepsilon_3$ for every $(m, s) \in X \cap (E \times I)$. Therefore, for $m \in E$, we have:

$$\left| \big(\rho(g)\omega\big)(m, s) - \omega(m, s) \right| \leqslant \begin{cases} \varepsilon_3 / \lambda_m(X)^{1/2} & \text{if } (m, s) \in X \\ 0 & \text{if } (m, s) \notin X \end{cases}$$

Therefore,

$$\int_{E \times I} \left| \big(\rho(g)\omega\big) - \omega \right|^2 d(\mu \times \lambda) \leqslant \varepsilon_3^2 < \varepsilon_2. \tag{5.11}$$

If F is any subset of M with $\mu(F) < 2\varepsilon_2$, then $\int_{F \times I} \omega^2 d(\mu \times \lambda) < 2\varepsilon_2$, because $\int \omega^2 d\lambda_m = 1$ for every $m \in M$. Then we must also have $\int_{F \times I} \big(\rho(g)\omega\big)^2 d(\mu \times \lambda) < 2\varepsilon_2$, because ρ is unitary, $g^{-1}(F \times I) = (g^{-1}F) \times I$, and μ is G-invariant. In particular, letting $F = M \setminus E$ and using the triangle inequality, we obtain

$$\int_{(M \setminus E) \times I} \left| \big(\rho(g)\omega\big) - \omega \right|^2 d(\mu \times \lambda) < 8\varepsilon_2. \tag{5.12}$$

Combining (5.10) and (5.11) yields

$$\|\rho(g)\omega - \omega\|^2 \leqslant \varepsilon_2 + 8\varepsilon_2 < \delta_1^2 = \big(\delta_1 \|\omega\|\big)^2.$$

This completes the proof of the claim. It also completes the proof of Theorem 5.2.

Remark 5.13. For a smooth manifold X and a point $x \in X$, let $\mathrm{Diff}^1_{\mathrm{Id}}(X; x)$ be the group of C^1 diffeomorphisms h of X, such that $h(x) = x$ and $Dh(x) = \mathrm{Id}$. It would be interesting to know whether Theorem 5.2 generalizes to the cocycles α: $G \times M \to \mathrm{Diff}^1_{\mathrm{Id}}(X; x)$, for $\dim X > 1$. It would also be interesting to know whether additional smoothness on the cocycle α yields additional smoothness on the function that implements the cohomology of α to a trivial cocycle.

6. Other Versions of the Main Theorem

The assumption that M is irreducible and ergodic is stronger than is necessary in Theorems 1.10 and 5.4. Namely, we may allow G to be a product of higher-rank normal subgroups whose ergodic components are irreducible (see 6.1). In particular, if no simple factor of G has real rank one, then there is no need for any ergodicity or irreducibility assumption on M (see 6.2).

There are also analogous results in the more general situation where G is allowed to be a product of semisimple algebraic groups over local fields (see 6.5), or a lattice in such a group (see 6.3 and 6.9). Thus, Theorem 1.3 generalizes to the situation where Γ is an S-arithmetic group (see 6.10 and 6.11).

The results also generalize to the case where G has $\mathrm{PSL}(2, \mathbb{R})$ as a factor, but the conclusion must be weakened (see 6.13 and 6.14).

COROLLARY 6.1. *Let*

- *G be a connected, semisimple, real Lie group, such that there is no continuous homomorphism from G onto* $\mathrm{PSL}(2, \mathbb{R})$*;*
- *M be a Borel G-space with invariant probability measure μ;*
- *α: $G \times M \to \mathrm{Homeo}(\mathbb{T})$ be a strict Borel cocycle; and*
- *$G_0, G_1, \ldots, G_r$ be connected, closed, normal subgroups of G, such that*
  - *$G = G_0 G_1 \cdots G_r$,*
  - *G_0 is compact,*
  - *$\mathbb{R}\text{-rank}(G_i) \geqslant 2$ for $i > 0$,*
  - *$[G_i, G_j] = e$ for all i and j, and*
  - *for each $i > 0$, almost every ergodic component of the action of G_i on M is irreducible.*

Then there is a G-invariant probability measure ν on $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ.

