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Tessellations of Homogeneous Spaces of Classical Groups of Real Rank Two*

ALESSANDRA IOZZI[1] and DAVE WITTE MORRIS[2]
[1]*Department of Mathematics, University of Maryland, College Park, MD 20910, U.S.A. Current address: FIM, ETH Zentrum, CH-8092 Zürich, Switzerland. e-mail: iozzi@math.ethz.ch*
[2]*Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, U.S.A. Current address: Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, AB, T1K 3M4, Canada. e-mail: dwitte@math.okstate.edu*

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Abstract. Let H be a closed, connected subgroup of a connected, simple Lie group G with finite center. The homogeneous space G/H has a *tessellation* if there is a discrete subgroup Γ of G, such that Γ acts properly discontinuously on G/H, and the double-coset space $\Gamma\backslash G/H$ is compact. Note that if either H or G/H is compact, then G/H has a tessellation; these are the obvious examples.

It is not difficult to see that if G has real rank one, then only the obvious homogeneous spaces have tessellations. Thus, the first interesting case is when G has real rank two. In particular, Kulkarni and Kobayashi constructed examples that are not obvious when $G = \mathrm{SO}(2,2n)^\circ$ or $\mathrm{SU}(2,2n)$. Oh and Witte constructed additional examples in both of these cases, and obtained a complete classification when $G = \mathrm{SO}(2,2n)^\circ$. We simplify the work of Oh-Witte, and extend it to obtain a complete classification when $G = \mathrm{SU}(2,2n)$. This includes the construction of another family of examples.

The main results are obtained from methods of Benoist and Kobayashi: we fix a Cartan decomposition $G = KA^+K$, and study the intersection $(KHK) \cap A^+$. Our exposition generally assumes only the standard theory of connected Lie groups, although basic properties of real algebraic groups are sometimes also employed; the specialized techniques that we use are developed from a fairly elementary level.

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1. Introduction

DEFINITION 1.1 ([KN, pp. 43–44]). A group Γ of homeomorphisms of a topological space M acts *properly discontinuously* on M if, for every compact subset C of M,

$$\{\gamma \in \Gamma \mid C \cap \gamma C \neq \emptyset\} \text{ is finite.}$$

Classically, a discrete group Γ of isometries of a Riemannian manifold M is a crystallographic group if Γ acts properly discontinuously on M, and the quotient

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$\Gamma\backslash M$ is compact. The Γ-translates of any fundamental domain for $\Gamma\backslash M$ form a tessellation of M.

These notions generalize to any homogeneous space, even without an invariant metric.

DEFINITION 1.2. Let

- G be a Lie group and
- H be a closed subgroup of G.

A discrete subgroup Γ of G is a *crystallographic group* for G/H if

(1) Γ acts properly discontinuously on G/H; and
(2) $\Gamma\backslash G/H$ is compact.

We say that G/H has a *tessellation* if there exists a crystallographic group Γ for G/H.

Crystallographic groups and the corresponding tessellations have been studied for many groups G. (A brief recent introduction to the subject is given in [Kb8].) The classical Bieberbach Theorems [Cha, Chap. 1] deal with the case where G is the group of isometries of Euclidean space $\mathbb{R}^n = G/H$. As another example, the Auslander Conjecture [Abe, AMS, FG, Mr1, Tom] asserts that if G is the group of all affine transformations of $\mathbb{R}^n$, then every crystallographic group has a solvable subgroup of finite index. In addition, the case where G is solvable has been discussed in [Wit].

In this paper, we focus on the case where G is a simple Lie group, such as $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}(m, n)$, or $\mathrm{SU}(m, n)$.

STANDING ASSUMPTIONS 1.3. *Throughout this paper:*

(1) *G is a linear, semisimple Lie group with only finitely many connected components; and*
(2) *H is a closed subgroup of G with only finitely many connected components.*

Remark 1.4. Because H/H° is finite (hence compact), it is easy to see that G/H has a tessellation if and only if G/H° has a tessellation. Also, if G/H has a tessellation, then G°/H° has a tessellation. Furthermore, the converse holds in many situations. (See Section 2D for a discussion of this issue.) Thus, there is usually no harm in assuming that both G and H are connected; we will feel free to do so whenever it is convenient. On the other hand, because $\mathrm{SO}(m, n)$ is usually not connected (it usually has two components [Hel, Lemma 10.2.4, p. 451]), it would be somewhat awkward to make this a blanket assumption.

EXAMPLE 1.5. There are two classical cases in which G/H is well known to have a tessellation.

(1) If G/H is compact, then we may let $\Gamma = e$ (or any finite subgroup of G).

(2) If H is compact, then we may let Γ be any cocompact lattice in G. (A. Borel [Br1] proved that every connected, simple Lie group has a cocompact lattice.)

Thus, the existence of a tessellation is an interesting question only when neither H nor G/H is compact. (In this case, any crystallographic group Γ must be infinite, and cannot be a lattice in G.)

Given G (satisfying 1.3(1)), we would like to find all the subgroups H (satisfying 1.3(2)), such that G/H has a tessellation. This seems to be a difficult problem in general. (See the surveys [Kb6] and [Lab] for a discussion of the many partial results that have been obtained, mainly under the additional assumption that H is reductive.) However, it can be solved in certain cases of low real rank. In particular, as we will now briefly explain, the problem is very easy if $\mathbb{R}\text{-rank}\, G = 0$ or 1. Most of this paper is devoted to solving the problem for certain cases where $\mathbb{R}\text{-rank}\, G = 2$.

If $\mathbb{R}\text{-rank}\, G = 0$ (that is, if G is compact), then G/H must be compact (and H must also be compact), so G/H has a tessellation, but this is not interesting. If $\mathbb{R}\text{-rank}\, G = 1$, then there are some interesting homogeneous spaces, but it turns out that none of them have tessellations.

EXAMPLE 1.6. $G = \mathrm{SL}(2, \mathbb{R})$ is transitive on $\mathbb{R}^2 - \{0\}$, so $\mathbb{R}^2 - \{0\}$ is a homogeneous space for G. It does not have a tessellation, for reasons that we now explain.

Let C be the unit circle, so C is a compact subset of $\mathbb{R}^2 - \{0\}$.

We claim that $C \cap gC \neq \emptyset$, for every $g \in G$ (cf. Figure 1.1). To see this, note that, because $\det g = 1$, the ellipse bounded by gC has the same area as the disk bounded by C, so gC cannot be contained in the interior of the disk bounded by C, and cannot contain C in its interior. Thus, gC must be partly inside C and partly outside, so gC must cross C, as claimed.

Let Γ be any discrete subgroup of G. The preceding paragraph implies that $C \cap \gamma C \neq \emptyset$, for every $\gamma \in \Gamma$. If Γ acts properly discontinuously on $\mathbb{R}^2 - \{0\}$, then, because C is compact, this implies that Γ is finite. So the quotient $\Gamma \backslash (\mathbb{R}^2 - \{0\})$ is not compact. Therefore Γ is not a crystallographic group. We have shown that no subgroup of G is a crystallographic group, so we conclude that $\mathbb{R}^2 - \{0\}$ does not have a tessellation.

This example illustrates the *Calabi–Markus Phenomenon*: if there is a compact subset C of G/H, such that $C \cap gC \neq \emptyset$, for every $g \in G$, then no infinite subgroup of G acts properly discontinuously on G/H (see 2.8). Thus, G/H does not have a essellation, unless G/H is compact (see 2.9).

We will see in Section 2 that the following proposition can be proved quite easily from basic properties of the Cartan projection.

PROPOSITION 2.16′ (cf. [Kb4, Lemma 3.2]). *If $\mathbb{R}\text{-rank}\, G = 1$, and H is not compact, then there is a compact subset C of G/H, such that $C \cap gC \neq \emptyset$, for every $g \in G$.*

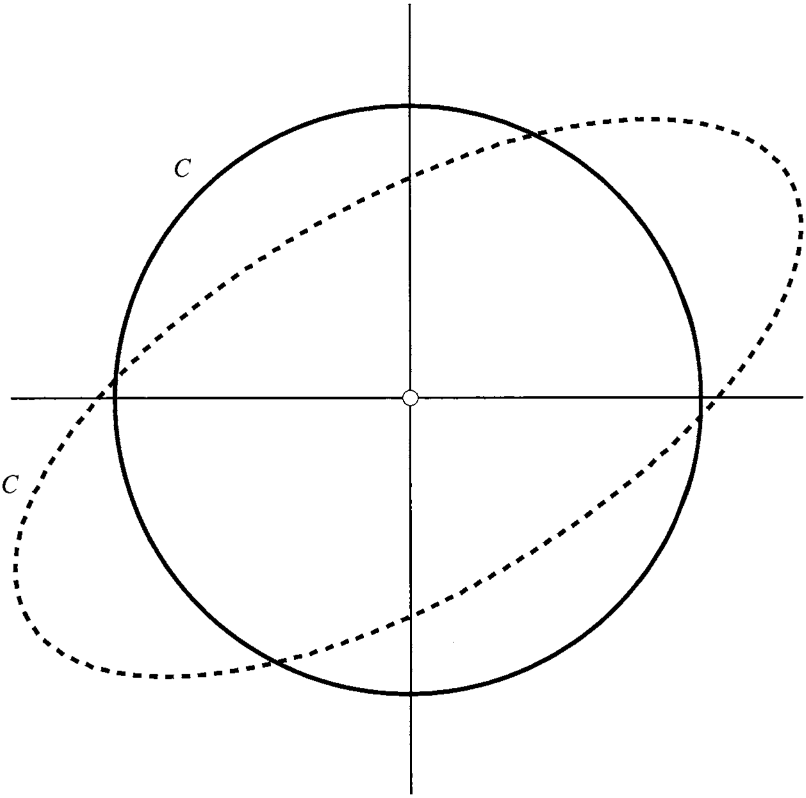


Figure 1.1. The Calabi–Markus Phenomenon (Example 1.6): $C \cap gC \neq \emptyset$, for every $g \in \mathrm{SL}(2, \mathbb{R})$, so no infinite subgroup of $\mathrm{SL}(2, \mathbb{R})$ acts properly discontinuously.

COROLLARY 1.7 (Kulkarni). *If* $\mathbb{R}$-rank $G = 1$, *and neither H nor G/H is compact, then G/H does not have a tessellation.*

We now consider groups of real rank two. The obvious example is $\mathrm{SL}(3, \mathbb{R})$, but, in this case, once again, none of the interesting homogeneous spaces have tessellations. Moreover, the same is true when real numbers are replaced by complex numbers or quaternions. The case where $\dim H = 1$ relies on beautiful methods of Benoist and Labourie [BL] or Margulis [Mr2], which we describe in Section 5.

THEOREM 1.8 (Benoist, Kobayashi, Margulis, Oh and Witte, see Section 6). *If*

$$G = \mathrm{SL}(3, \mathbb{R}), \mathrm{SL}(3, \mathbb{C}), \textit{ or } \mathrm{SL}(3, \mathbb{H}),$$

and neither H nor G/H is compact, then G/H does not have a tessellation.

It is important to note that some interesting homogeneous spaces do have tessellations. To apprehend this, it may be helpful to first look at the following rather trivial example.

EXAMPLE 1.9. Suppose $G = L \times H$, and let Γ be a cocompact lattice in L. Then Γ acts properly discontinuously on $L \cong G/H$, and $\Gamma \backslash G/H \cong \Gamma \backslash L$ is compact. So G/H has a tessellation.

The following easy lemma generalizes this example to the situation where G is a more general product of L and H, not necessarily a direct product.

LEMMA 1.10. *Let H and L be closed subgroups of G, such that*

- $G = LH$,
- *$L \cap H$ is compact; and*
- *L has a cocompact lattice Γ.*

Then G/H has a tessellation. (Namely, Γ is a crystallographic group for G/H.)

Proof. Because $G = LH$, we know that L is transitive on G/H (with stabilizer $L \cap H$), so G/H is L-equivariantly homeomorphic to $L/(L \cap H)$. Since Γ is a crystallographic group for $L/(L \cap H)$ (see 1.5(2)), this implies that Γ is a crystallographic group for G/H, as desired. ∎

For $G = \mathrm{SO}(2, n)$ or $\mathrm{SU}(2, n)$, this lemma leads to some interesting examples found by Kulkarni [Kul, Theorem 6.1] and Kobayashi [Kbl, Proposition 4.9].

EXAMPLE 1.11 (Kulkarni, Kobayashi). There are natural embeddings

$$\mathrm{SO}(1, n) \hookrightarrow \mathrm{SO}(2, n) \qquad \text{and} \qquad \mathrm{SU}(1, n) \hookrightarrow \mathrm{SU}(2, n).$$

Furthermore, identifying $\mathbb{C}^{1+m}$ with $\mathbb{R}^{2+2m}$ yields an embedding

$$\mathrm{SU}(1, m) \hookrightarrow \mathrm{SO}(2, 2m).$$

Similarly, identifying $\mathbb{H}^{1+m}$ with $\mathbb{C}^{2+2m}$ yields an embedding

$$\mathrm{Sp}(1, m) \hookrightarrow \mathrm{SU}(2, 2m).$$

Thus, we may think of $\mathrm{SO}(1, 2m)$ and $\mathrm{SU}(1, m)$ as subgroups of $\mathrm{SO}(2, 2m)$; and we may think of $\mathrm{SU}(1, 2m)$ and $\mathrm{Sp}(1, m)$ as subgroups of $\mathrm{SU}(2, 2m)$.

With the above understanding, we see that $\mathrm{SO}(1, 2m)$ is the stabilizer of a vector of norm $+1$. Since $\mathrm{SU}(1, m)$ is transitive on the set of such vectors, we have

$$\mathrm{SO}(2, 2m) = \mathrm{SO}(1, 2m)\, \mathrm{SU}(1, m).$$

Similarly,

$$\mathrm{SU}(2, 2m) = \mathrm{SU}(1, 2m)\, \mathrm{Sp}(1, m).$$

Then Lemma 1.10 implies that each of the following four homogeneous spaces has a tessellation:

- $\mathrm{SO}(2, 2m)/\mathrm{SO}(1, 2m)$,
- $\mathrm{SO}(2, 2m)/\mathrm{SU}(1, m)$,
- $\mathrm{SU}(2, 2m)/\mathrm{SU}(1, 2m)$, and
- $\mathrm{SU}(2, 2m)/\mathrm{Sp}(1, m)$.

Remark 1.12. When discussing $\mathrm{SO}(2,n)$ or $\mathrm{SU}(2,n)$, we always assume $n > 2$. This causes no harm, because $\mathrm{SO}(2,2)$ is locally isomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ [Hel, (x), p. 520], and $\mathrm{SU}(2,2)$ is locally isomorphic to $\mathrm{SO}(2,4)$ [Hel, (vi), p. 519].

When n is even, H. Oh and D. Witte [OW3] provided a complete description of all the (closed, connected) subgroups H, such that $\mathrm{SO}(2,n)/H$ has a tessellation, but some cases remain open when n is odd.

In this paper, we extend the work of Oh and Witte to obtain analogous results for homogeneous spaces of $G = \mathrm{SU}(2,n)$. We also give a much shorter proof of the main results of [OW3]. Our method is the same as in [OW3], but the proofs in [OW3] rely on a list [OW1] of all the homogeneous spaces of $\mathrm{SO}(2,n)$ that admit a proper action of a noncompact subgroup of $\mathrm{SO}(2,n)$. (The list was obtained by very tedious case-by-case analysis. It was extended to homogeneous spaces of $\mathrm{SU}(2,n)$ in [IW].) The following proposition (1.13) provides an *a priori* lower bound on $\dim H$, and it turns out that the classification of the interesting subgroups of large dimension can be achieved fairly easily (see Section 10). This is the main reason that we are able to give reasonably short complete proofs of our theorems. The Cartan projection (described in Section 2) is our main tool.

PROPOSITION 1.13 (see 4.12, 8.21, and 7.15). *Suppose $G = \mathrm{SO}(2,n)$ or $\mathrm{SU}(2,n)$, and let H be a closed, connected, noncompact subgroup of G. If G/H has a tessellation, then*

$$\dim H \geqslant \begin{cases} n & \text{if } G = \mathrm{SO}(2,n) \text{ and } n \text{ is even;} \\ n-1 & \text{if } G = \mathrm{SO}(2,n) \text{ and } n \text{ is odd;} \\ 2n & \text{if } G = \mathrm{SU}(2,n) \text{ and } n \text{ is even;} \\ 2n-2 & \text{if } G = \mathrm{SU}(2,n) \text{ and } n \text{ is odd.} \end{cases}$$

The same techniques should yield significant results for homogeneous spaces of the other classical simple groups of real rank two (namely, $\mathrm{Sp}(2,n)$, $\mathrm{SO}(5,\mathbb{C})$, $\mathrm{SO}(5,\mathbb{H})$), two real forms of the exceptional group E_6, and two forms of the exceptional group G_2), although the calculations seem to be difficult. On the other hand, the groups of higher real rank require different ideas.

Once one knows that a tessellation of G/H exists, it would be interesting to find *all* of the crystallographic groups for G/H and, for each crystallographic group, describe the possible tessellations. These are much more delicate questions, which we do not address at all. (Goldman [Gol], Salein [Sal], Kobayashi [Kb7], and Zeghib [Zeg] have interesting results in some special cases.)

In the remainder of this introduction, we state the specific results for homogeneous spaces of $\mathrm{SO}(2,n)$ and $\mathrm{SU}(2,n)$.

NOTATION 1.14 ([Iwa, p. 533]). For any connected Lie group H, let

$$d(H) = \dim H - \dim K_H,$$

where K_H is any maximal compact subgroup of H. This is well defined, because all the maximal compact subgroups of H are conjugate [Hc2, Theorem 15.3.1(iii), pp. 180–181].

EXAMPLE 1.15. If H is semisimple, we have the Iwasawa decomposition $H = K_H A_H N_H$ [Hel, Theorem 6.5.1, pp. 270–271], from which it is obvious that $d(H) = \dim(A_H N_H)$.

This yields the following calculations (see 7.5 and 7.15):

- $d(\mathrm{SO}(1,n)) = n$.
- $d(\mathrm{SO}(2,n)) = 2n$.
- $d(\mathrm{SU}(1,n)) = 2n$.
- $d(\mathrm{SU}(2,n)) = 4n$.
- $d(\mathrm{Sp}(1,n)) = 4n$.

Remark 1.16. If $H \subset AN$ (for some Iwasawa decomposition $G = KAN$ of G), then $d(H) = \dim H$ (see 3.18 and 3.15(3)).

NOTATION 1.17 ([Kb5, Definition 2.1.1]). For subgroups H_1 and H_2 of G, we write $H_1 \sim H_2$ if there is a compact subset C of G, such that $H_1 \subset CH_2C$ and $H_2 \subset CH_1C$.

Remark 1.18. Note that d is not invariant under the equivalence relation $\sim$. For example, the Cartan decomposition $G = KAK$ implies that $G \sim A$, but we have $d(A) = \dim A \neq \dim(AN) = d(G)$.

The following two theorems state a version of the main results for even n.

THEOREM 11.5′ ([OW3]). *Assume $G = \mathrm{SO}(2, 2m)$, and let H be a closed, connected, subgroup of G, such that neither H nor G/H is compact.*

The homogeneous space G/H has a tessellation if and only if

(1) *$d(H) = 2m$; and*
(2) *either $H \sim \mathrm{SO}(1, 2m)$ or $H \sim \mathrm{SU}(1, m)$.*

THEOREM 11.5″. *Assume $G = \mathrm{SU}(2, 2m)$, and let H be a closed, connected, subgroup of G, such that neither H nor G/H is compact.*

The homogeneous space G/H has a tessellation if and only if

(1) *$d(H) = 4m$; and*
(2) *either $H \sim \mathrm{SU}(1, 2m)$ or $H \sim \mathrm{Sp}(1, m)$.*

The subgroups H that arise in Theorems 11.5′ and 11.5″ can also be described more explicitly (cf. 11.2′ and 11.2″ below).