Furthermore, if G has Kazhdan's property (T), and $\alpha(G \times M) \subset \mathrm{Diff}^1(\mathbb{T})$, then ν can be taken to be equivalent to $\mu \times \lambda$.

Proof. If $G = G_0$, then G is compact, so we obtain a G-invariant measure ν on $M \times_\alpha \mathbb{T}$ simply by averaging over G. Thus, we may assume $r > 0$. Let

- $G^* = G_0 G_1 \cdots G_{r-1}$;
- α^* be the restriction of α to $G^* \times M$;

- α_r be the restriction of α to $G_r \times M$; and
- $\mathcal{A} = \{\psi\colon M \to \mathrm{Prob}(\mathbb{T}) \cup \psi \text{ is } \alpha^*\text{-equivariant}\}$.

By induction on r, we may assume that there is a G^*-invariant probability measure ν^* on $M \times_\alpha^* \mathbb{T}$, such that the projection of ν^* to M is μ. Therefore, $\mathcal{A}$ is nonempty.

Let P be a minimal parabolic subgroup of G. The space of measurable functions from $G_r/(P \cap G_r)$ to $\mathcal{A}$ is an affine G_r-space over $G_r/(P \cap G_r)$, so, because $P \cap G_r$ is amenable, there is a β-equivariant Borel map $\Phi\colon G_r/(P \cap G_r) \to \mathcal{A}$. Then, defining $\Psi\colon \big(G_r/(P \cap G_r)\big) \times M \to \mathrm{Prob}(\mathbb{T})$ by $\Psi(x, m) = \Phi(x)(m)$, we see that Ψ is α_r-equivariant.

For each $m \in M$, define $\Psi_m\colon G/P \to \mathrm{Prob}(\mathbb{T})$ by $\Psi_m(x) = \Psi(x, m)$. By assumption, each ergodic component of the action of G_r on M is irreducible. Thus, Theorem 4.3 implies that Ψ_m is essentially constant, for a.e. $m \in M$. Thus, Ψ induces an essentially well-defined Borel map $\bar{\Psi}\colon M \to \mathrm{Prob}(\mathbb{T})$. By construction, $\bar{\Psi}$ is both α^*-equivariant and α_r-equivariant, so $\bar{\Psi}$ is α-equivariant. Therefore, $\nu = \int_M \big(m \times \bar{\Psi}(m)\big) d\mu(m)$ is a G-invariant measure on $M \times_\alpha \mathbb{T}$. By construction, it projects to μ on M. □

COROLLARY 6.2. *Let*

- *G be a connected, semisimple, real Lie group, such that G has no factors of real rank one;*
- *M be a Borel G-space with invariant probability measure μ; and*
- *$\alpha\colon G \times M \to \mathrm{Homeo}(\mathbb{T})$ be a strict Borel cocycle.*

Then there is a G-invariant probability measure ν on $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ.

Furthermore, if $\alpha(G \times M) \subset \mathrm{Diff}^1(\mathbb{T})$, then ν can be taken to be equivalent to $\mu \times \lambda$.

COROLLARY 6.3. *Let*

- *G be a connected, semisimple, real Lie group, such that*
  - *$\mathbb{R}\text{-rank}\, G \geqslant 2$, and*
  - *there is no continuous homomorphism from G onto $\mathrm{PSL}(2, \mathbb{R})$;*
- *Γ be an irreducible lattice in G;*
- *M be an irreducible ergodic Γ-space with finite invariant measure μ_M; and*
- *$\alpha\colon \Gamma \times M \to \mathrm{Homeo}(\mathbb{T})$ be a Borel cocycle.*

Then there is a Γ-invariant probability measure ν on $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ.

Furthermore, if Γ has Kazhdan's property (T), and $\alpha(\Gamma \times M) \subset \mathrm{Diff}^1(\mathbb{T})$, then ν can be taken to be equivalent to $\mu \times \lambda$.

Proof. (a standard argument). Let $\sigma\colon G/\Gamma \to G$ be a Borel section (i.e., $\sigma(g\Gamma) \in g\Gamma$ for every $g\Gamma \in G/\Gamma$), and define $\gamma\colon G \times G/\Gamma \to \Gamma$ by $\sigma(gx) = g\sigma(x)\gamma(g, x)^{-1}$ for $g \in G$

and $x \in G/\Gamma$. Then γ is a Borel cocycle for the action of G on G/Γ.

Let $\hat{M} = \mathrm{Ind}_{\Gamma}^{G}(M) = (G/\Gamma) \times_{\gamma} M$ be the G-space induced from the Γ-space M. Then $\hat{M}$ is an irreducible, ergodic G-space. Define a Borel cocycle $\hat{\alpha}$: $G \times \hat{M} \to \mathrm{Homeo}(\mathbb{T})$ by $\hat{\alpha}\big(g, (x, m)\big) = \alpha\big(\gamma(g, x), m\big)$.

From Theorem 1.10, we know that there is a G-invariant probability measure $\hat{\nu}$ on $\hat{M} \times_{\hat{\alpha}} \mathbb{T}$, such that $\hat{\nu}$ projects to $\rho \times \mu$ on $(G/\Gamma) \times M$, where ρ is the G-invariant probability measure on G/Γ. We may write $\hat{\nu} = \int_G / \Gamma \hat{\nu}_x d\rho(x)$, where $\hat{\nu}_x \in \mathrm{Prob}(x \times M \times \mathbb{T})$.

Fix a.e. $g \in G$. The measure $\hat{\nu}_g \Gamma$ is $g\Gamma g^{-1}$-invariant, and projects to μ on M. Define ν by $g_*(e\Gamma \times \nu) = \nu_g \Gamma$.

Remark 6.4. For G as in Theorem 6.5, the definitions of irreducible lattice and irreducible action given in Section 1 (see 1.1 and 1.7) must be modified to refer to 'non-discrete' normal subgroups instead of 'connected' normal subgroups.

THEOREM 6.5. *Let*

- *S be a finite set of local fields (not necessarily of characteristic zero);*
- *$\mathbf{G}_F$ be a connected, semisimple algebraic group over F, for each $F \in S$;*
- *G be a closed, cocompact, normal subgroup of $\prod_{F \in S} \mathbf{G}_F(F)$;*
- *M be an irreducible ergodic G-space with finite invariant measure μ; and*
- *α: $G \times M \to \mathrm{Homeo}(\mathbb{T})$ be a strict Borel cocycle.*

Assume

(a) $\sum_{F \in S} F\text{-rank}(\mathbf{G}_F) \geqslant 2$; and
(b) the identity component G° has no continuous homomorphism onto $\mathrm{PSL}(2, \mathbb{R})$.

Then there is a G-invariant probability measure ν on $M \times_{\alpha} \mathbb{T}$, such that the projection of ν to M is μ.

Furthermore, if G has Kazhdan's property (T), and $\alpha(G \times M) \subset \mathrm{Diff}^1(\mathbb{T})$, then ν can be taken to be equivalent to $\mu \times \lambda$.

Remarks on the proof. Most of the arguments of Sections 4 and 5 apply with only minor changes.

As a replacement for the Moore Ergodicity Theorem 2.11, we note that the proof of [Ma, Thm. 7.2, p. 105] yields a version of this result that applies to the general groups G under consideration, in the special case where H contains a nontrivial split torus of G. This suffices for our purposes.

When F is nonarchimedean, we use Lemma 6.8 below in place of Theorem 4.5. (The proof of this lemma relies on an argument of É. Ghys [Gh, pp. 219–220] that was not needed in Section 4 or Section 5.) The existence of a subgroup L satisfying the hypotheses of this lemma follows from Lemma 6.6 below. (Because $\mathrm{SL}(2, F)$ has no infinite, proper, normal subgroups [Ma, Cor. 2.3.2, p. 53], we know that $\zeta\big(\mathrm{SL}(2, F)\big) \subset G$.)

LEMMA 6.6 ([Ti, Prop. 3.1(13)]). *Let*

- $\mathbf{G}$ *be a semisimple algebraic group over a field* F;
- $\mathbf{A}$ *be a maximal* F*-split torus of* $\mathbf{G}$; *and*
- α *be an* F*-root of* $\mathbf{G}$ *(with respect to* $\mathbf{A}$*), such that* 2α *is not an* F*-root.*

Then there is a nontrivial F*-homomorphism* $\zeta\colon \mathrm{SL}(2,\cdot) \to \mathbf{G}$, *such that*

$$\zeta\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathbf{U}_\alpha(F), \quad \zeta\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \mathbf{U}_{-\alpha}(F), \quad \zeta\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathbf{A}(F),$$

for all $x \in F$ *and* $a \in F \setminus \{0\}$.