Kobayashi [Kb7, 1.4] conjectured that if H is reductive and it is impossible to construct a tessellation of G/H by using a certain more sophisticated version of Lemma 1.10 (see 9.1), then G/H does not have a tessellation. The following lists three special cases of this general conjecture.

CONJECTURE 1.19. *The homogeneous spaces*

(a) $\mathrm{SO}(2, 2m+1)/\mathrm{SU}(1, m)$,
(b) $\mathrm{SU}(2, 2m+1)/\mathrm{Sp}(1, m)$, *and*
(c) $\mathrm{SU}(2, 2m+1)/\mathrm{SU}(1, 2m+1)$

do not have tessellations.

If this conjecture is true, then, for odd n, there is no interesting example of a homogeneous space of $\mathrm{SO}(2, n)$ or $\mathrm{SU}(2, n)$ that has a tessellation.

THEOREM 11.1′ ([OW3, Theorem 1.7], [IW]). *Assume*

$$G = \mathrm{SO}(2, 2m+1) \text{ or } \mathrm{SU}(2, 2m+1),$$

and let H be any closed, connected subgroup of G, such that neither H nor G/H is compact.

If Conjecture 1.19 *is true, then G/H does not have a tessellation.*

The proof of Theorem 11.1′ assumes the following special case proved by Kulkarni [Kul, Corollary 2.10]. In short, Kulkarni noted that the Euler characteristic of $\Gamma\backslash G/H$ must both vanish (because the Euler characteristic of G/H vanishes) and not vanish (by the Gauss–Bonnet Theorem). (Other results in the same spirit, obtaining a contradiction from the study of characteristic classes of $\Gamma\backslash G/H$, appear in [KO].)

THEOREM 1.20 (Kulkarni). *If n is odd, then $\mathrm{SO}(2, n)/\mathrm{SO}(1, n)$ does not have a tessellation.*

Let us give a more explicit description of the closed, connected subgroups H of $\mathrm{SO}(2, 2m)$ or $\mathrm{SU}(2, 2m)$, such that G/H has a tessellation. This shows that if n is even, then the Kulkarni–Kobayashi examples (1.11) and certain deformations are essentially the only interesting homogeneous spaces of $\mathrm{SO}(2, n)$ or $\mathrm{SU}(2, n)$ that have tessellations.

NOTATION 1.21 [Hel, Theorem 6.5.1, pp. 270–271], [Hc2, p. 180]. Fix an Iwasawa decomposition $G = KAN$. Thus,

- K is a maximal compact subgroup,
- A is the identity component of a maximal split torus, and
- N is a maximal unipotent subgroup.

The following two results are stated only for subgroups of AN, because the general case reduces to this (see 3.5). The reason is basically that H contains a connected, cocompact subgroup that is conjugate to a subgroup of AN. (Clearly, if H' is any cocompact subgroup of H, then G/H has a tessellation if and only if G/H' has a tessellation.) This is not quite true in general, but the following lemma provides a satisfactory substitute, by showing that it becomes true after enlarging H by a compact amount.

LEMMA 3.5′. *After replacing H by a conjugate subgroup, there is a closed, connected subgroup H^* of G, such that H^*/H and $H^*/(AN \cap H^*)^\circ$ are compact, where $(AN \cap H^*)^\circ$ denotes the identity component of $AN \cap H^*$.*

THEOREM 11.2′ (Oh-Witte [OW3, Theorem 1.7]). *Assume $G = \mathrm{SO}(2, 2m)$, and let H be a closed, connected, nontrivial, proper subgroup of AN.*

The homogeneous space G/H has a tessellation if and only if H is conjugate to a subgroup H', such that either

(1) *$H' = \mathrm{SO}(1, 2m) \cap AN$; or*
(2) *H' belongs to a certain family $\{H_B\}$ of deformations of $\mathrm{SU}(1, m) \cap AN$, described explicitly in Theorem 9.7 (with $\mathbb{F} = \mathbb{R}$).*

THEOREM 11.2″. *Assume $G = \mathrm{SU}(2, 2m)$, and let H be a closed, connected, nontrivial, proper subgroup of AN.*

The homogeneous space G/H has a tessellation if and only if H is conjugate to a subgroup H', such that either

(1) *H' belongs to a certain family $\{H_{|c|}\}$ of deformations of $\mathrm{SU}(1, 2m) \cap AN$, described explicitly in Theorem 9.14; or*
(2) *H' belongs to a certain family $\{H_B\}$ of deformations of $\mathrm{Sp}(1, m) \cap AN$, described explicitly in Theorem 9.7 (with $\mathbb{F} = \mathbb{C}$).*

2. Cartan Projection and Cartan Decomposition Subgroups

The main problem in this paper is to determine whether or not a homogeneous space G/H has a tessellation. This requires some method to determine whether or not a given discrete subgroup Γ of G acts properly discontinuously on G/H. Y. Benoist and T. Kobayashi (independently) demonstrated that the Cartan projection μ is an effective tool to study this question. It is the foundation of almost all of our work in later sections.

In this section, we introduce the Cartan projection, and describe some of its basic properties. First, however, we recall the notion of a proper action (a generalization of properly discontinuous actions) and of a Cartan decomposition subgroup. At the end of the section, we use the Cartan projection to briefly discuss the question of when there is a loss of generality in assuming that G is connected.

2A. PROPER ACTIONS

DEFINITION 2.1 ([Kb6], Definition 2.11, [Pal], Definition 1.2.2, (6)). A topological group L of homeomorphisms of a topological space M acts *properly* on M if, for every compact subset C of M,

$\{ g \in L \mid C \cap gC \neq \emptyset \}$ is compact.

Remark 2.2. It is important to note that a *discrete* group of homeomorphisms of M acts properly on M if and only if it acts properly discontinuously on M.

For the special case where $M = G/H$ is a homogeneous space, the following lemma restates the definition of a proper action in more group-theoretic terms.

LEMMA 2.3 ([Kb5, Obs. 2.1.3]). *A closed subgroup L of G acts properly on G/H if and only if, for every compact subset C of G, the intersection $L \cap (CHC)$ is compact.*

Proof. If C is any compact subset of G, then $\bar{C} = CH/H$ is a compact subset of G/H; furthermore, any compact subset of G/H is contained in one of the form $\bar{C}$. We have

$$\begin{aligned}\{g \in L \mid \bar{C} \cap g\bar{C} \neq \emptyset\} &= \{g \in L \mid (CH) \cap (gCH) \neq \emptyset\} \\ &= \{g \in L \mid g \in (CH)(CH)^{-1}\} \\ &= L \cap (CHC^{-1}). \end{aligned}$$ □

This has the following well-known, easy consequence.

COROLLARY 2.4 (cf. [Kb5, Lemma 2.2(2)]). *Suppose H, H_1, L, and L_1 are closed subgroups of G. If*

- *L acts properly on G/H, and*
- *there is a compact subset C of G, such that $H_1 \subset CHC$ and $L_1 \subset CLC$,*

then L_1 acts properly on G/H_1.

2B. CARTAN-DECOMPOSITION SUBGROUPS

The following definition describes the subgroups to which the Calabi–Markus Phenomenon applies (cf. Example 1.6).

DEFINITION 2.5. We say that H is a *Cartan decomposition subgroup* of G if $H \sim G$ (see Notation 1.17).

Remark 2.6. From the Cartan decomposition $G = KAK$, we know that A is a Cartan-decomposition subgroup.

Remark 2.7. Any conjugate of a Cartan decomposition subgroup is a Cartan decomposition subgroup.

LEMMA 2.8 (Calabi–Markus Phenomenon, cf. [Kul, proof of Theorem A.1.2]). *If H is a Cartan-decomposition subgroup of G, and Γ is a discrete subgroup of G that acts properly discontinuously on G/H, then Γ is finite.*

Proof. Because H is a Cartan-decomposition subgroup, there is a compact subset C of G, such that $CHC = G$. However, from Lemma 2.3, we know that $\Gamma \cap (CHC)$ is finite. Therefore

$$\Gamma = \Gamma \cap G = \Gamma \cap (CHC)$$

is finite. □

The following well-known, easy fact is a direct consequence of the Calabi–Markus Phenomenon. It is an important first step toward determining which homogeneous spaces have tessellations.

COROLLARY 2.9. *If H is a Cartan-decomposition subgroup of G, such that G/H is not compact, then G/H does not have a tessellation.*

2C. THE CARTAN PROJECTION

NOTATION 2.10 ([Hel, Section 9.1, p. 402]).

- If G is connected, let A^+ be the (closed) positive Weyl chamber of A in which the roots occurring in the Lie algebra of N are positive (cf. 1.21). Thus, A^+ is a fundamental domain for the action of the (real) Weyl group of G on A.
- In the general case, let A^+ be a closed, convex fundamental domain for the action of the (real) Weyl group of G on A, such that A^+ is contained in the (closed) positive Weyl chamber of A in which the roots occurring in the Lie algebra of N are positive.

DEFINITION 2.11 ([Hel, Theorem 9.1.1, p. 402], [Ben, Kb6]). For each element g of G, the Cartan decomposition $G = KA^+K$ implies that there is an element a of A^+ with $g \bullet KaK$. In fact, the element a is unique, so there is a well-defined function $\mu\colon G \to A^+$ given by $g \bullet K\mu(g)K$.

We remark that the function μ is continuous and proper (that is, the inverse image of any compact set is compact).

The following crucial result of Y. Benoist provides a uniform estimate on the variation of μ over disks of bounded radius. (A related result was proved, independently and simultaneously, by T. Kobayashi [Kb5, Theorem 3.4].) The proof is both elementary and elegant. However, it requires a bit of notation, so we postpone it to Section 8B (and, for concreteness, we will assume that G is either $SO(2, n)$ or $SU(2, n)$ in the proof).

PROPOSITION 2.12 ([Ben, Proposition 5.1]). *For any compact subset C of G, there is a compact subset C' of A, such that $\mu(CgC) \subset \mu(g)C'$, for all $g \bullet G$.*

NOTATION 2.13. For subsets U and V of A^+, we write $U \approx V$ if there is a compact subset C of A, such that $U \subset VC$ and $V \subset UC$.

COROLLARY 2.14 ([Ben, Proposition 5.1], Kobayashi [Kb5, Theorem 1.1]). *For any subgroups H_1 and H_2 of G, we have $H_1 \sim H_2$ if and only if $\mu(H_1) \approx \mu(H_2)$.*

Proof. ($\Rightarrow$) Let C be a compact subset of G, such that $H_1 \subset CH_2C$ and $H_2 \subset CH_1C$. Choose a corresponding compact subset C' of A, as in Proposition 2.12. Then

$$\mu(H_1) \subset \mu(CH_2C) \subset \mu(H_2)C'$$

and, similarly, $\mu(H_2) \subset \mu(H_1)C'$.

($\Leftarrow$) Let C be a compact subset of A, such that $\mu(H_1) \subset \mu(H_2)C$ and $\mu(H_2) \subset \mu(H_1)C$. Then

$$H_1 \subset K\mu(H_1)K \subset K(\mu(H_2)C)K \subset K((KH_2K)C)K$$

and, similarly, $H_2 \subset KH_1(KCK)$. □

The special case where $H_2 = G$ (and H_1 is closed and almost connected) can be restated as follows.

COROLLARY 2.15 (Benoist, Kobayashi). *H is a Cartan decomposition subgroup of G if and only if $\mu(H) \approx A^+$.*

COROLLARY 2.16 (cf. [Kb4, Lemma 3.2]). *Assume that* $\mathbb{R}$-rank $G = 1$. *The subgroup H is a Cartan decomposition subgroup of G if and only if H is noncompact.*

Proof. ($\Leftarrow$) We have $\mu(e) = e$, and, because μ is a proper map, we have $\mu(h) \to \infty$ as $h \to \infty$ in H. Because $\mathbb{R}$-rank $G = 1$, we know that A^+ is homeomorphic to the half-line $[0, \infty)$ (with the point e in A^+ corresponding to the endpoint 0 of the half-line), so, by continuity, it must be the case that $\mu(H) = A^+$. Then Corollary 2.15 implies that H is a Cartan decomposition subgroup, but we provide the following direct proof that avoids any appeal to Proposition 2.12.

From the definition of μ, we have $KHK = K\mu(H)K$. Therefore

$$KHK = K\mu(H)K = KA^+K = G,$$

so H is a Cartan decomposition subgroup (by taking $C = K$ in Definition 1.17). □

By using Lemma 2.3, the proof of Corollary 2.14 also establishes the following.

COROLLARY 2.17 ([Ben, Proposition 1.5], [Kb5, Corollary 3.5]). *Suppose H and L are closed subgroups of G. The subgroup L acts properly on G/H if and only if $\mu(L) \cap \mu(H)C$ is compact, for every compact subset C of A.*

2D. DISCONNECTED GROUPS

As was mentioned in Remark 1.4, we may assume, without loss of generality, that H is connected. However, it may not be possible to assume that G is connected, because, although there are no known examples, it is possible that the following question has an affirmative answer.

QUESTION 2.18. Does there exist a homogeneous space G/H (satisfying Assumption 1.3), such that G°/H° has a tessellation, but G/H does not have a tessellation?

If Γ is a crystallographic group for G°/H°, then it is easy to see that $\Gamma\backslash G/H$ is compact. However, the following example shows that Γ may not act properly discontinuously on G/H.

EXAMPLE 2.19. Let

- $L = H = \mathrm{SL}(2, \mathbb{R})$,
- σ be the automorphism of $L \times H$ that interchanges the two factors (that is, $\sigma(x, y) = (y, x)$),
- $G = (L \times H) \rtimes \langle\sigma\rangle$ (semidirect product), and
- Γ be a cocompact lattice in L (cf. 1.5(2)).

Then $H = H^\circ$, and Γ is a crystallographic group for $G^\circ/H = (L \times H)/H$ (see Example 1.9).

However, $\Gamma \subset L = \sigma^{-1}H\sigma$, so Γ does not act properly on G/H (see 2.3 with $C = \{\sigma, \sigma^{-1}\}$).

Even so, G/H does have a tessellation, because the diagonal embedding

$$\Delta(\Gamma) = \{(\gamma, \gamma) \in L \times H \mid \gamma \in \Gamma\}$$

is a crystallographic group for G/H. Thus, this example does not provide an answer to Question 2.18.

In this example, σ represents an element of the Weyl group of G that does not belong to the Weyl group of G°. The following proposition shows that this is a crucial ingredient in the construction.

PROPOSITION 2.20. *Let Γ be a crystallographic group for G°/H°.*

If the (real) Weyl group of G is same as the (real) Weyl group of G°, then Γ is a crystallographic group for G/H.

Proof. By assumption, we may choose the same fundamental domain A^+ for the Weyl groups of G and G°. Let $\mu\colon G \to A^+$ and $\mu^\circ\colon G^\circ \to A^+$ be the Cartan projections; then μ° is the restriction of μ to G°. For simplicity, assume, without loss of generality, that $H \subset G^\circ$ (for example, assume H is connected). Then, for any compact subset C of A, we have

$$\mu(\Gamma) \cap \mu(H)C = \mu^\circ(\Gamma) \cap \mu^\circ(H)C$$

is finite (see 2.17). Thus, Γ acts properly discontinuously on G/H (see 2.17), as desired. ■

A. Borel and J. Tits [BT, Corollary 14.6, p. 147] proved that if G is Zariski connected, then every element of the Weyl group of G has a representative in G°. Also, any element of the Weyl group must act as an automorphism of the root system. Thus, we have the following corollary.

COROLLARY 2.21. *Let Γ be a crystallographic group for G°/H°.*
If either

- *G is Zariski connected, or*
- *every automorphism of the real root system of G° belongs to the Weyl group of the root system,*

then Γ is a crystallographic group for G/H.

EXAMPLE 2.22. 1) If $G = \mathrm{SO}(2, n)$, then G is Zariski connected (because $\mathrm{SO}(n+2, \mathbb{C})$ is connected [GdW, Theorem 2.1.9, p. 60]), so G/H has a tessellation if and only if G°/H° has a tessellation.

2) More generally, if $G^\circ = \mathrm{SO}(2, n)^\circ$ or $\mathrm{SU}(2, n)$ (with $n \geqslant 3$), then every automorphism of the real root system of G° belongs to the Weyl group of the root system (cf. Figure 7.1), so G/H has a tessellation if and only if G°/H° has a tessellation.

EXAMPLE 2.23. If $G = \mathrm{SL}(3, \mathbb{R}) \rtimes \langle\sigma\rangle$, where σ is the Cartan involution of $\mathrm{SL}(3, \mathbb{R})$, then σ represents an element of the Weyl group of G that does not belong to the Weyl group of G°, so the proposition does not apply to G. However, this does not matter: if neither H nor G/H is compact, then Theorem 1.8 implies that G°/H° has no tessellations, so G/H has no tessellations either.

3. Preliminaries on Subgroups of AN

This section recalls a technical result that often allows us to assume that H is a subgroup of AN. It also recalls some basic topological properties of such subgroups, and also recalls a simple observation relating these subgroups to the root spaces of the Lie algebra $\mathfrak{g}$.

3A. REDUCTION TO SUBGROUPS OF AN

DEFINITION 3.1 ([Hel, Theorem 9.7.2, p. 431]). An element g of G is:

- *hyperbolic* if g is conjugate to an element of A;
- *unipotent* if g is conjugate to an element of N;
- *elliptic* if g is conjugate to an element of K.

LEMMA 3.2 (Real Jordan Decomposition [Hel, Lemma 9.7.1, p. 430]). *Each $g \in G$ has a unique decomposition in the form $g = auc$, such that*

- *a is hyperbolic, u is unipotent, and c is elliptic; and*
- *a, u, and c all commute with each other.*

Remark 3.3. If $g = auc$ is the Real Jordan Decomposition of some element g of G, then a, u, and c commute, not only with each other, but also with any element of G that commutes with g. This is because the Real Jordan Decomposition of $h^{-1}gh$ is

$$h^{-1}gh = (h^{-1}ah)(h^{-1}uh)(h^{-1}ch):$$

if $h^{-1}gh = g$, then the uniqueness of the Real Jordan Decomposition of g implies $h^{-1}ah = a$, $h^{-1}uh = u$, and $h^{-1}gh = c$.

The following observation is a generalization of the fact that a collection of commuting triangularizable matrices can be simultaneously triangularized.

LEMMA 3.4 (cf. proof of [Hm1, Theorem 17.6]). *If H is abelian (or, more generally, solvable), and is generated by hyperbolic and/or unipotent elements, then H is conjugate to a subgroup of AN.*

Because of the following result, we usually assume $H \subset AN$ (by replacing H with a conjugate of H').

LEMMA 3.5 (cf. [OW1, Lemma 2.9]). *If H is connected, then there is a closed, connected subgroup H' of G and a compact, connected subgroup C of G, such that*

(1) *H' is conjugate to a subgroup of AN;*
(2) *$CH = CH'$ is a subgroup of G; and*
(3) *$d(H') = d(H)$ (see Notation 1.14).*

Moreover, it is easy to see from (2) that the homogeneous space G/H has a tessellation if and only if G/H' has a tessellation.

Idea of Proof. First, let us note that every connected subgroup of AN is closed (see 3.15(1) and 3.18), so we do not need to show that H' is closed.

Second, let us note that (3) is a consequence of (1) and (2). To see this, let K^* be a maximal compact subgroup of CH that contains C. Then a standard argument shows that $K^* \cap H$ is a maximal compact subgroup of H. (Because all maximal compact subgroups of CH are conjugate, there is some $g \bullet CH$, such that $(g^{-1}K^*g) \cap H$ is a maximal compact subgroup of H that contains $K^* \cap H$. Since $C \subset K^*$, we know that C normalizes K^*, so we may assume $g \in H$; thus, g normalizes H. Then $g^{-1}(K^* \cap H)g = (g^{-1}K^*g) \cap H$ contains $K^* \cap H$. Because $K^* \cap H$ is compact, this implies that g normalizes $K^* \cap H$. So $K^* \cap H = (g^{-1}K^*g) \cap H$ is a maximal compact subgroup of H.) Therefore

$$\dim(K^*H/K^*) = \dim(H/(K^* \cap H)) = \dim H - \dim(K^* \cap H) = d(H).$$

Similarly, $\dim(K^*H'/K) = d(H')$. Since $K^*H = CH = CH' = K^*H'$, we conclude that $d(H') = d(H)$, as desired.

Case 1. *Assume H is semisimple.* We have an Iwasawa decomposition $H = K_H A_H N_H$; let $H' = A_H N_H$ and $C = K_H$.

Case 2. *Assume $H = \{h^t\}$ is a one-parameter subgroup.* Let

- $h^t = a^t u^t c^t$ be the Real Jordan Decomposition of h^t (see 3.2);
- $H' = \{a^t u^t\}$; and
- $C = \overline{\{c^t\}}$ be the closure of $\{c^t\}$.

(Lemma 3.4 implies that H' is conjugate to a subgroup of AN.)

Case 3. *Assume H is Abelian.* We may write H as a product of one-parameter subgroups:

$$H = \{\, h_1^{t_1} h_2^{t_2} \cdots h_r^{t_r} \mid t_1, \ldots, t_r \in \mathbb{R}\}.$$

Let $h_j^t = a_j^t u_j^t c_j^t$ be the Real Jordan Decomposition of h_j^t (see 3.2). Note that $a_j^{t_j}$, $u_j^{t_j}$, and $c_j^{t_j}$ commute, not only with each other, but also with every $a_k^{t_k}$, $u_k^{t_k}$, and $c_k^{t_k}$ (see 3.3). Let

$$H' = \{\, (a_1^{t_1} u_1^{t_1})(a_2^{t_2} u_2^{t_2}) \cdots (a_k^{t_r} u_k^{t_r}) \mid t_1, \ldots, t_r \in \mathbb{R}\,\},$$

and let $C = \overline{\{c_1^t\}} \cdots \overline{\{c_1^t\}}$. (Lemma 3.4 implies that H' is conjugate to a subgroup of AN.)

Case 4. *The general case.* From the Levi decomposition [Jac, p. 91], we know that there is a connected, semisimple subgroup L of H and a connected, solvable, normal subgroup R of H, such that $H = LR$ (and $L \cap R$ is finite). Let $U = [H, R]$, so U is a connected, normal subgroup of H, and U is conjugate to a subgroup of N (cf. [Jac, Corollary 2.7.1, p. 51]). By modding out U, we (essentially) reduce to the direct product of Cases 1 and 3. □

Remark 3.6. For H and H' as in Lemma 3.5, Proposition 3.20 (and 3.18) implies that if $H' \neq AN$, then AN/H' is not compact; also, Proposition 3.15(3) (and 3.18) implies that if $H' \neq e$, then H' is not compact. Therefore,

- $H' = AN$ if and only if G/H is compact; and
- $H' = e$ if and only if H is compact.

Thus, if neither H nor G/H is compact, then H' is a nontrivial, proper subgroup of AN.

3B. TOPOLOGY OF SOLVABLE GROUPS AND THEIR HOMOGENEOUS SPACES

Everything is this subsection is well known, though somewhat scattered in the literature. The main results are Propositions 3.15 and 3.20, which, together with Corollary 3.18, show that connected subgroups of AN and their homogeneous spaces are very well behaved topologically. Corollary 3.19, on the homology of very simple quotient spaces, is also used in later sections.

We begin with the easy case of Abelian groups. This lemma generalizes almost verbatim to solvable groups (see 3.15), but the proof in that generality is not as trivial.

LEMMA 3.7 ([Var, Theorem 3.6.2, p. 196]). *Let R be a 1-connected, Abelian Lie group.*

(1) *If H is a connected subgroup of R, then H is closed, simply connected, and isomorphic to $\mathbb{R}^k$, for some k.*
(2) *If H and L are connected subgroups of R, then $H \cap L$ is connected.*
(3) *If C is a compact subgroup of R, then C is trivial.*

Proof. Because R is Abelian and 1-connected, the exponential map is a Lie group isomorphism from the additive group of the Lie algebra $\mathfrak{r}$ onto R.

(1) Let $k = \dim H$. Because the exponential map is a Lie group isomorphism (hence a diffeomorphism), and because $\mathfrak{h}$ is a closed k-submanifold of $\mathfrak{r}$, we know that $\exp(\mathfrak{h})$ is a closed k-submanifold of R. Of course, $\exp(\mathfrak{h})$ is contained in H, which is also a k-submanifold of R. Because the dimensions are the same, we know that $\exp(\mathfrak{h})$ is open in H. Also, because $\exp(\mathfrak{h})$ is closed in R, we know that $\exp(\mathfrak{h})$ is closed in H. Therefore
$$\exp(\mathfrak{h}) = H \tag{3.8}$$
(because H is connected). Finally, we know that $\exp|_{\mathfrak{h}}$ is a diffeomorphism from its domain $\mathfrak{h} \simeq \mathbb{R}^k$ onto its image H.
(2) From (3.8), we have $\exp(\mathfrak{h}) = H$ and, similarly, $\exp(\mathfrak{l}) = L$. Also, because exp is bijective, we have $\exp \mathfrak{h} \cap \exp \mathfrak{l} = \exp(\mathfrak{h} \cap \mathfrak{l})$. Therefore
$$H \cap L = \exp \mathfrak{h} \cap \exp \mathfrak{l} = \exp(\mathfrak{h} \cap \mathfrak{l})$$
is connected.
(3) Because $\mathbb{R}^k$ is not compact (for $k > 0$), we know, from 3.7(1), that C° is trivial; so C is finite. Since $R \cong (\mathfrak{r}, +) \cong \mathbb{R}^d$ has no elements of finite order, we conclude that C is trivial. □

As is usual in the theory of solvable groups, the main results of this section are proved by induction, based on modding out some normal subgroup L. To be effective, this method requires an understanding of the quotient space R/L. The information we need (even if L is not normal) comes from the following elementary observation, because R is a principal L-bundle over R/L.

LEMMA 3.9. *Let P be a principal H-bundle over a manifold M.*

(1) *If H is diffeomorphic to $\mathbb{R}^n$, then*
   (a) *P is H-equivariantly diffeomorphic to $M \times H$, so*
   (b) *P is homotopy equivalent to M.*
(2) *If M is diffeomorphic to $\mathbb{R}^n$, then*
   (a) *P is H-equivariantly diffeomorphic to $M \times H$, so*
   (b) *P is homotopy equivalent to H.*

Proof. Any principal bundle with a section is trivial [Hus, Corollary 4.8.3, p. 48]. If either the fiber or the base is contractible, then there is no obstruction to constructing a section [Hus, Theorem 2.7.1(H1), p. 21], so P is trivial: $P \simeq M \times H$. (The diffeomorphism can be taken to be H-equivariant, with respect to the natural H-action on $M \times H$, given by $(m, h)h' = (m, hh')$.) Then the conclusions on homotopy equivalence follow from the fact that $\mathbb{R}^n$ is contractible (that is, homotopically trivial). ∎

We recall the long exact sequence of the fibration $H \to R \to R/H$:

LEMMA 3.10 ([Whi, Corollary IV.8.6, p. 187]). *Let H be a closed subgroup of a Lie group R. There is a (natural) long exact sequence of homotopy groups:*

$$\cdots \to \pi_1(H) \to \pi_1(R) \to \pi_1(R/H) \to \pi_0(H) \to \pi_0(R) \to \pi_0(R/H) \to 0.$$

COROLLARY 3.11. *Let H be a closed subgroup of a 1-connected Lie group R. The homogeneous space R/H is simply connected if and only if H is connected.*

Proof. Because R is 1-connected, we have $\pi_1(R) = \pi_0(R) = 0$, so, from (3.10), we know that the sequence

$$0 \to \pi_1(R/H) \to \pi_0(H) \to 0$$

is exact. Thus, $\pi_1(R/H) \cong \pi_0(H)$, so the desired conclusion is immediate. ∎

As a step toward Proposition 3.15, we prove two special cases that describe the topology of normal subgroups.

LEMMA 3.12. *If R is a 1-connected, solvable Lie group, then R is diffeomorphic to $\mathbb{R}^d$, for some d.*

Proof. We may assume the group R is non-Abelian (otherwise, the desired conclusion is given by Lemma 3.7(1)). Then, because R is solvable, there is a nontrivial, connected, proper, closed, normal subgroup L of R. Since R/L is simply connected (see 3.11), and $\dim(R/L) < \dim R$, we may assume, by induction on $\dim R$, that R/L is diffeomorphic to some $\mathbb{R}^{d_1}$. Therefore

(a) R is diffeomorphic to $(R/L) \times L$ and
(b) L is homotopy equivalent to R

(see 3.9(2)). Because R is 1-connected, (b) implies that L is 1-connected; hence, L is a 1-connected, solvable Lie group, so we may assume, by induction on $\dim R$, that L is diffeomorphic to some $\mathbb{R}^{d_2}$. Thus, (a) implies that R is diffeomorphic to $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \simeq \mathbb{R}^{d_1+d_2}$, as desired. ∎

COROLLARY 3.13 (of proof). *If R is a 1-connected, solvable Lie group, then every connected, closed, normal subgroup of R is 1-connected.*

The following proposition is a nearly complete generalization of Lemma 3.7 to the class of solvable groups. There are two exceptions:

(1) Of course, subgroups of a solvable group may not be Abelian, so the conclusion in 3.7(1) that H is isomorphic to some $\mathbb{R}^k$ must be weakened to the conclusion that H is diffeomorphic to some $\mathbb{R}^k$.
(2) The intersection of connected subgroups is not always connected (see 3.14), so we add the restriction that L is normal to 3.7(2). (We remark that no such restriction is necessary if $R \subset AN$, because the exponential map is a diffeomorphism from $\mathfrak{r}$ onto R in this case [Dix, Sai].)

EXAMPLE 3.14. Let

$$R = \left\{ \begin{pmatrix} e^{2\pi i t} & x+iy & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix} \middle| \, t, x, y \in \mathbb{R} \right\}, \quad g^t = \begin{pmatrix} e^{2\pi i t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix},$$

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h^t = u^{-1} g^t u = \begin{pmatrix} e^{2\pi i t} & e^{2\pi i t} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Then R, being diffeomorphic to $\mathbb{R}^3$, is 1-connected; and $\{g^t\}$ and $\{h^t\}$ are connected subgroups. But

$$\{g^t\} \cap \{h^t\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^n \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\}$$

is not connected.

PROPOSITION 3.15 ([Hc2, Theorems 12.2.2 and 12.2.3, pp. 137–138]). *Let R be a 1-connected, solvable Lie group.*

(1) *If H is a connected subgroup of R, then H is closed, simply connected, and diffeomorphic to some $\mathbb{R}^d$.*
(2) *If H and L are connected subgroups of R, and L is normal, then $H \cap L$ is connected.*
(3) *If C is a compact subgroup of R, then C is trivial.*

Proof. (2) We may assume L is nontrivial, so $\dim(R/L) < \dim R$. Thus, by induction on $\dim R$, using (1), we may assume that HL/L is a closed, simply connected subgroup of R/L. Then, since $H/(H \cap L)$ is homeomorphic to HL/L, we see that

$$H/(H \cap L) \text{ is simply connected,} \tag{3.16}$$

so Lemma 3.11 implies that $H \cap L$ is connected.

(1) Because R is solvable, there is a connected, closed, proper, normal subgroup L of R, such that R/L is Abelian. We know that L is 1-connected (see 3.13), so, by induction on $\dim R$, we may assume that every connected subgroup of L is closed

and simply connected. From (2), we know that $H \cap L$ is connected, so we conclude that $H \cap L$ is closed, and

$$\pi_1(H \cap L) = 0. \tag{3.17}$$

From (3.10) (with H in the place of R, and L in the place of H), together with (3.16) and (3.17), we conclude that $\pi_1(H) = 0$; that is, H is simply connected. So (3.12) implies H is diffeomorphic to some $\mathbb{R}^d$.

Because both HL/L and $H \cap L$ are closed, it is not difficult to see that H is closed.

(3) Because R is solvable, there is a connected, closed, proper, normal subgroup L of R, such that R/L is Abelian. We know that R/L is 1-connected (see 3.11), so R/L has no nontrivial, compact subgroups (see 3.7(3)); thus, we must have $C \subset L$. Therefore, C is a compact subgroup of L. Then, since L is 1-connected (see 3.13), we may conclude, by induction on $\dim R$, that C is trivial. ■

COROLLARY 3.18. *AN is a 1-connected, solvable Lie group.*

Proof. Because G is linear, it is a subgroup of some $\mathrm{GL}(n, \mathbb{R})$. Replacing G by a conjugate, we may assume that AN is contained in the group B of upper triangular matrices with positive diagonal entries (cf. 3.4). The matrix entries provide an obvious diffeomorphism from B onto $(\mathbb{R}^+)^n \times \mathbb{R}^{n(n-1)/2} \simeq \mathbb{R}^{n(n+1)/2}$, so B is 1-connected. Thus, Proposition 3.15(1) implies that AN is simply connected. □

The following observation will be used in Sections 4 and 9.

COROLLARY 3.19. *Let F be a connected subgroup of AN, and suppose we have a proper, C^∞ action of F on a manifold M. Then M and M/F have the same homology.*

Proof. Because the action is proper, we know that the stabilizer of each point of M is compact. However, F has no nontrivial compact subgroups (see 3.15(3)). Thus, the action is free.

Because the action is free, proper, and C^∞, it is easy to see that the manifold M is a principal fiber bundle over the quotient M/F [Pal, Theorem 1.1.3]. Furthermore, the fiber F of the bundle is contractible (see 3.15(1)), so Lemma 3.9(1) implies that M homotopy equivalent to M/F. Therefore, the spaces M and M/F have the same homology. □

For the special case where M/F is a homogeneous space of a solvable group, the following more detailed result describes the topology of M/F, not just its homology.

PROPOSITION 3.20 (Mostow [Mos, Proposition 11.2]). *If H is any connected subgroup of a 1-connected, solvable Lie group R, then R/H is diffeomorphic to the Euclidean space $\mathbb{R}^d$, for some d.*

Proof. Because R is solvable, it has a nontrivial, connected, closed, Abelian, normal subgroup L. Since L is Abelian and $H \cap L$ is connected (see 3.15(2)), we know that $L/(H \cap L)$ is a 1-connected Abelian group (see 3.11), so it is isomorphic to some $\mathbb{R}^{d_1}$ (see 3.7(1)).

We know H is closed (see 3.15(1)). Also, since L is nontrivial, we have $\dim(R/L) < \dim R$, so we may assume, by induction on $\dim R$, that

$$R/(HL) \simeq (R/L)/(HL/L)$$

is diffeomorphic to some $\mathbb{R}^{d_2}$.

Now R is a principal HL-bundle over $R/(HL)$. Because $R/(HL) \simeq \mathbb{R}^{d_2}$, this bundle is trivial (see 3.9(2)): R is HL-equivariantly diffeomorphic to $R/(HL) \times HL$. Then

$$R/H \simeq R/(HL) \times HL/H \simeq R/(HL) \times L/(H \cap L) \simeq \mathbb{R}^{d_2} \times \mathbb{R}^{d_1} = \mathbb{R}^{d_1+d_2},$$

as desired. ■

3C. T-INVARIANT SUBSPACES OF $\mathfrak{a} + \mathfrak{n}$

The following well-known observation puts an important restriction on the subspaces of $\mathfrak{a} + \mathfrak{n}$ that are normalized by a torus. It is an ingredient in our case-by-case analysis of all possible subgroups of AN in Sections 10 and 11.

LEMMA 3.21. *Let*

- Φ^+ *be the set of weights of A on $\mathfrak{n}$ (in other words, the set of all positive real roots of G);*
- *T be a subgroup of A;*
- $\omega \in \Phi^+ \cup \{0\}$;
- $\mathfrak{n}^{=\omega} = \bigoplus_{\sigma|_T = \omega|_T} \mathfrak{n}_\sigma$, *where the sum is over all $\sigma \in \Phi^+ \cup \{0\}$, such that the restriction of σ to T is the same as the restriction of ω to T;*
- $\mathfrak{n}^{\neq\omega} = \bigoplus_{\sigma|_T \neq \omega|_T} \mathfrak{n}_\sigma$, *where the sum is over all $\sigma \in \Phi^+ \cup \{0\}$, such that the restriction of σ to T is not the same as the restriction of ω to T.*

If $\mathfrak{u}$ is any $\mathbb{R}$-subspace of $\mathfrak{a} + \mathfrak{n}$ normalized by T, then $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{n}^{=\omega}) \oplus (\mathfrak{u} \cap \mathfrak{n}^{\neq\omega})$.

Proof. Since $T \subset A$, we know that the elements of $\mathrm{Ad}_G T$ are simultaneously diagonalizable (over $\mathbb{R}$), so their restrictions to the invariant subspace $\mathfrak{u}$ are also simultaneously diagonalizable (cf. [ZS, Theorems 26 and 27 in Section 3.12, pp. 167–168]). Thus, $\mathfrak{u}$ is a direct sum of weight spaces:

$$\mathfrak{u} = \bigoplus_{\psi \in \Psi} \mathfrak{u}_\psi.$$

For each weight ψ of T on $\mathfrak{u}$, we have

$$\mathfrak{u}_\psi = \mathfrak{u} \cap \mathfrak{n}_\psi = \mathfrak{u} \cap \mathfrak{n}^{=\psi},$$

so

$$\mathfrak{u}_{\omega|_T} = \mathfrak{u} \cap \mathfrak{n}^{=\omega}$$

and

$$\bigoplus_{\psi \neq \omega|_T} \mathfrak{u}_\psi = \mathfrak{u} \cap \mathfrak{n}^{\neq\omega}.$$

The conclusion follows. □

4. Lower Bound on the Dimension of H

In this section, we prove Corollary 4.12, an *a priori* lower bound on $\dim H$. On the way, we recall a result of T. Kobayashi that will also be used several times in later sections, and we establish that crystallographic groups have only one end.

4A. T. KOBAYASHI'S DIMENSION THEOREM

The following theorem is essentially due to T. Kobayashi. (Kobayashi assumed that H is reductive, but H. Oh and D. Witte [OW3, Theorem 3.4] pointed out that, by using Lemma 3.5, this restriction can be eliminated.) The proof here is based on Kobayashi's original argument and the modifications of Oh and Witte, but uses less sophisticated topology. Namely, instead of group cohomology and the spectral sequence of a covering space, we use only some basic properties of homology groups of manifolds (including Lemma 3.19). These comments also apply to Theorem 9.1.

THEOREM 4.1 ([Kb3, Theorem 1.5], [Kb1, Theorem 4.7]). *Let H and H_1 be closed, connected subgroups of G, and assume there is a crystallographic group Γ for G/H, such that Γ acts properly discontinuously on G/H_1. Then:*

(1) *We have $d(H_1) \leqslant d(H)$.*
(2) *If $d(H_1) \geqslant d(H)$, then $\Gamma\backslash G/H_1$ is compact, so G/H_1 has a tessellation.*

Proof. By Lemma 3.5, we may assume $H, H_1 \subset AN$. (So $d(H) = \dim H$ and $d(H_1) = \dim H_1$ (see 1.16).)