We will use the following elementary observation in the proof of Lemma 6.8.

LEMMA 6.7. *Let*

- F be a local field;
- $L = \mathrm{SL}(2, F)$;
- P be a proper parabolic subgroup of L; and
- a, b, c be three distinct elements of L/P.

If $F \neq \mathbb{R}$, then there exist $y_0, \dots, y_n \bullet T$, such that $y_0 = b$, $y_n = a$, and $(y_i - 1, y_i, c) \in L(a, b, c)$ for $i = 1, \dots, n$.

Proof. It is well known that there is an identification of L/P with $F \cup \{\infty\}$, so that we have the standard action of L on $F \cup \{\infty\}$ by linear-fractional transformations. Then, because $\mathrm{GL}(2, F)$ is triply transitive on $F \cup \{\infty\}$ and normalizes L, we may assume $a = 0$, $b = 1$, and $c = \infty$. Because $F \neq \mathbb{R}$, we may choose $t_0, \dots, t_n \bullet F$ such that $t_0 = 1$ and $t_0^2 + \cdots + t_n^2 = 0$. Let $y_i = t_0^2 + \cdots + t_i^2$. Then

$$(y_{i-1}, y_i, c) = \begin{pmatrix} t_{i-1} & y_{i-1}/t_{i-1} \\ 0 & 1/t_{i-1} \end{pmatrix}(0, 1, \infty) \in L(a, b, c),$$

as desired. □

LEMMA 6.8. *Let* G, M, *and* α *be as in Theorem 0.3, and let* $P = G \cap \prod_F \in \mathcal{S}\mathbf{P}_F(F)$, *where* $\mathbf{P}_F$ *is a minimal parabolic subgroup of* $\mathbf{G}_F$, *for each* $F \in \mathcal{S}$.

Suppose $\Psi\colon (G/P) \times M \to \mathbb{T}_k$ *is an* α*-equivariant Borel map. For each* $m \in M$, *define* $\Psi'_m\colon G \to \mathbb{T}_k$ *by* $\Psi'_m(g) = \Psi(gP, m)$.

Suppose $F \neq \mathbb{R}$, *and* L *is a closed subgroup of* G, *such that*

- $C_P(L)$ *contains a nontrivial split torus of* G;
- L *is the image of an* F*-morphism* $\zeta\colon \mathrm{SL}(2,\cdot) \to \mathbf{G}_F$ *with finite kernel; and*
- $\zeta^{-1}(L \cap P)$ *is a parabolic subgroup of* $\mathrm{SL}(2, F)$.

Then Ψ'_m *is essentially right* L*-invariant, for a.e.* $m \in M$.

Proof. Define X as in the proof of Theorem 4.5, and fix some $(a, b, c) \in X$. It follows from Lemma 6.7 that there exist $y_0, \ldots, y_n \in G/P$, such that $y_0 = b$, $y_n = a$, and $(y_i - 1, y_i, c) \in G(a, b, c)$ for $i = 1, \ldots, n$.

Case 1. Assume that $k = 1$, and that $\Psi'_m(x_1)$, $\Psi'_m(x_2)$, and $\Psi'_m(x_3)$ are distinct, for almost all $m \in M$ and $(x_1, x_2, x_3) \in G(a, b, c)$. From the Moore Ergodicity Theorem, we know that G is ergodic on $G(a, b, c) \times M$. Thus, for almost every $m \in M$ and $g \in G$, we conclude that

$$\left(\Psi'_m(gy_{i-1}), \Psi'_m(gy_i), \Psi'_m(gc)\right) \quad \text{and} \quad \left(\Psi'_m(ga), \Psi'_m(gb), \Psi'_m(gc)\right)$$

have the same orientation, for $i = 1, \ldots, n$. Thus, by induction, we see that

$$\left(\Psi'_m(gy_0), \Psi'_m(gy_i), \Psi'_m(gc)\right) \quad \text{and} \quad \left(\Psi'_m(ga), \Psi'_m(gb), \Psi'_m(gc)\right)$$

have the same orientation. In particular, by letting $i = n$, we see that

$$\left(\Psi'_m(gb), \Psi'_m(ga), \Psi'_m(gc)\right) \quad \text{and} \quad \left(\Psi'_m(ga), \Psi'_m(gb), \Psi'_m(gc)\right)$$

have the same orientation. This is a contradiction.