From Lemma 3.19, we know that $\Gamma\backslash G$ and $\Gamma\backslash G/H_1$ have the same homology. Therefore

$$\max\{\, k \mid \mathcal{H}_k(\Gamma\backslash G) \neq 0\} = \max\{\, k \mid \mathcal{H}_k(\Gamma\backslash G/H_1) \neq 0\} \leqslant \dim G/H_1,$$

with equality if and only if $\Gamma\backslash G/H_1$ is compact [Dol, Corollary 8.3.4, p. 260]. Similarly, we have

$$\max\{\, k \mid \mathcal{H}_k(\Gamma\backslash G) \neq 0\} = \dim G/H.$$

Combining these two statements, we conclude (1) that $\dim G/H \leqslant \dim G/H_1$ and, furthermore, (2) that equality holds if and only if $\Gamma\backslash G/H_1$ is compact. ∎

COROLLARY 4.2 (Kobayashi). *Let H and H_1 be closed, connected subgroups of G, such that $d(H_1) > d(H)$. If there is a compact subset C of A, such that $\mu(H_1) \subset \mu(H)C$, then G/H does not have a tessellation.*

Proof. Suppose Γ is a crystallographic group for G/H. (This will lead to a contradiction.) Because Γ acts properly discontinuously on G/H, the assumption on $\mu(H_1)$ implies that Γ also acts properly discontinuously on G/H_1 (cf. 2.17). So Theorem 4.1(1) yields a contradiction. □

4B. CRYSTALLOGRAPHIC GROUPS HAVE ONLY ONE END

It is easy to see that crystallographic groups are finitely generated; we now show that they have only one end (see 4.9).

DEFINITION 4.3 (cf. [Gro, 0.2.A_2', p. 4]). Let F be a finite generating set for an (infinite) group Γ. We say that Γ has *only one end* if, for every partition $\Gamma = A_1 \cup A_2 \cup C$ of Γ into three disjoint sets A_1, A_2, and C, such that A_1 and A_2 are infinite, but C is finite, there exists $\gamma \in A_1$ and $f \in F \cup F^{-1}$, such that $\gamma f \in A_2$. (This does not depend on the choice of the generating set F.)

The following observation is a straightforward reformulation of Definition 4.3 (obtained by letting $A_2 = \Gamma \smallsetminus (A_1 \cup C')$ and $C = C' \smallsetminus A_1$).

LEMMA 4.4. *Let F be a finite generating set for an infinite group Γ that has only one end. If A_1 and C' are subsets of Γ, such that*

- *A_1 is infinite,*
- *C' is finite, and*
- *$A_1 f \in A_1 \cup C'$, for every $f \in F \cup F^{-1}$,*

then the complement $\Gamma \smallsetminus A_1$ is finite.

Remark 4.5 (cf. [Coh, pp. 25–26, p. 32, and Proposition 2.14]). Definition 4.3 is often stated in the language of Cayley graphs: The *Cayley graph* of Γ, with respect to the generating set F, is the graph $\mathrm{Cay}(\Gamma; F)$ whose vertex set V and edge set E are given by:

$$V = \Gamma;$$
$$E = \{ (\gamma, \gamma f) \mid \gamma \in \Gamma, f \in F \cup F^{-1} \}.$$

The group Γ has only one end if and only if, for every finite subset C of Γ, the graph $\mathrm{Cay}(\Gamma; F) \setminus C$ has only one infinite component.

The following lemma is not difficult, but, unfortunately, we do not have a proof that is both short and elementary.

LEMMA 4.6 (see proof of Lemma 10.7(1)). *If $HN = AN$, then, for some $x \in N$, the conjugate $x^{-1}Hx$ is normalized by A.*

COROLLARY 4.7. *If $d(G) - d(H) \leqslant 1$, and G/H is not compact, then G/H does not have a tessellation.*

Proof. It suffices to show that H is a Cartan decomposition subgroup of G (see 2.9). We may assume, without loss of generality, that $H \subset AN$ (see 3.5) then

$$\dim H + 1 = d(H) + 1 \geqslant d(G) = \dim(AN)$$

(see 1.16 and 1.15). A theorem of B. Kostant [Kos, Theorem 5.1] implies that N is a Cartan decomposition subgroup, so we may assume $N \not\subset H$; then $\dim(H \cap N) \leqslant \dim N - 1$. Therefore

$$\dim(HN) = \dim H + \dim N - \dim(H \cap N) \geqslant \dim H + 1 \geqslant \dim(AN).$$

Hence $HN = AN$, so, from Lemma 4.6, we see that, after replacing H by a conjugate subgroup, we may assume that H is normalized by A. Then, letting $\omega = 0$ and $T = A$ in (3.21), we see that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{a}) + (\mathfrak{h} \cap \mathfrak{n})$. Since $HN = AN$, we have $\mathfrak{h} + \mathfrak{n} = \mathfrak{a} + \mathfrak{n}$, so this implies that $\mathfrak{a} \subset \mathfrak{h}$; therefore H contains A. Since A is a Cartan decomposition subgroup (see 2.6), this implies H is a Cartan decomposition subgroup, as desired. □

DEFINITION 4.8 (cf. [Gro, 0.2.A_2', p. 4]). A topological space M is *connected at* ∞ if every compact subset $\mathcal{C}$ is contained in a compact subset $\mathcal{C}'$, such that the complement $M \setminus \mathcal{C}'$ is connected.

PROPOSITION 4.9. *If* Γ *is a crystallographic group for* G/H, *then* Γ *is finitely generated and has only one end.*

Proof. Assume, without loss of generality, that $H \subset AN$ (see 3.5). Then H is torsion free, so Γ must act freely on G/H; therefore $\Gamma\backslash G/H$ is a compact manifold (rather than an orbifold). Because Γ is essentially the fundamental group of $\Gamma\backslash G/H$ (specifically, $\Gamma \cong \pi_1(\Gamma\backslash G/H)/\pi_1(G/H)$), and the fundamental group of any compact manifold is finitely generated [Rag, Theorem 6.16, p. 95] we know that Γ is finitely generated.

From the Iwasawa decomposition $G = KAN$, we see that G/H is homeomorphic to $K \times (AN/H)$, and Proposition 3.20 asserts that AN/H is homeomorphic to $\mathbb{R}^d$, for some d. Obviously, we must have $d = \dim(AN) - \dim H$, and we may assume G/H is not compact (otherwise, Γ is finite, so the desired conclusion is obvious), so Corollary 4.7 implies that $d > 1$. Thus, we conclude that G/H is connected at ∞.

To complete the proof, we use a standard argument (cf. [Gro 0.2.C_1, p. 5]) to show that, because G/H is connected at ∞ and $\Gamma\backslash G/H$ is compact, the group Γ has only one end. To begin, note that there is a compact subset $\mathcal{C}$ of G/H, such that $\Gamma\mathcal{C} = G/H$. Let

$$F_0 = \{f \in \Gamma \mid \mathcal{C} \cap f\mathcal{C} \neq \emptyset\}$$

(cf. [PR, (ii), p. 195]). Because Γ acts properly discontinuously on G/H, we know that F_0 is finite; let F be a finite generating set for Γ, such that $F_0 \subset F$.

Suppose $\Gamma = A_1 \cup A_2 \cup C$, with $|A_1| = |A_2| = \infty$ and $|C| < |\infty|$. (We wish to show there exist $\gamma \in A_1$ and $f \in F$, such that $\gamma f \in A_2$; this establishes that Γ has only one end.) Because G/H is connected at ∞, there is a compact subset $\mathcal{C}'$ of G/H, containing $C\mathcal{C}$, such that $(G/H) \setminus \mathcal{C}'$ is connected. Because $C\mathcal{C} \subset \mathcal{C}'$, we have

$$(G/H) \setminus \mathcal{C}' = (\Gamma\mathcal{C}) \setminus \mathcal{C}' \subset A_1\mathcal{C} \cup A_2\mathcal{C}.$$

Because Γ acts properly discontinuously on G/H, we know $A_1\mathcal{C}$ and $A_2\mathcal{C}$ are closed (and neither is contained in $\mathcal{C}'$), so connectivity implies that $A_1\mathcal{C} \cap A_2\mathcal{C} \neq \emptyset$: there exist $\gamma \in A_1$ and $\gamma' \bullet A_2$, such that $\gamma\mathcal{C} \cap \gamma'\mathcal{C} \neq \emptyset$. Let $f = \gamma^{-1}\gamma'$; then $\gamma \bullet A_1$, $\gamma f = \gamma' \bullet A_2$, and

$$\mathcal{C} \cap f\mathcal{C} = \gamma^{-1}(\gamma\mathcal{C} \cap \gamma'\mathcal{C}) \neq \emptyset,$$

so $f \bullet F_0 \subset F$, as desired. ∎

4C. WALLS OF A^+ AND A LOWER BOUND ON $d(H)$

PROPOSITION 4.10. *Assume* $\mathbb{R}$-rank $G = 2$. *Let*

- L_1 *and* L_2 *be the two walls of* A^+, *and*
- Γ *be a crystallographic group for* G/H.

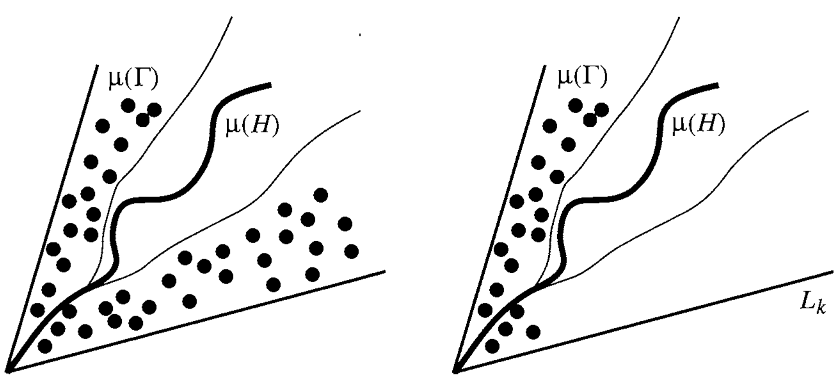
If H is not compact, then there exists $k \bullet \{1, 2\}$, *such that, for every compact subset C of A, the intersection $\mu(\Gamma) \cap L_k C$ is finite.*

Proof. (cf. Figure 4.1). Suppose there is a compact subset C of A, such that each of $\mu(\Gamma) \cap L_1 C$ and $\mu(\Gamma) \cap L_2 C$ is infinite. (This will lead to a contradiction.) Let F be a (symmetric) finite generating set for Γ (see 4.9). We may assume C is so large that $\mu(\gamma F) \subset \mu(\gamma)C$ for every $\gamma \in \Gamma$ (see 2.12). We may also assume that C is convex and symmetric.

Because Γ acts properly on G/H, there is a compact subset $\mathcal{C}$ of A, such that $\mu(H)C \cap \mu(\Gamma) \subset \mathcal{C}$ (see 2.17). Furthermore, we may assume that $\mu(L_1)C \cap \mu(L_2)C \subset \mathcal{C}$.

Let

- $M = \bigcup_{\gamma \in \Gamma} \mu(\gamma)C \setminus \mathcal{C}$,
- M_1 be the union of all the connected components of M that contain a point of L_1, and
- $A_1 = \Gamma \cap \mu^{-1}(M_1)$.



(a) (b)

Figure 4.1. Proposition 4.10: (a) $\mu(\Gamma)$ cannot be on both sides of $\mu(H)$, because Γ has only one end. (b) Therefore, $\mu(\Gamma)$ stays away from L_k.

Then A_1 is infinite (because $\mu(\Gamma) \cap L_1 C$ is infinite). Also, for any $\gamma \in A_1$ and $f \in F$, we have $\mu(\gamma f) \in \mu(\gamma)C$, so $\gamma f \in A_1 \cup \mu^{-1}(\mathcal{C})$. Since Γ has only one end (see 4.9), this implies $\Gamma \smallsetminus A_1$ is finite (see 4.4). Because $\mu(\Gamma) \cap L_2 C$ is infinite, we conclude that $M_1 \cap L_2 \neq \emptyset$.

Because $\mu(H)$ separates L_1 from L_2, and every connected component of M_1 contains a point of L_1, we conclude that $\mu(H) \cap M_1 \neq \emptyset$. This contradicts the fact that $\mu(H)C \cap \mu(\Gamma) \subset \mathcal{C}$. ∎

COROLLARY 4.11. *Assume* $\mathbb{R}$-rank $G = 2$. *Let*

(1) L_1 *and* L_2 *be the two walls of* A^+; *and*
(2) H_1 *and* H_2 *be closed, connected, nontrivial subgroups of* G,

such that

$$\mu(H_k) \approx L_k$$

for $k = 1, 2$. *If* H *is not compact, then any crystallographic group for* G/H *acts properly discontinuously on either* G/H_1 *or* G/H_2.

Proof. Suppose Γ acts properly discontinuously on *neither* G/H_1 *nor* G/H_2. (This will lead to a contradiction.) From Proposition 2.17, we know there is a compact subset C of A, such that each of $\mu(\Gamma) \cap \mu(H_1)C$ and $\mu(\Gamma) \cap \mu(H_2)C$ is infinite. Then, since $\mu(H_k) \approx L_k$, we may assume (by enlarging C) that each of $\mu(\Gamma) \cap L_1 C$ and $\mu(\Gamma) \cap L_2 C$ is infinite. This contradicts the conclusion of Proposition 4.10. □

COROLLARY 4.12. *Assume* $\mathbb{R}$-rank $G = 2$. *Let*

(1) L_1 *and* L_2 *be the two walls of* A^+; *and*
(2) H_1 *and* H_2 *be closed, connected, nontrivial subgroups of* G;

such that

$$\mu(H_k) \approx L_k$$

for $k = 1, 2$. *If* G/H *has a tessellation, and* H *is not compact, then*

$$d(H) \geqslant \min\{d(H_1), d(H_2)\}.$$

Proof. The desired conclusion is obtained by combining Corollary 4.11 with Theorem 4.1(1). □

Remark 4.13. For $G = \mathrm{SL}(3, \mathbb{R})$, there does not exist a connected subgroup H_k, such that $\mu(H_k) \approx L_k$ (in the notation of the proof of (6.3), note that $\tau(L_k) \not\approx L_k$). Thus, Corollary 4.12 does not provide a lower bound on $d(H)$ in this case.

5. One-Dimensional Subgroups

Although the following conjecture does not seem to have been stated previously in the literature, it is perhaps implicit in [OW3].

CONJECTURE 5.1. *If $d(H) = 1$, then G/H does not have a tessellation.*

In this section, we use known results to establish that the conjecture is valid in two cases: if either $\mathbb{R}$-rank $G \leqslant 2$ (see 5.9) or G is almost simple (see 5.13). Each of these illustrates a general theorem: for groups of real rank two, the conjecture follows from a theorem of Y. Benoist and F. Labourie that is based on differential geometry; H. Oh and D. Witte observed that, for simple groups, the conjecture follows from a theorem of G. A. Margulis that is based on unitary representation theory.

The following example is the only case of Conjecture 5.1 that is needed in later sections. (It is used in the proof of Theorem 1.8.) Because $\mathbb{R}$-rank $(\mathrm{SL}(3, \mathbb{F})) = 2$ and $\mathrm{SL}(3, \mathbb{F})$ is almost simple, this example is covered both by the theorem of Benoist-Labourie and by the theorem of Margulis, but it would be interesting to have an easy proof.

PROPOSITION 5.2 (see 5.9 or 5.13). *Assume $G = \mathrm{SL}(3, \mathbb{F})$, for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let*

$$H_1 = \left\{ \left. \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} \right| t \in \mathbb{R} \right\} \subset G. \tag{5.3}$$

Then G/H_1 does not have a tessellation.

Let us begin with an easy observation.

LEMMA 5.4. *If $d(H) = 1$ and $\mathbb{R}$-rank $G = 1$, then G/H does not have a tessellation.*

Proof. We may assume $H \subset AN$ (see 3.5). From (2.16), we know that H is a Cartan-decomposition subgroup, so Lemma 2.9 implies that G/H must be compact; thus, the trivial group e is a crystallographic group for G/H. However, since

$$d(G) = \dim A + \dim N \geqslant 1 + 1 > 1 = d(H)$$

(see 1.15), and e acts properly discontinuously on G/G, this contradicts Theorem 4.1(1). □

Remark 5.5. The proof of Lemma 5.4 shows that the dimension of every connected, cocompact subgroup of G is at least $d(G)$. This is a result of M. Goto and H.-C. Wang [GtW, (1.2), p. 263].

5A. THE THEOREM OF BENOIST AND LABOURIE

THEOREM 5.6 ([BL, Corollary 3]). *If*

- *H is reductive,*
- *neither H nor G/H is compact, and*
- *H contains a nontrivial element of A in its center,*

then G/H does not have a tessellation.

The following special case is sufficient for our needs.

COROLLARY 5.7. *If H is a one-dimensional subgroup of A, then G/H does not have a tessellation.*

NOTATION 5.8 ([Ben, p. 320]). Let τ be the opposition involution in A^+; that is, for $a \in A^+$, $\tau(a) = \mu(a^{-1})$ is the unique element of A^+ that is conjugate (under an element of the Weyl group) to a^{-1}. Thus, for all $h \in G$, we have

$$\mu(h^{-1}) = \tau(\mu(h)).$$

See (6.1) for an explicit description of the opposition involution in $G = \mathrm{SL}(3, \mathbb{R})$. For some groups, such as $G = \mathrm{SO}(2, n)$, we have $\mu(h^{-1}) = \mu(h)$ for all $h \in G$ (see 8.14); in such a case, the opposition involution is simply the identity map on A^+.

COROLLARY 5.9. *If $d(H) = 1$ and $\mathbb{R}$-rank $G \leqslant 2$, then G/H does not have a tessellation.*

Proof. Suppose Γ is a crystallographic group for G/H. (This will lead to a contradiction.)

From (5.4), we know $\mathbb{R}$-rank $G = 2$. Let L_1 and L_2 be the two walls of A^+ and, for $k \in \{1, 2\}$, let $H_k = L_k \cup L_k^{-1}$. Because L_k is a ray (that is, a one-parameter semigroup), it is clear that H_k is a subgroup of A.

From Proposition 4.10, we know that there is some $k \bullet \{1, 2\}$, such that

$$\mu(\Gamma) \cap L_k C \text{ is finite,}$$

for every compact subset C of A. Since $\Gamma = \Gamma^{-1}$, we have $\tau(\mu(\Gamma)) = \mu(\Gamma)$, so this implies that

$$\mu(\Gamma) \cap \tau(L_k) C \text{ is finite,}$$

for every compact subset C of A. Also, because $L_k \subset A^+$, we have $\mu(L_k) = L_k$, so

$$\mu(H_k) = \mu(L_k \cup L_k^{-1}) = \mu(L_k) \cup \tau(\mu(L_k)) = L_k \cup \tau(L_k).$$

Therefore

$$\mu(\Gamma) \cap \mu(H_k) C = (\mu(\Gamma) \cap L_k C) \cup (\mu(\Gamma) \cap \tau(L_k) C) \text{ is finite,}$$

for every compact subset C of A. Hence, Corollary 2.17 implies that Γ acts properly discontinuously on G/H_k. Then, because $d(H) = 1 = d(H_k)$ (see 1.16), Theorem 4.1(2) implies that G/H_k has a tessellation. This contradicts Corollary 5.7. □

5B. THE THEOREM OF MARGULIS

DEFINITION 5.10 ([Mr2, Definition 2.2, Remark 2.2]). The subgroup H is *tempered* in G if there exists a (positive) function $f \bullet L^1(H)$ (with respect to a left-invariant Haar measure on H), such that whenever

- π is a unitary representation of G, and
- 0 is the only vector that is fixed by every element of $\pi(G^\circ)$,

we have

$$|\langle \pi(h)\phi \mid \psi \rangle| \leqslant f(h)\|\phi\|\|\psi\|$$

for all $h \in H$ and all K-fixed vectors ϕ and ψ.

For many examples of tempered subgroups of simple Lie groups, see [Oh]. We need only the following example, which is a consequence of important work of Howe [How, Corollary 7.2 and Section 7] and Cowling [Cow, Theorem 2.4.2] on the decay of matrix coefficients (see also [KS, Section 3, p. 140]). (The assumption that $\mathbb{R}\text{-rank}\, G \geqslant 2$ can be relaxed: it suffices to assume that G is not locally isomorphic to $\mathrm{SO}(1, n)$ or $\mathrm{SU}(1, n)$.)

LEMMA 5.11 ([OW3, Proposition 3.7]). *Assume G is simple, and* $\mathbb{R}\text{-rank}\, G \geqslant 2$. *If H is a one-parameter subgroup of AN, then either*

(1) *H is tempered; or*
(2) *$H \subset N$.*

THEOREM 5.12 ([Mr2, Theorem 3.1]). *If H is noncompact and tempered, then G/H does not have a tessellation.*

COROLLARY 5.13 ([OW3, Proposition 3.7]). *If $d(H) = 1$ and G is simple, then G/H does not have a tessellation.*

Proof. We may assume $H \subset AN$ (see 3.5), so $\dim H = d(H) = 1$ (see 1.16).

If $\mathbb{R}\text{-rank}\, G < 2$, then Lemma 5.4 applies.

If $H \subset N$, then Lemma 5.14 below applies.

If $\mathbb{R}\text{-rank}\, G \geqslant 2$ and $H \not\subset N$, then Lemma 5.11 implies that H is tempered, so Theorem 5.12 applies. □

LEMMA 5.14. *If $d(H) = 1$ and $H \subset N$, then G/H does not have a tessellation.*

Proof [OW3, Proposition 3.7]. We have $\dim H = d(H) = 1$ (see 1.16), so H is a connected, one-dimensional, unipotent subgroup. Hence, the Jacobson–Morosov Lemma [Hel, Theorem 9.7.4, p. 432] implies that there exists a connected, closed subgroup H_1 of G, such that H_1 contains H, and H_1 is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$. Then H is a Cartan decomposition subgroup of H_1 (see 2.16), so there is a compact subset C of A, such that $\mu(H_1) \subset \mu(H)C$ (see 2.12). Also, we have $d(H_1) = 2 > 1 = d(H)$. Therefore, Theorem 4.2 applies. □

6. Homogeneous Spaces of $\mathrm{SL}(3, \mathbb{R})$, $\mathrm{SL}(3, \mathbb{C})$, and $\mathrm{SL}(3, \mathbb{H})$

Benoist [Ben, Corollary 1] and Margulis (unpublished) proved (independently) that $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(2, \mathbb{R})$ does not have a tessellation. Much earlier, Kobayashi (cf. [Kb2, Example 7]) had shown that the conclusion is true if $\mathbb{R}$ replaced with either $\mathbb{C}$ or $\mathbb{H}$. Using Benoist's method, Oh and Witte [OW3, Proposition 1.10] generalized the Benoist-Margulis result by replacing $\mathrm{SL}(2,\mathbb{R})$ with any closed, connected subgroup H, such that neither H nor $\mathrm{SL}(3,\mathbb{R})/H$ is compact. (The same argument applies when $\mathbb{R}$ is replaced with either $\mathbb{C}$ or $\mathbb{H}$, so the Kobayashi result also generalizes.) However, the proof of Benoist (which applies in a more general context) relies on a somewhat lengthy argument to establish one particular lemma. Here, we adapt Benoist's method to obtain a short proof of Theorem 1.8 that avoids any appeal to the lemma.

NOTATION 6.1. Assume $G = \mathrm{SL}(3,\mathbb{F})$, for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

- Let τ be the opposition involution in A^+ (see 5.8);
- Let $B^+ = \{a \in A^+ \mid \tau(a) = a\}$.

More concretely, we have

$$A^+ = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle| \begin{array}{c} a_1, a_2, a_3 \in \mathbb{R}^+, \\ a_1 a_2 a_3 = 1, \\ a_1 \geqslant a_2 \geqslant a_3 \end{array} \right\};$$

$$\tau \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = \begin{pmatrix} a_3^{-1} & 0 & 0 \\ 0 & a_2^{-1} & 0 \\ 0 & 0 & a_1^{-1} \end{pmatrix};$$

$$B^+ = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \middle| \, a \geqslant 1 \right\}.$$

LEMMA 6.2. *If $G = \mathrm{SL}(3,\mathbb{F})$ and $d(H) = 1$, then G/H does not have a tessellation.*

Proof. Since $\mathbb{R}$-rank $G = 2$, the desired conclusion follows from Corollary 5.9; since G is simple, it also follows from Corollary 5.13. However, we give a proof that requires only the special case described in Proposition 5.2, rather than the full strength of (5.9) or (5.13).

Suppose Γ is a crystallographic group for G/H. (This will lead to a contradiction.) Let L_1 and L_2 be the two walls of A^+. From Proposition 4.10, we know that there exists $k \in \{1, 2\}$, such that $\mu(\Gamma)C \cap L_k$ is finite, for every compact subset C of A.

Because $\Gamma^{-1} = \Gamma$, we have $\tau(\mu(\Gamma)) = \mu(\Gamma)$. On the other hand, τ interchanges L_1 and L_2. Thus, the preceding paragraph implies that $\mu(\Gamma)C \cap (L_1 \cup L_2)$ is finite, for every compact subset C of A.

For H_1 as in (5.3), we have $\mu(H_1) = L_1 \cup L_2$, so the conclusion of the preceding paragraph implies that Γ acts properly discontinuously on G/H_1 (see 2.17). Now Theorem 4.1(2) implies $\Gamma \backslash G/H_1$ is compact; thus, G/H_1 has a tessellation. This contradicts Proposition 5.2. $\square$

For completeness, we include the proof of the following simple proposition.

PROPOSITION 6.3 ([OW3, Proposition 7.3]). *Assume* $G = \mathrm{SL}(3, \mathbb{F})$. *If* H *is a closed, connected subgroup of* AN *with* $\dim H \geqslant 2$, *then* $B^+ \subset \mu(H)$.

Proof (cf. Figure 6.2 and proof of 8.19). Since $H \subset AN$ and $\dim H \geqslant 2$, it is easy to construct a continuous, proper map Φ: $[0, 1] \times \mathbb{R}^+ \to H$ such that $\Phi(1, t) = \Phi(0, t)^{-1}$, for all $t \in \mathbb{R}^+$ (cf. Figure 6.1). For example, choose two linearly independent elements u and v of $\mathfrak{h}$, and define

$$\Phi(s, t) = \exp(t \cos(\pi s)u + t \sin(\pi s)v).$$

If we identify A with its Lie algebra $\mathfrak{a}$, then A^+ is a convex cone in $\mathfrak{a}$ and the opposition involution τ is the reflection in A^+ across the ray B^+. Thus, for any $a \in A^+$,

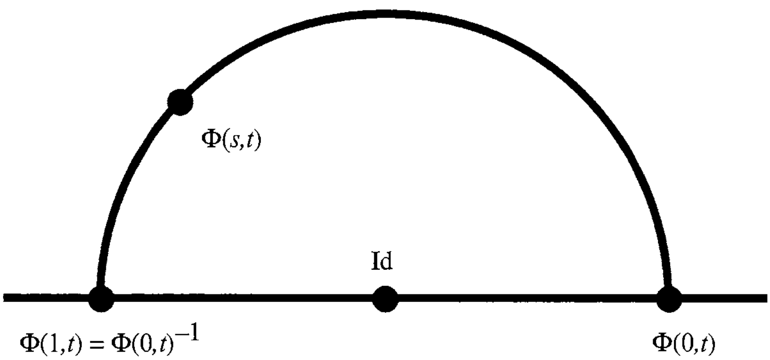


Figure 6.1. Construction of $\Phi(s, t)$.

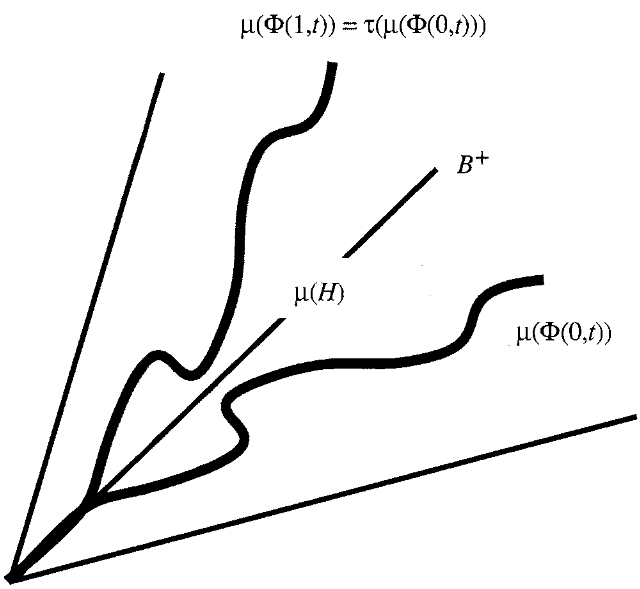


Figure 6.2. Proposition 6.3: $\mu(H)$ is on both sides of B^+, so it must contains B^+.

the points a and $\tau(a)$ are on opposite sides of B^+, so any continuous curve in A^+ from a to $\tau(a)$ must intersect B^+. In particular, for each $t \in \mathbb{R}^+$, the curve

$$\{\mu(\Phi(s, t)) \mid 0 \leqslant s \leqslant 1\}$$

from $\mu(\Phi(0, t))$ to $\mu(\Phi(1, t))$ must intersect B^+. Thus, we see, from an elementary continuity argument, that $\mu\left[\Phi\left([0, 1] \times \mathbb{R}^+\right)\right]$ contains B^+. Therefore, B^+ is contained in $\mu(H)$. ∎

Proof of Theorem 1.8 (cf. Figure 6.3 and proof of 4.12). Suppose Γ is a crystallographic group for G/H. (This will lead to a contradiction.) We may assume $H \subset AN$ (see 3.5).

Let F be a (symmetric) finite generating set for Γ, and choose a compact, convex, symmetric subset C of A so large that $\mu(\gamma F) \subset \mu(\gamma)C$ for every $\gamma \in \Gamma$ (see 2.12).

From Lemma 6.2, we know that $\dim H \geqslant 2$, so Proposition 6.3 implies that $B^+ \subset \mu(H)$. Then, because Γ acts properly on G/H, we conclude that $\mu(\Gamma) \cap B^+C$ is finite (see 2.17). Since μ is a proper map, this implies that $\Gamma \cap \mu^{-1}(B^+C)$ is finite.

Let A_1 and A_2 be the two components of $A^+ \backslash B^+$. Because $\Gamma^{-1} = \Gamma$, we know that $\tau(\mu(\Gamma)) = \mu(\Gamma)$. Then, because τ interchanges A_1 and A_2, we conclude that

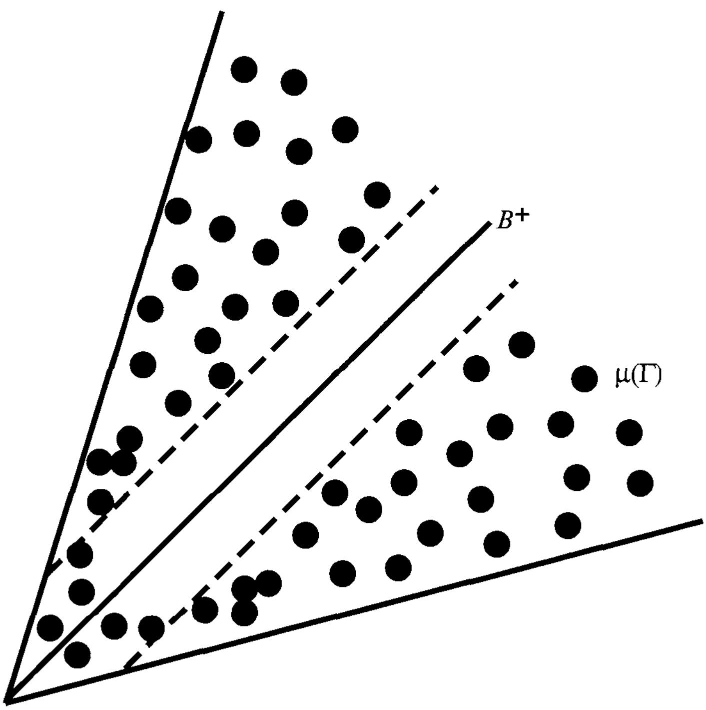


Figure 6.3. Proof of Theorem 1.8: $\mu(\Gamma)$ stays away from B^+, because $B^+ \subset \mu(H)$. Also, half of $\mu(\Gamma)$ is on each side of B^+, because $\tau(\mu(\Gamma)) = \mu(\Gamma)$. This contradicts the fact that Γ has only one end.

$\tau(\mu(\Gamma) \cap A_1) = \mu(\Gamma) \cap A_2$. Therefore, $\mu(\Gamma) \cap A_1$ and $\mu(\Gamma) \cap A_2$ have the same cardinality, so they must both be infinite. So

$$\text{each of } \Gamma \cap \mu^{-1}(A_1) \quad \text{and} \quad \Gamma \cap \mu^{-1}(A_2) \text{ is infinite.}$$

Because Γ has only one end (see 4.9), this implies there exist

$$\gamma \in \left(\Gamma \cap \mu^{-1}(A_1)\right) \smallsetminus \mu^{-1}(B^+C),$$

such that

$$\gamma f \in \left(\Gamma \cap \mu^{-1}(A_2)\right) \smallsetminus \mu^{-1}(B^+C), \tag{6.4}$$

for some $f \in F$. Then $\mu(\gamma) \in A_1$, $\mu(\gamma f) \in A_2$, and

$$\mu(\gamma f) \in \mu(\gamma F) \subset \mu(\gamma)C.$$

Using the fact that C is symmetric and the fact that C contains the identity element e, we conclude that

$$\mu(\gamma) \in (\mu(\gamma f)C) \cap A_1 \quad \text{and} \quad \mu(\gamma f) \in (\mu(\gamma f)C) \cap A_2;$$

therefore $\mu(\gamma f)C$ intersects both A_1 and A_2. Since B^+ separates A_1 from A_2, and C is connected, this implies that $\mu(\gamma f)C$ intersects B^+; hence $\mu(\gamma f) \in B^+C$. This contradicts the fact that $\gamma f \notin \mu^{-1}(B^+C)$ (see 6.4). □

7. Explicit Coordinates on SO(2, *n*) and SU(2, *n*)

From this point on, we focus almost entirely on $\mathrm{SO}(2,n)$ and $\mathrm{SU}(2,n)$. (The only exception is that some of the examples constructed in Section 9 are for other groups.) In this section, we define the group $\mathrm{SU}(2,n;\mathbb{F})$, which allows us to provide a fairly unified treatment of $\mathrm{SO}(2,n)$ and $\mathrm{SU}(2,n)$ in later sections.

7A. THE GROUP SU(2, *n*; 𝔽)

NOTATION 7.1.

- We use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.
- Let $q = \dim_{\mathbb{R}} \mathbb{F}$, so $q \in \{1, 2\}$.
- We use $\mathbb{F}_{\mathrm{imag}}$ to denote the purely imaginary elements of $\mathbb{F}$, so

$$\mathbb{F}_{\mathrm{imag}} = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{R}, \\ i\mathbb{R} & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

- For $\phi \in \mathbb{F}$, there exist unique $\operatorname{Re}\phi \in \mathbb{R}$ and $\operatorname{Im}\phi \in \mathbb{F}_{\mathrm{imag}}$, such that $\phi = \operatorname{Re}\phi + \operatorname{Im}\phi$. (*Warning:* in our notation, the imaginary part of $a + bi$ is bi, *not* b.)

- For $\phi \in \mathbb{F}$, we use $\bar{\phi}$ to denote the conjugate $\operatorname{Re}\phi - \operatorname{Im}\phi$ of ϕ. (If $\mathbb{F} = \mathbb{R}$, then $\bar{\phi} = \phi$.)
- For a row vector $x \in \mathbb{F}^{n-2}$, or, more generally, for any matrix x with entries in $\mathbb{F}$, we use $x^\dagger$ to denote the conjugate-transpose of x.

NOTATION 7.2. For

$$J = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & \mathrm{Id} & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathrm{SL}(n+2, \mathbb{F}),$$

we define

$$\mathrm{SU}(2, n; \mathbb{F}) = \{ g \in \mathrm{SL}(n+2, \mathbb{F}) \mid gJg^\dagger = J \}$$

and

$$\mathfrak{su}(2, n; \mathbb{F}) = \{ u \in \mathfrak{sl}(n+2, \mathbb{F}) \mid uJ + Ju^\dagger = 0 \}.$$

Then:

- $\mathrm{SU}(2, n; \mathbb{R})$ is a realization of $\mathrm{SO}(2, n)$,
- $\mathrm{SU}(2, n; \mathbb{C})$ is a realization of $\mathrm{SU}(2, n)$, and
- $\mathfrak{su}(2, n; \mathbb{F})$ is the Lie algebra of $\mathrm{SU}(2, n; \mathbb{F})$.

We choose

- A to consist of the diagonal matrices in $\mathrm{SU}(2, n; \mathbb{F})$ that have nonnegative real entries,
- N to consist of the upper-triangular matrices in $\mathrm{SU}(2, n; \mathbb{F})$ with only 1's on the diagonal, and
- $K = \mathrm{SU}(2, n; \mathbb{F}) \cap \mathrm{SU}(n+2)$.

A straightforward matrix calculation shows that the Lie algebra of AN is

$$\mathfrak{a} + \mathfrak{n} = \left\{ \begin{pmatrix} t_1 & \phi & x & \eta & \mathrm{x} \\ 0 & t_2 & y & \mathrm{y} & -\bar{\eta} \\ 0 & 0 & 0 & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & -t_2 & -\bar{\phi} \\ 0 & 0 & 0 & 0 & -t_1 \end{pmatrix} \middle| \begin{array}{c} t_1, t_2 \in \mathbb{R}, \\ \phi, \eta \in \mathbb{F}, \\ x, y \in \mathbb{F}^{n-2}, \\ \mathrm{x}, \mathrm{y} \in \mathbb{F}_{\mathrm{imag}} \end{array} \right\}. \tag{7.3}$$

Remark 7.4. From (7.3), we see that the first two rows of any element of $\mathfrak{a} + \mathfrak{n}$ are sufficient to determine the entire matrix. In fact, it is also not necessary to specify the last entry of the second row of the matrix.

Remark 7.5. From (1.15) and (7.3), we see that $d(SU(2,n;\mathbb{F})) = \dim(\mathfrak{a}+\mathfrak{n}) = 2qn$.

NOTATION 7.6. Because N is simply connected and nilpotent, the exponential map is a diffeomorphism from $\mathfrak{n}$ to N (indeed, its inverse, the logarithm map, is a polynomial [Hc1, Theorem 8.1.1, p. 107]), so each element of N has a unique representation in the form $\exp u$ with $u \in \mathfrak{n}$. Thus, each element h of N determines corresponding values of ϕ, x, y, η, x and y (with $t_1 = t_2 = 0$). We write

$$\phi_h, x_h, y_h, \eta_h, \mathsf{x}_h, \mathsf{y}_h$$

for these values.

NOTATION 7.7. We let α and β be the simple real roots of $SU(2,n;\mathbb{F})$, defined by

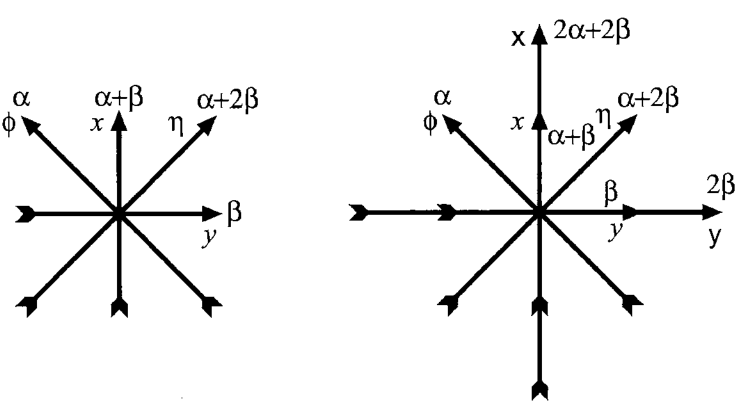
$$\alpha(a) = a_{1,1}/a_{2,2} \quad \text{and} \quad \beta(a) = a_{2,2},$$

for a (diagonal) element a of A. Thus, the positive real roots (see Figure 7.1) are

$$\begin{array}{ll} \alpha, \beta, \alpha+\beta, \alpha+2\beta, & \text{if } \mathbb{F} = \mathbb{R}, \\ \alpha, \beta, \alpha+\beta, \alpha+2\beta, 2\beta, 2\alpha+2\beta & \text{if } \mathbb{F} = \mathbb{C}. \end{array}$$

Concretely:

- the root space $\mathfrak{n}_\alpha$ is the ϕ-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{n}_\beta$ is the y-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{n}_{\alpha+\beta}$ is the x-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{n}_{\alpha+2\beta}$ is the η-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{n}_{2\beta}$ is the y-subspace in $\mathfrak{n}$ (this is 0 if $\mathbb{F} = \mathbb{R}$), and
- the root space $\mathfrak{n}_{2\alpha+2\beta}$ is the x-subspace in $\mathfrak{n}$ (this is 0 if $\mathbb{F} = \mathbb{R}$).