Case 2. The general case. If Ψ'_m is not essentially right L-invariant, then the argument of [Gh, pp. 219–220] shows that, after replacing Ψ with a different α-equivariant Borel function, we may assume, for almost every $m \in M$ and $(x_1, x_2, x_3) \in G(a, b, c)$, that $\Psi'_m(x_1)$ is disjoint from $\Psi'_m(x_2)$, and the sets $\Psi'_m(x_1)$ and $\Psi'_m(x_2)$ alternate around the circle.

Now, if three pairwise disjoint k-element subsets B_1, B_2, B_3 of $\mathbb{T}$ are pairwise alternating around the circle, we say that (B_1, B_2, B_3) is *positively oriented* if there is a positively oriented arc of the circle from a point of B_1 to a point of B_2 that does not contain any point of B_3 [Gh, bot. of p. 220]. This relation has the properties of a circular order, so we obtain a contradiction by applying the same argument as in Case 1. □

The proof of Corollary 6.3 yields the following as a corollary of Theorem 6.5.

COROLLARY 6.9. *Let*

- *G be as in Theorem* 6.5 (*including assumptions* (a) *and* (b));
- *Γ be an irreducible lattice in G;*
- *M be an irreducible ergodic Γ-space with finite invariant measure μ; and*
- *α: $\Gamma \times M \to \mathrm{Homeo}(\mathbb{T})$ be a strict Borel cocycle.*

Then there is a Γ-invariant probability measure ν on $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ.

Furthermore, if Γ has Kazhdan's property (T), and $\alpha(\Gamma \times M) \subset \mathrm{Diff}^1(\mathbb{T})$, then ν can be taken to be equivalent to $\mu \times \lambda$.

The following generalization of Theorems 1.3 and 1.9 is the special case of Corollary 6.9 in which M is a single point. This result is essentially due to M. Burger and N. Monod [BM1, BM2, BM3], but a few isolated cases are not covered by their theorems. (On the other hand, some of their results apply in a more general setting where G need not be a product of algebraic groups, or Lie groups.) In the final conclusion of this corollary, we do not assume Γ has Kazhdan's property (T), because Thurston's Theorem 5.1 can be applied if Γ is finitely generated. Furthermore, this restriction to finitely generated lattices may be superfluous: we do not know an example of an irreducible lattice in such a group that is not finitely generated.

COROLLARY 6.10. *Let*

- G *be as in Theorem* 6.5 (*including assumptions* (a) *and* (b)); *and*
- Γ *be an irreducible lattice in* G.

Then every continuous action of Γ *on* $\mathbb{T}$ *has a finite orbit.*

Furthermore, if Γ *is finitely generated, then every homomorphism from* Γ *to* $\mathrm{Diff}^1(\mathbb{T})$ *has finite image.*

Ghys' Theorems 1.3 and 1.9 are essentially the special case of the following corollary in which E is a number field and S consists only of the infinite places. (More generally, if E is a number field and $\sum_{s\in S_\infty} E_s\text{-rank}(\mathbf{G}) \geqslant 2$, then Conclusion (b) of this corollary is a consequence of Ghys' Theorem. Namely, Theorem 1.3 applies to the subgroup $\mathbf{G}(\mathcal{O})$ of Γ, and then the Margulis Finiteness Theorem [Ma, Thm. IV.4.10, p. 167] implies that the image of Γ is finite.) Note that Assumption (a) implies Γ is finitely generated [Ma, Thm. III.5.7(c), p. 131], [Be, Thm. 1a].

COROLLARY 6.11. *Let*

- E *be a global field;*
- S *be a nonempty, finite set of places of* E, *including all of the infinite places;*
- $\mathbf{G}$ *be a connected, almost simple algebraic group over* E;
- $\mathcal{O}(S)$ *be the ring of* S*-integers in* E; *and*
- Γ *be a finite-index subgroup of* $\mathbf{G}\big(\mathcal{O}(S)\big)$.