(a) (b)

Figure 7.1. The real root systems of (a) $SU(2,n;\mathbb{R}) = SO(2,n)$ and (b) $SU(2,n;\mathbb{C}) = SU(2,n)$.

DEFINITION 7.8. Let

$$\begin{aligned}\mathfrak{d} &= \mathfrak{n}_{\alpha+2\beta} + \mathfrak{n}_{2\alpha+2\beta} + \mathfrak{n}_{2\beta}\\ &= \{z \in \mathfrak{n} \mid \phi_z = 0, x_z = y_z = 0\}\\ &= \left\{ \left(\begin{matrix} 0 & 0 & 0 & \eta & \mathsf{x} \\ & 0 & 0 & \mathsf{y} & -\bar{\eta} \\ & & \cdots & & \end{matrix} \right) \middle| \begin{matrix} \eta \in \mathbb{F}, \\ \mathsf{x}, \mathsf{y} \in \mathbb{F}_{\text{imag}} \end{matrix} \right\}.\end{aligned}$$

and, for a given Lie algebra $\mathfrak{h} \subset \mathfrak{n}$,

$$\mathfrak{d}_{\mathfrak{h}} = \mathfrak{d} \cap \mathfrak{h}.$$

Note that if $\phi_u = 0$ for every $u \in \mathfrak{h}$, then $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{d}_{\mathfrak{h}}$ and $\mathfrak{d}_{\mathfrak{h}}$ is contained in the center of $\mathfrak{h}$ (cf 7.22).

Remark 7.9. By definition (see 2.10), we have

$$A^+ = \{a \in A \mid \alpha(a) \geqslant 1, \beta(a) \geqslant 1\}.$$

Therefore, from the definition of α and β (see 7.7), we see that

$$A^+ = \{a \in A \mid a_{1,1} \geqslant a_{2,2} \geqslant 1\}. \tag{7.10}$$

Remark 7.11. For $\mathbb{F} = \mathbb{H}$, the division algebra of real quaternions, the group $\mathrm{SU}(2, n; \mathbb{H})$ is a realization of $\mathrm{Sp}(2, n)$. Most of the work in this paper carries over, but the upper bound on $\dim H$ given in Theorem 10.14 is not sharp in this case (and it does not seem to be easy to improve this result to obtain a sharp bound). Thus, we have not obtained any interesting conclusions about the nonexistence of tessellations of homogeneous spaces of $\mathrm{Sp}(2, n)$.

7B. THE SUBGROUPS $\mathrm{SU}(1, n; \mathbb{F})$ AND $\mathrm{Sp}(1, m; \mathbb{F})$

We now describe how the four important families of homogeneous spaces of Example 1.11 are realized in terms of $\mathrm{SU}(2, n; \mathbb{F})$.

DEFINITION 7.12. Let

- $\mathrm{SU}(1, n; \mathbb{R}) = \mathrm{SO}(1, n)$;
- $\mathrm{Sp}(1, n; \mathbb{R}) = \mathrm{SU}(1, n)$;
- $\mathrm{SU}(1, n; \mathbb{C}) = \mathrm{SU}(1, n)$; and
- $\mathrm{Sp}(1, n; \mathbb{C}) = \mathrm{Sp}(1, n)$.

Then, for an appropriate choice of the embeddings in Example 1.11, we have

$$\mathfrak{su}(1, n; \mathbb{F}) \cap (\mathfrak{a} + \mathfrak{n}) = \left\{ \left(\begin{matrix} t & \phi & x & \phi & \mathsf{x} \\ 0 & 0 & 0 & 0 & -\bar{\phi} \\ & & \cdots & & \end{matrix} \right) \middle| \begin{matrix} t \in \mathbb{R}, \\ \phi \in \mathbb{F}, \\ x \in \mathbb{F}^{n-2}, \\ \mathsf{x} \in \mathbb{F}_{\text{imag}} \end{matrix} \right\}. \tag{7.13}$$

and (if $2m \leqslant n$) we have

$$\mathfrak{sp}(1,m;\mathbb{F})\cap(\mathfrak{a}+\mathfrak{n}) = \left\{ \begin{pmatrix} t & 0 & x_1 & x_2 & x_3 & x_4 & \cdots & x_{2m-3} & x_{2m-2} & 0 & \cdots & 0 & \eta & \mathsf{x} \\ 0 & t & -\overline{x_2} & \overline{x_1} & -\overline{x_4} & \overline{x_3} & \cdots & -\overline{x_{2m-2}} & \overline{x_{2m-3}} & 0 & \cdots & 0 & -\mathsf{x} & -\bar{\eta} \\ & & & & & & \cdots & & & & & & & \end{pmatrix} \,\middle|\, \begin{array}{c} t\in\mathbb{R}, \\ x_j\in\mathbb{F}, \\ \eta\in\mathbb{F}, \\ \mathsf{x}\in\mathbb{F}_{\mathrm{imag}} \end{array} \right\}. \tag{7.14}$$

Remark 7.15. From (1.15), (7.13) and (7.14), we see that

- $d(\mathrm{SU}(1,n;\mathbb{F})) = \dim(\mathfrak{su}(1,n;\mathbb{F}) \cap (\mathfrak{a}+\mathfrak{n})) = qn$ and
- $d(\mathrm{Sp}(1,m;\mathbb{F})) = \dim(\mathfrak{sp}(1,m;\mathbb{F}) \cap (\mathfrak{a}+\mathfrak{n})) = 2qm.$

7C. FORMULAS FOR EXPONENTIALS AND BRACKETS

The arguments in later sections often require the calculation of exp u, for some $u \in \mathfrak{n}$, or of $[u,v]$, for some $u, v \in \mathfrak{n}$. We now provide these calculations for the reader's convenience.

Remark 7.16. For

$$u = \begin{pmatrix} 0 & \phi & x & \eta & \mathsf{x} \\ 0 & 0 & y & \mathsf{y} & -\bar{\eta} \\ 0 & 0 & 0 & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & 0 & -\bar{\phi} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{n},$$

we have

$$\exp(u) = \begin{pmatrix} 1 & \phi & x+\frac{1}{2}\phi y & \eta-\frac{1}{2}xy^\dagger+\frac{1}{2}\phi\mathsf{y}-\frac{1}{6}\phi|y|^2 & \begin{array}{c} -\frac{1}{2}|x|^2-\mathrm{Re}(\phi\bar{\eta})+\frac{1}{24}|\phi|^2|y|^2 \\ +\left(\mathsf{x}-\frac{1}{6}\phi\mathsf{y}\bar{\phi}+\frac{1}{3}\mathrm{Im}(xy^\dagger\bar{\phi})\right) \end{array} \\ 0 & 1 & y & \mathsf{y}-\frac{1}{2}|y|^2 & -\bar{\eta}-\frac{1}{2}yx^\dagger-\frac{1}{2}\mathsf{y}\bar{\phi}+\frac{1}{6}|y|^2\bar{\phi} \\ 0 & 0 & \mathrm{Id} & -y^\dagger & -x^\dagger+\frac{1}{2}y^\dagger\bar{\phi} \\ 0 & 0 & 0 & 1 & -\phi^\dagger \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.17}$$

When $\phi = 0$, this simplifies to

$$\exp(u) = \begin{pmatrix} 1 & 0 & x & \eta-\frac{1}{2}xy^\dagger & \mathsf{x}-\frac{1}{2}|x|^2 \\ 0 & 1 & y & \mathsf{y}-\frac{1}{2}|y|^2 & -\bar{\eta}-\frac{1}{2}yx^\dagger \\ 0 & 0 & \mathrm{Id} & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.18}$$

Similarly, when $y = 0$, we have

$$\exp(u) = \begin{pmatrix} 1 & \phi & x & \eta + \frac{1}{2}\phi\mathsf{y} & \begin{matrix} -\frac{1}{2}|x|^2 - \mathrm{Re}(\phi\bar{\eta}) \\ + \left(\mathsf{x} - \frac{1}{6}\phi\mathsf{y}\bar{\phi}\right) \end{matrix} \\ 0 & 1 & 0 & \mathsf{y} & -\bar{\eta} - \frac{1}{2}\mathsf{y}\bar{\phi} \\ 0 & 0 & \mathrm{Id} & 0 & -x^{\dagger} \\ 0 & 0 & 0 & 1 & -\phi^{\dagger} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.19}$$

Remark 7.20. For

$$u = \begin{pmatrix} 0 & \phi & x & \eta & \mathsf{x} \\ & 0 & y & \mathsf{y} & -\bar{\eta} \\ & & \cdots & & \end{pmatrix} \quad \text{and} \quad \tilde{u} = \begin{pmatrix} 0 & \tilde{\phi} & \tilde{x} & \tilde{\eta} & \tilde{\mathsf{x}} \\ & 0 & \tilde{y} & \tilde{\mathsf{y}} & -\bar{\tilde{\eta}} \\ & & \cdots & & \end{pmatrix}, \tag{7.21}$$

we have

$$[u, \tilde{u}] = \begin{pmatrix} 0 & 0 & \phi\tilde{y} - \tilde{\phi}y & -x\tilde{y}^{\dagger} + \tilde{x}y^{\dagger} + \phi\tilde{\mathsf{y}} - \tilde{\phi}\mathsf{y} & -2\,\mathrm{Im}(x\tilde{x}^{\dagger} + \phi\bar{\tilde{\eta}} - \tilde{\phi}\bar{\eta}) \\ & 0 & 0 & -2\,\mathrm{Im}(y\tilde{y}^{\dagger}) & \tilde{y}x^{\dagger} - y\tilde{x}^{\dagger} + \tilde{\mathsf{y}}\bar{\phi} - \mathsf{y}\bar{\tilde{\phi}} \\ & & & \cdots & \end{pmatrix}. \tag{7.22}$$

Remark 7.23. [Var, Eq. (2.13.8), p. 104]. For $u, v \bullet \mathfrak{h}$, we have

$$\exp(-v)u\exp(v) = u + [u, v] + \frac{1}{2}\big[[u, v], v\big] + \frac{1}{3!}\big[[[u, v], v], v\big] + \cdots.$$

Combining this with (7.22) allows us to calculate the effect of conjugating by an element of N.

For example, suppose $u \bullet \mathfrak{n}$, with $\phi_u = 0$ and $y_u = 0$, and suppose $v \bullet \mathfrak{n}_{\alpha+\beta}$. We see, from (7.22), that $\phi_{[u,v]} = 0$ and that $x_{[u,v]} = y_{[u,v]} = 0$, so $\big[[u, v], v\big] = 0$ (see 7.22). Therefore

$$\exp(-v)u\exp(v) = u + [u, v].$$

8. Calculating the Cartan Projection

Benoist [Ben. Lemma 2.4] showed that calculating values of the Cartan projection μ is no more difficult than calculating the norm of a matrix (see 8.11). In this section, we describe this elegant method and some of its consequences, in the special case $G = \mathrm{SU}(2, n; \mathbb{F})$.

8.1. STANDING ASSUMPTIONS

Throughout this section, we assume $G = \mathrm{SU}(2, n; \mathbb{F})$.

8A. THE BASIC DEFINITIONS

NOTATION 8.2. We employ the usual Big Oh and little oh notation: for functions f_1, f_2 on a subset X of G, we say

$$f_1 = \mathrm{O}(f_2) \quad \text{for } h \in X$$

if there is a constant C, such that, for all $h \in X$ with $\|h\|$ large, we have $\|f_1(h)\| \leqslant C\|f_2(h)\|$. (The values of each f_j are assumed to belong to some finite-dimensional normed vector space, typically either $\mathbb{C}$ or a space of complex matrices. Which particular norm is used does not matter, because all norms are equivalent up to a bounded factor.) We say

$$f_1 = \mathrm{o}(f_2) \quad \text{for } h \in X$$

if $\|f_1(h)\|/\|f_2(h)\| \to 0$ as $h \to \infty$. (We use $h \to \infty$ to mean $\|h\| \to \infty$.) Also, we write $f_1 \asymp f_2$ if $f_1 = \mathrm{O}(f_2)$ and $f_2 = \mathrm{O}(f_1)$.

We use the following norm on $\mathrm{SU}(2, n; \mathbb{F})$, because it is easy to calculate. The reader is free to make a different choice, at the expense of changing $=$ to $\asymp$ in a few of the calculations.

DEFINITION 8.3. For $h \in \mathrm{SU}(2, n; \mathbb{F})$, we define $\|h\|$ to be the maximum absolute value among the matrix entries of h. That is,

$$\|h\| = \max_{1 \leqslant j,k \leqslant n+2} |h_{j,k}|.$$

DEFINITION 8.4. Define $\rho\colon \mathrm{SU}(2, n; \mathbb{F}) \to \mathrm{GL}(\mathbb{F}^{n+2} \wedge \mathbb{F}^{n+2})$ by $\rho(h) = h \wedge h$, so ρ is the second exterior power of the standard representation of $\mathrm{SU}(2, n; \mathbb{F})$. Thus, we may define $\|\rho(h)\|$ to be the maximum absolute value among the determinants of all the 2×2 submatrices of the matrix h. That is,

$$\|\rho(h)\| = \max_{1 \leqslant j,k,\ell,m \leqslant n+2} \left| \det\begin{pmatrix} h_{j,k} & h_{j,\ell} \\ h_{m,k} & h_{m,\ell} \end{pmatrix} \right|.$$

From (7.17), (7.18), and (7.19), it is clear that the 2×2 minor in the top right corner is often larger than the other 2×2 minors, so we give it a special name.

DEFINITION 8.5. For $h \in \mathrm{Mat}_{n+2}(\mathbb{F})$, define

$$\Delta(h) = \det\begin{pmatrix} h_{1,n+1} & h_{1,n+2} \\ h_{2,n+1} & h_{2,n+2} \end{pmatrix}.$$

8B. Y. BENOIST'S METHOD FOR USING MATRIX NORMS TO CALCULATE μ

LEMMA 8.6. *For $a \in A^+$, we have $\|a\| = a_{1,1}$ and $\|\rho(a)\| = a_{1,1}a_{2,2}$.*

Proof. From (7.3), we see that

$$a_{j,j} = \begin{cases} 1 & \text{if } 3 \leqslant j \leqslant n, \\ 1/a_{2,2} & \text{if } j = n+1, \\ 1/a_{1,1} & \text{if } j = n+2. \end{cases} \tag{8.7}$$

Thus, from (7.10), we see that

$$a_{1,1} \geqslant a_{2,2} \geqslant a_{j,j}$$

for $j \geqslant 3$ (and, since a is diagonal, we have $a_{j,k} = 0$ for $j \neq k$). Therefore, the desired conclusions follow from the definitions of $\|a\|$ and $\|\rho(a)\|$. □

PROPOSITION 8.8 ([Ben, Lemma 2.4]). *We have*

$$\mu(h) \asymp h, \tag{8.9}$$

$$\rho\big(\mu(h)\big) \asymp \rho(h), \tag{8.10}$$

$$\mu(h)_{1,1} \asymp \|h\|, \text{ and } \mu(h)_{2,2} \asymp \|\rho(h)\|/\|h\|, \tag{8.11}$$

for $h \in \mathrm{SU}(2,n;\mathbb{F})$.

Proof. Choose $k_1, k_2 \in K$, such that $\mu(h) = k_1 h k_2$. Because $\|xy\| = \mathrm{O}(\|x\|\|y\|)$ for $x, y \in \mathrm{SU}(2,n;\mathbb{F})$, and $\max_{k\in K}\|k\| < \infty$ (since K is compact), we have

$$\|\mu(h)\| = \|k_1 h k_2\| = \mathrm{O}(\|h\|)$$

and

$$\|h\| = \|k_1^{-1}\mu(h)k_2^{-1}\| = \mathrm{O}(\|\mu(h)\|),$$

so (8.9) holds. Similarly, we have

$$\|\rho\big(\mu(h)\big)\| = \|\rho(k_1)\rho(h)\rho(k_2)\| \asymp \|\rho(h)\|,$$

so (8.10) holds.

For $a \in A^+$, we know, from (8.6), that $a_{1,1} = \|a\|$ and $a_{2,2} = \|\rho(a)\|/a_{1,1}$. Thus, letting $a = \mu(h)$, and using (8.9) and (8.10), we see that

$$\mu(h)_{1,1} = \|\mu(h)\| \asymp \|h\|$$

and

$$\mu(h)_{2,2} = \frac{\|\rho\big(\mu(h)\big)\|}{\mu(h)_{1,1}} \asymp \frac{\|\rho(h)\|}{\|h\|},$$

as desired. ■

Remark 8.12. Proposition 8.8 generalizes to any reductive group G [Ben, Lemma 2.3]. However, one may need to use a different representation in the place of ρ. In fact, if $\mathbb{R}$-rank $G = r$, then r representations of G are needed; for $G = \mathrm{SU}(2,n;\mathbb{F})$, we have $\mathbb{R}$-rank $G = 2$, and the two representations we use are ρ and the identity representation $I(h) = h$.

COROLLARY 8.13. *Let* $g_n \to \infty$ *and* $h_n \to \infty$ *be two sequences of elements of* $\mathrm{SU}(2, n; \mathbb{F})$. *We have*

$$g_n \asymp h_n \quad \text{and} \quad \rho(g_n) \asymp \rho(h_n)$$

if and only if there is a compact subset C *of* A, *such that, for all* $n \in \mathbb{Z}^+$, *we have* $\mu(g_n) \in \mu(h_n)C$.

Proof. ($\Rightarrow$) Let $a = \mu(h_n)^{-1}\mu(g_n)$. From (8.11), we see that $\mu(g_n)_{j,j} \asymp \mu(h_n)_{j,j}$ for $j \in \{1, 2\}$, so, using (8.7), we have

$$a_{j,j} = \frac{\mu(g_n)_{j,j}}{\mu(h_n)_{j,j}} = \begin{cases} \mathrm{O}(1) & \text{if } 1 \leqslant j \leqslant 2; \\ 1/1 = 1 & \text{if } 3 \leqslant j \leqslant n; \\ \mu(h_n)_{2,2}/\mu(g_n)_{2,2} = \mathrm{O}(1) & \text{if } j = n+1; \\ \mu(h_n)_{1,1}/\mu(g_n)_{1,1} = \mathrm{O}(1) & \text{if } j = n+2. \end{cases}$$

Therefore $a = \mathrm{O}(1)$, as desired.

($\Leftarrow$) Because C is compact, we have

$$\mu(g_n) \asymp \mu(h_n) \quad \text{and} \quad \rho\big(\mu(g_n)\big) \asymp \rho\big(\mu(h_n)\big)$$

(cf. proof of (8.9) and (8.10)). Then the desired conclusions follow from (8.9) and (8.10). □

Proof of Proposition 2.12 *for* $G = \mathrm{SU}(2, n; \mathbb{F})$. Because C is compact, we have $g' \asymp g$ and $\rho(g') \asymp \rho(g)$ for any $g' \in CgC$ (cf. proof of (8.9) and (8.10)). Thus, the desired conclusion follows from Corollary 8.13. ■

Because of Proposition 8.8, we will often need to calculate $\|h\|$ and $\|\rho(h)\|$. The following observation and its corollary sometimes simplifies the work, by allowing us to replace h with h^{-1}.

LEMMA 8.14. *We have* $\mu(h^{-1}) = \mu(h)$ *for* $h \in \mathrm{SU}(2, n; \mathbb{F})$.

Proof. Define J as in (7.2), and choose $k_1, k_2 \in K$, such that $\mu(h) = k_1 h k_2$. For any $a \in A^+$, we see, using (7.3) or (8.7), that $Ja^{-1}J = a$, so

$$(Jk_2^{-1})h^{-1}(k_1^{-1}J) = J\,\mu(h)^{-1}\,J = \mu(h).$$

Note that $\det J = 1$. Also, we have $J^2 = \mathrm{Id}$ and $J^\dagger = J$, so it is obvious that $JJJ^\dagger = J$ and $JJ^\dagger = \mathrm{Id}$. Therefore

$$J \in \mathrm{SU}(2, n; \mathbb{F}) \cap \mathrm{SU}(n+2) = K.$$

Thus, from the definition of μ, we conclude that $\mu(h^{-1}) = \mu(h)$, as desired. □

The following corollary is obtained by combining Lemma 8.14 with Corollary 8.13.

COROLLARY 8.15. *We have* $h^{-1} \asymp h$ *and* $\rho(h^{-1}) \asymp \rho(h)$ *for* $h \in \mathrm{SU}(2, n; \mathbb{F})$.

8C. THE WALLS OF A^+

NOTATION 8.16. For $k \in \{1, 2\}$, set

$$L_k = \{\, a \in A^+ \mid a_{2,2} = a_{1,1}^{k-1} \,\}. \tag{8.17}$$

From (7.10), we see that L_1 and L_2 are the two walls of A^+. From (8.6), we have

$$\rho(a) \asymp \|a\|^k \quad \text{for } a \in L_k. \tag{8.18}$$

We reproduce the proof of the following result, because it is both short and instructive. (Although we have no need for it here, let us point out that the converse of this proposition also holds, and that there is no need to assume $H \subset AN$.) Because of this proposition (and Corollary 2.9), Section 10 will study the existence of curves h^t, such that $h^t \asymp \|h^t\|^k$, for $k \in \{1, 2\}$.

PROPOSITION 8.19 ([OW1, Proposition 3.24]). *Let H be a closed, connected subgroup of AN in* $\mathrm{SU}(2, n; \mathbb{F})$. *If, for each $k \in \{1, 2\}$, there is a continuous curve h^t in H, such that $\rho(h^t) \asymp \|h^t\|^k \to \infty$ as $t \to \infty$, then H is a Cartan-decomposition subgroup.*

Proof (cf. Figure 8.1 and proof of Proposition 6.3). By hypothesis, there is a continuous, proper map Φ: $\{1, 2\} \times \mathbb{R}^+ \to H$, such that $\rho(\Phi(k, t)) \asymp \|\Phi(k, t)\|^k$. Because $H \subset AN$, we know that H is homeomorphic to some Euclidean space $\mathbb{R}^m$ (see 3.15(1)).

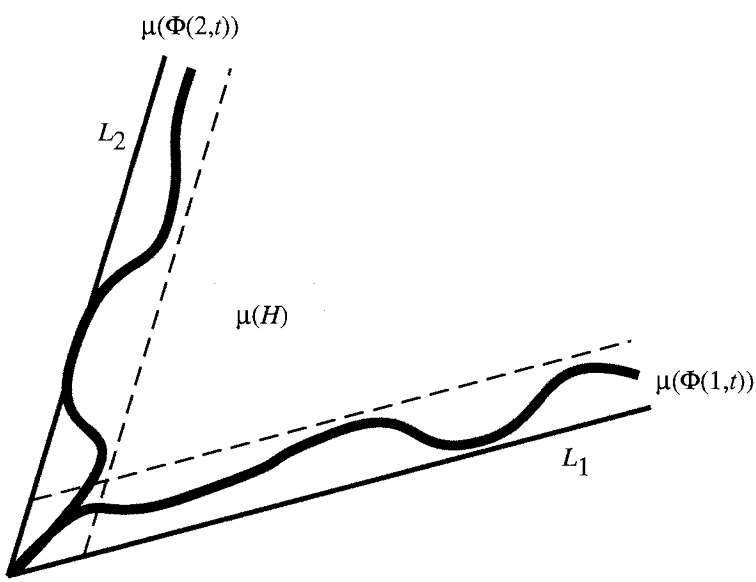


Figure 8.1. Proposition 8.19: if $\mu(H)$ contains a curve near each well of A^+, then it also contains the interior.

Suppose, for the moment, that $\dim H = 1$. (This will lead to a contradiction.) We know that $\rho(h) \asymp h$ for $h \bullet \Phi(1, \mathbb{R}^+)$. Because $h^{-1} \asymp h$ and $\rho(h^{-1}) \asymp \rho(h)$ (see 8.15), we must also have $\rho(h) \asymp h$ for $h \in \Phi(1, \mathbb{R}^+)^{-1}$. There is no harm in assuming $\phi(1, 0) = \mathrm{Id}$; then $\Phi(1, \mathbb{R}^+) \cup \Phi(1, \mathbb{R}^+)^{-1} = H$ (because $\dim H = 1$), so we conclude that $\rho(h) \asymp h$ for all $h \in H$. This contradicts the fact that $\rho(h) \asymp \|h\|^2$ for $h \in \Phi(2, \mathbb{R}^+)$.

We may now assume $\dim H \geqslant 2$. Then, because H is homeomorphic to $\mathbb{R}^m$, it is easy to extend Φ to a continuous and proper map Φ': $[1, 2] \times \mathbb{R}^+ \to H$. From (8.18) and (8.13), we know that the curve $\mu\big(\Phi'(k, t)\big)$ stays within a bounded distance from the wall L_k; say $\operatorname{dist}\big[\big(\Phi'(k, t)\big), L_k\big] < C$ for all t. We may assume C is large enough that $\operatorname{dist}\big(\Phi'(s, 1), e\big) < C$ for all $s \bullet [1, 2]$. Then an elementary homotopy argument shows that $\mu\big[\Phi'\big([1, 2] \times \mathbb{R}^+\big)\big]$ contains

$$\{a \in A^+ \mid \operatorname{dist}(a, L_1 \cup L_2) > C\},$$

so $\mu\big[\Phi'\big([1, 2] \times \mathbb{R}^+\big)\big] \approx A^+$. Because $\mu(H) \supset \mu\big[\Phi'\big([1, 2] \times \mathbb{R}^+\big)\big]$, we conclude from Theorem 2.15 that H is a Cartan decomposition subgroup. $\square$

Remark 8.20. When $\mathbb{R}$-rank $G = 1$, the Weyl chamber A^+ has only one point at infinity. Thus, if H is any noncompact subgroup, then the closure of $\mu(H)$ must contain this point at infinity. This is why it is easy to prove that any noncompact subgroup of G is a Cartan decomposition subgroup (see 2.16).

The idea of Proposition 8.19 is that if $\mathbb{R}$-rank $G = 2$, then the points at ∞ of the Weyl chamber A^+ form a closed interval. If the closure of $\mu(H)$ contains the two endpoints of this interval, then, by continuity, it must also contain all the points in between.

Unfortunately, we have no good substitute for this proposition when $\mathbb{R}$-rank $G > 2$. The points at ∞ of A^+ form a closed disk (topologically speaking). It is easy to define a map f from one disk to another, such that the image of f contains the entire boundary sphere, but does not contain the interior of the disk. Thus, it does not suffice to show only that the closure of $\mu(H)$ contains the boundary of the disk at ∞; rather, one needs additional homotopical information to guarantee that no interior points are missed.

LEMMA 8.21. *Let $G = \mathrm{SU}(2, n; \mathbb{F})$, and fix some $m \leqslant n/2$. Then $\mu(\mathrm{SU}(1, n; \mathbb{F}))$ and $\mu(\mathrm{Sp}(1, m; \mathbb{F}))$ are the two walls of A^+.*

We have

(1) $\rho(h) \asymp h$ *for* $h \bullet \mathrm{SU}(1, n; \mathbb{F})$; *and*
(2) $\rho(h) \asymp \|h\|^2$ *for* $h \bullet \mathrm{Sp}(1, m; \mathbb{F})$.

Proof. Let $H = \mathrm{SU}(1, n; \mathbb{F})$ or $\mathrm{Sp}(1, m; \mathbb{F})$. Then $H \cap K$ is a maximal compact subgroup of H. From the Cartan decomposition

$$H = (K \cap H)(A \cap H)(K \cap H),$$

and the definition of μ, we conclude that $\mu(H) = \mu(A \cap H)$. In the notation of (8.17), we see (from Definition 7.12) that $H \cap A = L_k \cup L_k^{-1}$, where

$$k = \begin{cases} 1 & \text{if } H = \mathrm{SU}(1, n; \mathbb{F}); \\ 2 & \text{if } H = \mathrm{Sp}(1, m; \mathbb{F}). \end{cases}$$

Then, since $\mu(a^{-1}) = \mu(a)$ (see 8.14) and $\mu(a) = a$ for $a \in A^+$, we conclude that $\mu(H) = L_k$ is a wall of A^+. Furthermore, we have $\rho(a) \asymp \|a\|^k$ for $a \in \mu(H)$ (see 8.18), so $\rho(h) \asymp \|h\|^k$ for $h \in H$ (see 8.8). □

COROLLARY 8.22. *If there is a continuous curve $h^t \to \infty$ in H, such that $\rho(h^t) \asymp h^t$, then there is a compact subset C of G, such that* $\mathrm{SU}(1, n; \mathbb{F}) \subset CHC$.

Proof. For any (large) $g \in \mathrm{SU}(1, n; \mathbb{F})$, we see from continuity (more precisely, from the Intermediate Value Theorem) that there exists $t \in \mathbb{R}^+$, such that

$$\|h^t\| = \|g\|.$$

Then, by assumption and from 8.21(1), we have

$$\rho(h^t) \asymp h^t \asymp g \asymp \rho(g),$$

so there is a compact subset C' of A, such that

$$\mu\big(\mathrm{SU}(1, n; \mathbb{F})\big) \subset \{\, \mu(h^t) \mid t \in \mathbb{R}^+ \,\}C' \subset \mu(H)C'$$

(see 8.13). Therefore

$$\mathrm{SU}(1, n; \mathbb{F}) \subset K\mu(\mathrm{SU}(1, n; \mathbb{F}))\,K \subset K\mu(H)C'\,K \subset K(KHK)C'K,$$

as desired. □

The following corollary can be proved by a similar argument. (Recall that the equivalence relation $\sim$ is defined in (1.17).)

COROLLARY 8.23. *Assume H is not compact.*

(1) *We have $H \sim$* $\mathrm{SU}(1, n; \mathbb{F})$ *if and only if $\rho(h) \asymp h$ for $h \in H$.*
(2) *We have $H \sim$* $\mathrm{Sp}(1, m; \mathbb{F})$ *if and only if $\rho(h) \asymp \|h\|^2$ for $h \in H$.*

Because of Proposition 8.19, we will often want to show that a curve h^t satisfies $\rho(h^t) \asymp \|h\|^k$, for some $k \in \{1, 2\}$. The following lemma does half of the work.

LEMMA 8.24. *Let X be a subset of* $\mathrm{SU}(2, n; \mathbb{F})$.

(1) *If $\rho(h) = O(h)$ for $h \in X$, then $\rho(h) \asymp h$ for $h \in X$.*
(2) *If $\|h\|^2 = O(\rho(h))$ for $h \in X$, then $\rho(h) \asymp \|h\|^2$ for $h \in X$.*

Proof. From (8.6) and (7.10), we have

$$\|a\| = a_{1,1} \leqslant a_{1,1}\, a_{2,2} = \|\rho(a)\|$$

and

$$\|\rho(a)\| = a_{1,1}\, a_{2,2} \leqslant a_{1,1}^2 = \|a\|^2$$

for $a \in A^+$. Thus, letting $a = \mu(h)$, and using (8.9) and (8.10), we have

$$\|h\| \asymp \|\mu(h)\| \leqslant \|\rho\big(\mu(h)\big)\| \asymp \|\rho(h)\|$$

and

$$\|\rho(h)\| \asymp \|\rho(\mu(h))\| \leqslant \|\mu(h)\|^2 \asymp \|h\|^2,$$

so $h = \mathrm{O}\,(\rho(h))$ and $\rho(h) = \mathrm{O}\,(\|h\|^2)$. The desired conclusions follow. ■

For convenience, we record the following simple observation. (For the proof, cf. the proof of (8.9) and (8.10).)

LEMMA 8.25. *Let*

- $k \bullet \{1, 2\}$,
- $g \bullet G$, *and*
- $h^t \to \infty$ *be a continuous curve in* H.

If $\rho(h^t) \asymp \|h^t\|^k$, *then* $\rho(g^{-1}h^t g) \asymp \|g^{-1}h^t g\|^k$.

8D. HOMOGENEOUS FUNCTIONS OF THE SAME DEGREE

The following well-known, elementary observation is used frequently in the later sections.

LEMMA 8.26. *Let* V' *be a subspace of a finite-dimensional real vector space* V, *and let* $f_1\colon V \to W_1$ *and* $f_2\colon V \to W_2$ *be linear transformations.*

(1) *If* $f_1^{-1}(0) \cap V' = \{0\}$ *(or, more generally, if* $f_1^{-1}(0) \cap V' \subset f_2^{-1}(0)$*), then there is a linear transformation* $f\colon W_1 \to W_2$, *such that* $f_2(v) = f(f_1(v))$ *for all* $v \in V'$. *Therefore* $f_2 = \mathrm{O}\,(f_1)$ *on* V'.
(2) *If* $f_1^{-1}(0) \cap V' = f_2^{-1}(0) \cap V'$, *then* $f_1 \asymp f_2$ *on* V'.

Proof. (1) By passing to a subspace, we may assume $V' = V$. Then, by modding out $f_1^{-1}(0)$, we may assume f_1 is an isomorphism onto its image. Define $f'\colon f_1(V) \to W_2$ by $f'(w) = f_2(f_1^{-1}(w))$, and let $f\colon W_1 \to W_2$ be any extension of f'.

For $v \in V'$, we have

$$\|f_2(v)\| = \|f(f_1(v))\| \leqslant \|f\|\, \|f_1(v)\|,$$

so $f_2 = \mathrm{O}\,(f_1)$.

(2) From (1), we have $f_2 = \mathrm{O}\,(f_1)$ and $f_1 = \mathrm{O}\,(f_2)$, so $f_1 \asymp f_2$. □

EXAMPLE 8.27. Let $\mathfrak{h}$ be a real Lie subalgebra of $\mathfrak{a} + \mathfrak{n}$, and assume there does not exist a nonzero element u of $\mathfrak{h}$, such that $x_u = 0$ and $y_u = 0$. Then there exist $\mathbb{R}$-linear transformations $R, S\colon \mathbb{F}^{n-2} \to \mathbb{F}$, such that $\eta_u = R(x_u) + S(y_u)$ for all $u \bullet \mathfrak{h}$.

(Similarly, ϕ_u, x_u, and y_u are also functions of (x_u, y_u).) Furthermore, we have $u \asymp |x_u| + |y_u|$.

The following well-known result is a generalization of the fact that all norms on a finite-dimensional vector space are equivalent up to a bounded factor.

LEMMA 8.28. *If V is any finite-dimensional real vector space, and $f_1, f_2 \colon V \to \mathbb{R}$ are two continuous, homogeneous functions of the same degree, such that $f_1^{-1}(0) = f_2^{-1}(0) = \{0\}$, then $f_1 \asymp f_2$.*

Proof. By continuity, the function f_1/f_2 attains a nonzero minimum and a finite maximum on the unit sphere. Because f_1/f_2 is homogeneous of degree zero, these values bound f_1/f_2 on all of $V \setminus \{0\}$. □

9. Existence of Tessellations

In this section, we show how to construct several families of homogeneous spaces that have tessellations. All of these examples are based on a method of T. Kobayashi (see 9.1) that generalizes Example 1.11.

9A. THE GENERAL KULKARNI–KOBAYASHI CONSTRUCTION

As explained in the comments before Theorem 4.1, the following theorem is essentially due to Kobayashi.

THEOREM 9.1 ([Kb1, Theorem 4.7]). *If*

- *H and L are closed subgroups of G, with only finitely many connected components;*
- *L acts properly on G/H;*
- *$d(L) + d(H) = d(G)$; and*
- *there is a cocompact lattice Γ in L,*

then G/H has a tessellation. (Namely, Γ is a crystallographic group for G/H.)

Proof. Because Γ is a closed subgroup of L, we know that it acts properly on G/H (see 2.4). Thus, it suffices to show that $\Gamma \backslash G/H$ is compact.

From Lemma 3.5, we see that there is no harm in assuming $H \subset AN$, and that there is a closed, connected subgroup L' of G, such that

- L' is conjugate to a subgroup of AN,
- $d(L') + d(H) = d(G)$, and
- $L'C = LC$, for some compact subset C of G.

(Unfortunately, we cannot assume $L \subset AN$: we may not be able to replace L with L', because there may not be a cocompact lattice in L'. For example, there is not lattice in AN, because any group with a lattice must be unimodular [Rag, Remark 1.9, p. 21].)

It suffices to show that $L'\backslash G/H$ is compact. (Because $L' \subset LC$ is compact, and $\Gamma\backslash L$ is compact, this implies that $\Gamma\backslash G/H$ is compact, as desired.)

We know that L' acts properly on G/H (see 2.4), so $L' \times H$ acts properly on G, with quotient $L'\backslash G/H$. Therefore, Lemma 3.19 implies that $L'\backslash G/H$ has the same homology as G; in particular,

$$\mathcal{H}_{\dim K}(L'\backslash G/H) \cong \mathcal{H}_{\dim K}(G).$$

From the Iwasawa decomposition $G = KAN$, and because AN is homeomorphic to $\mathbb{R}^{d(G)}$ (see 3.18 and 3.12), we know that G is homeomorphic to $K \times \mathbb{R}^{d(G)}$. Since $\mathbb{R}^{d(G)}$ is contractible, this implies that G is homotopy equivalent to K, so G and K have the same homology; in particular,

$$\mathcal{H}_{\dim K}(G) = \mathcal{H}_{\dim K}(K) \neq 0.$$

Since

$$\begin{aligned}\dim(L'\backslash G/H) &= \dim G - \dim L' - \dim H \\ &= \dim G - (d(L') + d(H)) \\ &= \dim G - d(G) \\ &= \dim(KAN) - \dim(AN) \\ &= \dim K,\end{aligned}$$

this implies that the top-dimensional homology of the manifold $L'\backslash G/H$ is nontrivial. Therefore $L'\backslash G/H$ is compact [Dol, Corollary 8.3.4], as desired. $\square$

Our results for $G = \mathrm{SU}(2, 2m; \mathbb{F})$ are based on the following special case of the theorem. The converse of this corollary is proved in Section 11 (see 11.5).

Recall the equivalence relation $\sim$, introduced in Notation 1.17.

COROLLARY 9.2 ([Kb1, Proposition 4.9]). *Let H be a closed, connected subgroup of $G = \mathrm{SU}(2, 2m; \mathbb{F})$. If*

- *$d(H) = 2qm$; and*
- *either $H \sim \mathrm{SU}(1, 2m; \mathbb{F})$ or $H \sim \mathrm{Sp}(1, m; \mathbb{F})$,*

then G/H has a tessellation.