Assume

(a) $\sum_{s\in S} E_s\text{-rank}(\mathbf{G}) \geqslant 2$; *and*
(b) *for each Archimedean* $s \in S$, *there is no continuous homomorphism from* $\mathbf{G}(E_s)^\circ$ *onto* $\mathrm{PSL}(2, \mathbb{R})$.

Then

(a) *every continuous action of* Γ *on* $\mathbb{T}$ *has a finite orbit; and*
(b) *every homomorphism from* Γ *to* $\mathrm{Diff}^1(\mathbb{T})$ *has finite image.*

The following theorem is the main result of [Gh, Section 7], although it was not stated explicitly.

THEOREM 6.12. (Ghys [Gh, Section 7]). *Let*

- *G be a connected Lie group that is locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})^n$, *for some* $n > 0$;
- *Γ be a countable group;*
- *$\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{T})$ and $\iota\colon \Gamma \to G$ be homomorphisms;*
- *P be a parabolic subgroup of G; and*
- *$\Psi\colon G/P \to \mathbb{T}_k$ be a Γ-equivariant Borel map, for some $k \geqslant 1$.*

If $\iota(\Gamma)$ is ergodic on G/H, for every closed, noncompact subgroup H of G, then either $\phi(\Gamma)$ has a finite orbit, or there is a semiconjugacy as described in Corollary 6.13(2) *below.*

Although Theorem 6.12 assumes that G is connected, an examination of the proof shows that if G is a real algebraic group, then it holds under the weaker assumption that G is *Zariski* connected. This yields the following generalization of Corollary 6.10 that allows $\mathrm{PSL}(2, \mathbb{R})$ as a factor of G. This generalization was proved by É. Ghys [Gh, Thm. 1.2] for $\mathcal{S} \subset \{\mathbb{R}, \mathbb{C}\}$. To justify the stronger conclusion when $\phi(\Gamma) \subset \mathrm{Diff}^2(\mathbb{T})$, see [Gh, Prop. 10.2].

COROLLARY 6.13. *Let*

- *G be as in Theorem* 6.5, *except that we do not assume* (b) (*although we do assume* (a));
- *Γ be an irreducible lattice in G; and*
- *$\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{T})$ be a homomorphism.*

Then either

(1) *$\phi(\Gamma)$ has a finite orbit; or*
(2) *the restriction of ϕ to Γ is semiconjugate to a finite cover of the composition of the following:*
   (a) *the inclusion of Γ into G;*
   (b) *a continuous surjection $G \to \mathrm{PSL}(2, \mathbb{R})$; and*
   (c) *the standard action of* $\mathrm{PSL}(2, \mathbb{R})$ *on $\mathbb{T}$ by linear-fractional transformations.*

Furthermore, if $\phi(\Gamma) \subset \mathrm{Diff}^2(\mathbb{T})$, then any semiconjugacy as in (2) *above is actually a topological conjugacy.*

For completeness, we state the following generalization of Theorem 6.5. Its proof is completed by translating [Gh, Section 7] in a straightforward way from the setting of homomorphisms of lattices to the setting of Borel cocycles for ergodic G-actions.

COROLLARY 6.14. *Let G, M, μ, and α be as in Theorem* 6.5, *except that we do not assume* (b) (*although we do assume* (a)). *Assume $\alpha(g, m)$ is orientation preserving, for all $g \in G$ and $m \in M$* (*cf.* 4.2).

Then there is a probability measure ν on $M \times_\alpha \mathbb{T}$, such that the projection of ν to M is μ, and either

(1) *ν is G-invariant; or*
(2) *there exist*

(a) *a continuous surjection $\tau\colon G \to \mathrm{PSL}(2, \mathbb{R})$; and*
(b) *a G-equivariant, measure-preserving function*

$$f\colon (M \times_\alpha \mathbb{T}, \nu) \to (M \times_\tau \mathbb{T}, \mu \times \mathrm{Leb});$$

such that f is of the form $f(m, t) = \big(m, f_m(t)\big)$, where, for a.e. $m \in M$,

- *$f_m\colon \mathbb{T} \to \mathbb{T}$ is continuous, and*
- *any continuous lift $\tilde{f}_m\colon \mathbb{R} \to \mathbb{R}$ is increasing*

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