Proof. Let $L_+ = \mathrm{SU}(1, 2m; \mathbb{F})$ and $L_- = \mathrm{Sp}(1, m; \mathbb{F})$. By assumption, we have $H \sim L_\varepsilon$, for some $\varepsilon \in \{+, -\}$; let $L = L_{-\varepsilon}$. Because $\mu(L_+)$ and $\mu(L_-)$ are the two walls of A^+ (see 8.21), we know that $L = L_{-\varepsilon}$ acts properly on G/L_ε (see 2.17); since $H \sim L_\varepsilon$, this implies that L acts properly on G/H (see 2.4). Also, we have

$$d(L) + d(H) = 2qm + 2qm = d(G),$$

(see 7.15 and 1.15), and there is a cocompact lattice in L (cf. 1.5(2)). Thus, the desired conclusion follows from Theorem 9.1. $\square$

9B. DEFORMATIONS OF SO(2, 2m)/SU(1, m) AND SU(2, 2m)/Sp(1, m)

The homogeneous spaces described here were found by Oh and Witte [OW2, Theorems 4.1 and 4.6], [OW3, Theorem 1.5].

NOTATION 9.3. For any $\mathbb{R}$-linear B: $\mathbb{F}^{n-2} \to \mathbb{F}^{n-2}$, we define

$$\mathfrak{h}_B = \left\{ \begin{pmatrix} t & 0 & x & \eta & \mathsf{x} \\ & t & xB & -\mathsf{x} & -\overline{\eta} \\ & & \cdots & & \end{pmatrix} \middle| \begin{array}{l} t \in \mathbb{R}, \\ x \in \mathbb{F}^{n-2}, \\ \eta \in \mathbb{F}, \\ \mathsf{x} \in \mathbb{F}_{\text{imag}} \end{array} \right\} \subset \mathfrak{a} + \mathfrak{n}.$$

We write xB, rather than Bx, because x is a row vector.

It is easy to see, using, for instance, the formula for the bracket in (7.22), that if

$$\operatorname{Im}\big((vB)(wB)^{\dagger}\big) = -\operatorname{Im}(vw^{\dagger}) \quad \text{for every } v, w \in \mathbb{F}^{n-2}, \tag{9.4}$$

then $\mathfrak{h}_B$ is a real Lie subalgebra of $\mathfrak{a} + \mathfrak{n}$; we let H_B denote the corresponding connected Lie subgroup of AN.

From (1.16), we have

$$\begin{aligned} d(H_B) &= \dim \mathfrak{h}_B \\ &= \dim \mathbb{R} + \dim \mathbb{F}^{n-2} + \dim \mathbb{F} + \dim \mathbb{F}_{\text{imag}} \\ &= 1 + q(n-2) + q + (q-1) \\ &= qn. \end{aligned} \tag{9.5}$$

Remark 9.6. Assume $n = 2m$. By comparing (7.14) with (9.3), we see that there is a $\mathbb{R}$-linear map B_0: $\mathbb{F}^{2m-2} \to \mathbb{F}^{2m-2}$, such that $\mathfrak{sp}(1, m; \mathbb{F}) \cap (\mathfrak{a} + \mathfrak{n}) = H_{B_0}$ (and B_0 satisfies (9.4)). Thus, in general, H_B is a deformation of $\mathfrak{sp}(1, m; \mathbb{F}) \cap (\mathfrak{a} + \mathfrak{n})$.

THEOREM 9.7 ([OW2, Theorems 4.1 and 4.6]). *Let B: $\mathbb{F}^{2m-2} \to \mathbb{F}^{2m-2}$ be $\mathbb{R}$-linear. If*

- *Condition* (9.4) *holds, and*
- *$xB \notin \mathbb{F}x$, for every nonzero $x \in \mathbb{F}^{2m-2}$,*

then

(1) *$\rho(h) \asymp \|h\|^2$ for $h \in H_B$; and*
(2) *$\mathrm{SU}(2, 2m; \mathbb{F})/H_B$ has a tessellation.*

Proof. (1) Given $h \bullet H_B$, write $h = au$, with $a \bullet A$ and $u \bullet N$. We may assume that $a_{1,1} \geqslant 1$ (by replacing h with h^{-1} if necessary (see 8.15). It suffices to show $\|h\|^2 = O(a_{1,1}a_{2,2} + |\Delta(h)|)$ (for then $\|h\|^2 = O(\rho(h))$, so Lemma 8.24(2) applies).

Case 1. *Assume a is trivial.* From (7.18) and (9.3), we see that

$$h = O\big(|x_h|^2 + |\eta_h| + |\mathsf{x}_h|\big),$$

so

$$\|h\|^2 = \mathrm{O}\big(|x_h|^4 + |\eta_h|^2 + |\mathsf{x}_h|^2\big).$$

From (7.18) and (8.5), we have

$$-\mathrm{Re}\,\Delta(h) = \tfrac{1}{4}\big(|x_h|^2|y_h|^2 - |xy^\dagger|^2\big) + \big(|\eta_h|^2 + \mathsf{x}_h\mathsf{y}_h\big).$$

From (9.9), we see that $|x|^2|xB|^2 - |x(xB)^\dagger|^2 > 0$ for every nonzero $x \in \mathbb{F}^{2m-2}$, so Lemma 8.28 implies

$$|x_h|^4 \asymp |x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2.$$

Also, because $\mathsf{y}_h = -\mathsf{x}_h$ (and $\mathsf{x}_h \in \mathbb{F}_{\mathrm{imag}}$), we have

$$|\eta_h|^2 + \mathsf{x}_h\mathsf{y}_h = |\eta_h|^2 + |\mathsf{x}_h|^2 \geqslant 0.$$

Thus,

$$-\mathrm{Re}\,\Delta(h) \asymp |x_h|^4 + \big(|\eta_h|^2 + |\mathsf{x}_h|^2\big),$$

so $\|h\|^2 = \mathrm{O}\big(\mathrm{Re}\,\Delta(h)\big) = \mathrm{O}\big(\Delta(h)\big)$, as desired.

Case 2. *The general case.* From Case 1, we know $\|u\|^2 = \mathrm{O}\big(1 + |\Delta(u)|\big)$. Then, because $\|h\| \leqslant \|a\|\,\|u\| = a_{1,1}\|u\|$, we have

$$\|h\|^2 \leqslant a_{1,1}^2\|u\|^2 = \mathrm{O}\left(a_{1,1}^2(1 + |\Delta(u)|)\right) = \mathrm{O}\big(a_{1,1}^2 + a_{1,1}^2|\Delta(u)|\big).$$

Then, since $a_{1,1} = a_{2,2}$ and $\Delta(h) = a_{1,1}a_{2,2}\Delta(u)$, we conclude that

$$\|h\|^2 = \mathrm{O}\big(a_{1,1}a_{2,2} + |\Delta(h)|\big),$$

as desired.

(2) From (1) and 8.23(2), we see that $H_B \sim \mathrm{Sp}(1, m; \mathbb{F})$. Then, because $d(H_B) = q(2m)$ (see 9.5), Theorem 9.2 implies that $\mathrm{SU}(2, 2m; \mathbb{F})/H_B$ has a tessellation. □

LEMMA 9.8. *Let B: $\mathbb{F}^{n-2} \to \mathbb{F}^{n-2}$ be $\mathbb{R}$-linear. Condition* (9.4) *holds if and only if either*

(1) $\mathbb{F} = \mathbb{R}$; *or*
(2) $\mathbb{F} = \mathbb{C}$ *and $B' \in \mathrm{Sp}(2n-4; \mathbb{R})$, where $xB' = \overline{xB}$ and we use the natural identification of $\mathbb{C}^{n-2}$ with $\mathbb{R}^{2n-4}$.*

Proof. Case 1. *Assume $\mathbb{F} = \mathbb{R}$.* Because $\mathrm{Im}\, z = 0$ for every $z = 0$, it is obvious that (9.4) holds.

Case 2. Assume $\mathbb{F} = \mathbb{C}$. If (9.4) holds, then

$$\begin{aligned}\mathrm{Im}((vB')(wB')^\dagger) &= \mathrm{Im}\,((\overline{vB})(\overline{wB})^\dagger)\\ &= \mathrm{Im}\,\overline{((vB)(wB)^\dagger)}\\ &= -\mathrm{Im}\,((vB)(wB)^\dagger)\\ &= -(-\mathrm{Im}\,(vw^\dagger))\\ &= \mathrm{Im}\,(vw^\dagger),\end{aligned}$$

so B' is symplectic. The argument is reversible. □

Remark 9.9.

- For $\mathbb{F} = \mathbb{R}$, the assumption that $xB \notin \mathbb{F}x$ simply requires that B have no real eigenvalues.
- For $\mathbb{F} = \mathbb{C}$, we do not know a good description of the linear transformations B that satisfy $xB \notin \mathbb{F}x$, although it is easy to see that this is an open set (and not dense). A family of examples was constructed by Oh and Witte (see 9.10 below).
- If n is odd, then there does not exist $B\colon \mathbb{F}^{n-2} \to \mathbb{F}^{n-2}$ satisfying the assumption that $xB \notin \mathbb{F}x$. For $\mathbb{F} = \mathbb{R}$, this is simply the elementary fact that a linear transformation on an odd-dimensional real vector space must have a real eigenvalue. For $\mathbb{F} = \mathbb{C}$, see Step 1.2.1 of the proof of Proposition 10.12.
- If n is even, then, by varying B, one can obtain uncountably many pairwise nonconjugate subgroups H_B, such that $\mathrm{SU}(2, n; \mathbb{F})/H_B$ has a tessellation. For $\mathbb{F} = \mathbb{R}$, this is proved in [OW3, Theorem 1.5]). For $\mathbb{F} = \mathbb{C}$, a similar argument can be applied to the examples constructed in (9.10) below.

EXAMPLE 9.10 ([OW2, Theorem 4.6(1)]). Assume n is even, let $B' \bullet \mathrm{SO}(n-2; \mathbb{R})$, such that B' has no real eigenvalue, and define an $\mathbb{R}$-linear map $B\colon \mathbb{C}^{n-2} \to \mathbb{C}^{n-2}$ by $xB = \overline{x}B'$. Let us verify that B satisfies the conditions of Theorem 9.7 (for $\mathbb{F} = \mathbb{C}$).

Let $x_1, x_2, y_1, y_2 \in \mathbb{R}^{n-2}$. From the definition of B, and because $B' \in \mathrm{SO}(n-2; \mathbb{R})$, we have

$$\begin{aligned}
\mathrm{Im}(((x_1 + ix_2)B)((y_1 + iy_2)B)^{\dagger}) &= \mathrm{Im}\,(((x_1 - ix_2)B')((y_1 - iy_2)B')^{\dagger}) \\
&= i((x_1B')(y_2B')^{\dagger} - (x_2B')(y_1B')^{\dagger}) \\
&= i(x_1y_2^{\dagger} - x_2y_1^{\dagger}) \\
&= -\mathrm{Im}((x_1 + ix_2)(y_1 + iy_2)^{\dagger}).
\end{aligned}$$

Suppose $Bx = \lambda x$, for some $\lambda \in \mathbb{C}$. Because $B \bullet \mathrm{SO}(2n-4; \mathbb{R})$, we must have $|\lambda| = 1$. Then

$$B'(x + \overline{\lambda x}) = B'x + \overline{\lambda}B'\overline{x} = \overline{B'\overline{x}} + \overline{\lambda}Bx = \overline{Bx} + \overline{\lambda}(\lambda x) = \overline{\lambda x} + x.$$

Because B' has no real eigenvalues, we know that 1 is not an eigenvalue of B', so we conclude that $x + \overline{\lambda x} = 0$. Similarly, because -1 is not an eigenvalue of B', we see that $x - \overline{\lambda x} = 0$. Therefore

$$x = \tfrac{1}{2}\big((x + \overline{\lambda x}) + (x - \overline{\lambda x})\big) = \tfrac{1}{2}(0 + 0) = 0.$$

9C. DEFORMATIONS OF $\mathrm{SU}(2, 2m)/\mathrm{SU}(1, 2m)$

These examples are new for $\mathbb{F} = \mathbb{C}$, but provide nothing interesting for $\mathbb{F} = \mathbb{R}$ (see 9.13(1)).

NOTATION 9.11. For $c \in (0, 1]$, we define

$$\mathfrak{h}_{[c]} = \left\{ \begin{pmatrix} t & \phi & x & \operatorname{Re}\phi + c\operatorname{Im}\phi & \mathsf{x} \\ & 0 & 0 & 0 & * \\ & & \cdots & & \end{pmatrix} \,\middle|\, \begin{array}{l} t \in \mathbb{R}, \\ \phi \in \mathbb{F}, \\ x \in \mathbb{F}^{n-2}, \\ \mathsf{x} \in \mathbb{F}_{\mathrm{imag}} \end{array} \right\}.$$

It is easy to see, using, for instance, the formula for the bracket in (7.22), that $\mathfrak{h}_{[c]}$ is a real Lie subalgebra of $\mathfrak{a} + \mathfrak{n}$ (even without the assumption that $0 < c \leqslant 1$); we let $H_{[c]}$ be the corresponding connected Lie subgroup of AN.

From (1.16), we have

$$\begin{aligned} d(H_{[c]}) &= \dim \mathfrak{h}_{[c]} \\ &= \dim \mathbb{R} + \dim \mathbb{F} + \dim \mathbb{F}^{n-2} + \dim \mathbb{F}_{\mathrm{imag}} \\ &= 1 + q + q(n-2) + (q-1) \\ &= qn. \end{aligned}$$

Remark 9.13. Let $\mathfrak{su}(1, n; \mathbb{F})$ be embedded into $\mathfrak{su}(2, n; \mathbb{F})$ as in 7.13.

(1) If $\mathbb{F} = \mathbb{R}$, then c is irrelevant in the definition of $\mathfrak{h}_{[c]}$ (because $\operatorname{Im}\phi = 0$); therefore $\mathfrak{h}_{[c]} = \mathfrak{su}(1, n; \mathbb{R}) \cap (\mathfrak{a} + \mathfrak{n})$.

(2) If $\mathbb{F} = \mathbb{C}$, then $\mathfrak{h}_{[1]} = \mathfrak{su}(1, n; \mathbb{C}) \cap (\mathfrak{a} + \mathfrak{n})$.

Thus, in general, $\mathfrak{h}_{[c]}$ is either $\mathfrak{su}(1, n; \mathbb{F}) \cap (\mathfrak{a} + \mathfrak{n})$ or a deformation of it.

THEOREM 9.14. *Assume $\mathbb{F} = \mathbb{C}$, and $n = 2m$ is even. If $c \in (0, 1]$, then*

(1) *$\rho(h) \asymp h$ for $h \in H_{[c]}$; and*

(2) *$\mathrm{SU}(2, 2m; \mathbb{F})/H_{[c]}$ has a tessellation.*

Proof. (1) Given $h \in H_{[c]}$, it suffices to show that $\rho(h) = O(h)$ (see 8.24(1)). Write $h = au$, with $a \in A$ and $u \in N$. We may assume that $a_{1,1} \geqslant 1$ (by replacing h with h^{-1} if necessary (see 8.15)).

Let Q: $\mathbb{C} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C} \to \mathbb{R}$ be the real quadratic form

$$Q(\phi, x, \eta) = |x|^2 + 2\operatorname{Re}(\phi\overline{\eta}),$$

and let V be the $\mathbb{R}$-subspace of $\mathbb{C} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C}$ defined by

$$V = \left\{ (\phi, x, \eta) \,\middle|\, \begin{array}{l} \phi \in \mathbb{C}, \\ x \in \mathbb{C}^{n-2}, \\ \eta = \operatorname{Re}\phi + c\operatorname{Im}\phi \end{array} \right\}.$$

Step 1. *For $v \in V$, we have $Q(v) \asymp |\phi|^2 + |x|^2$.* For $(\phi, x, \eta) \in V \setminus \{0\}$, we have

$$\begin{aligned} Q(\phi, x, \eta) &= |x|^2 + 2\operatorname{Re}(\phi\overline{\eta}) \\ &= |x|^2 + 2\operatorname{Re}\big(\phi(\overline{\operatorname{Re}\phi + c\operatorname{Im}\phi})\big) \\ &= |x|^2 + 2(\operatorname{Re}\phi)^2 - 2c(\operatorname{Im}\phi)^2 \\ &> 0 \end{aligned}$$

(because $c > 0$ and $\operatorname{Im}\phi$ is purely imaginary). Thus, the restriction of Q to V is positive definite, so the desired conclusion follows from Lemma 8.28. ■

Step 2. *We have* $u_{1,n+2} \asymp \big(|\phi_u|^2 + |x_u|^2\big) + |\mathsf{x}_u|$. From (7.19) (with $\mathsf{y} = 0$), we have

$$\operatorname{Re} u_{1,n+2} = -\big(\tfrac{1}{2}|x_u|^2 + \operatorname{Re}(\phi_u \overline{\eta_u})\big) \asymp |x_u|^2 + 2\operatorname{Re}(\phi_u \overline{\eta_u})$$

and

$$\operatorname{Im} u_{1,n+2} = \mathsf{x}_u.$$

Then, from Step 1, we see that $\operatorname{Re} u_{1,n+2} \asymp |\phi_u|^2 + |x_u|^2$, so

$$u_{1,n+2} \asymp |\operatorname{Re} u_{1,n+2}| + |\operatorname{Im} u_{1,n+2}| \asymp \big(|\phi_u|^2 + |x_u|^2\big) + |\mathsf{x}_u|,$$

as desired. □

Step 3. *Completion of the Proof.* From Step 2, we have

$$h_{1,n+2} = a_{1,1} u_{1,n+2} \asymp a_{1,1}\big(|x_u| + |\phi_u|\big)^2 + a_{1,1}|\mathsf{x}_u|.$$

Also, from (7.19), we have

$$h_{jk} = \begin{cases} O(1) & \text{if } j \neq 1 \text{ and } k \neq n+2 \\ O(a_1(|\phi_u| + |x_u|)) & \text{if } j = 1 \text{ and } k \neq n+2 \\ O(|\phi_u| + |x_u|) & \text{if } j \neq 1 \text{ and } k = n+2. \end{cases}$$

Thus, it is easy to see that

$$\rho(h) = O\big(a_{1,1}|\mathsf{x}_u| + a_{1,1}(|\phi_u| + |x_u|)^2\big) = O(h_{1,n+2}) = O(h),$$

so the desired conclusion follows from Lemma 8.24(1).

(2) From (1) and 8.23(1), we see that $H_{[c]} \sim \mathrm{SU}(1,n)$. Then, because $d(H_{[c]}) = 2n$ (see 9.12), Theorem 9.2 implies that $\mathrm{SU}(2,2m;\mathbb{F})/H_{[c]}$ has a tessellation. □

Remark 9.15. Proposition 11.6 shows that if $\mathbb{F} = \mathbb{C}$, then $H_{[c]}$ is not conjugate to $H_{[c']}$ unless $c = c'$ (for $c, c' \in (0,1]$). Thus, Theorem 9.14(2) implies that, by varying c, one obtains uncountably many nonconjugate subgroups $H_{[c]}$, such that $\mathrm{SU}(2,2m)/H_{[c]}$ has a tessellation.

9D. THE PRODUCT OF TWO RANK-ONE GROUPS

PROPOSITION 9.16. *Let $G = G_1 \times G_2$ be the direct product of two connected, linear, almost simple Lie groups G_1 and G_2 of real rank one, with finite center, and let H be a nontrivial, closed, connected, proper subgroup of AN.*

The homogeneous space G/H has a tessellation if and only if, perhaps after interchanging G_1 and G_2, there is a continuous homomorphism $\sigma: AN \cap G_1 \to AN \cap G_2$, such that

$$H = \{ (h, \sigma(h)) \mid h \in AN \cap G_1 \}.$$

Proof. ($\Rightarrow$) We may assume $d(G_1) \geqslant d(G_2)$ (by interchanging G_1 and G_2 if necessary).

Case 1. *Assume $H \cap G_1 \neq e$ and $H \cap G_2 \neq e$.* For $j = 1, 2$, we know that $H \cap G_j$ is not compact (see 3.15(3)), so Corollary 2.16 implies that there is a compact subset C_j of G_j, such that $C_j(H \cap G_j)C_j = G_j$. Then, letting $C = C_1C_2$, we have $CHC = G$, so Proposition 2.9 implies that G/H does not have a tessellation. This is a contradiction. □

Case 2. *Assume $H \cap G_1 \neq e$ and $H \cap G_2 = e$.* From Corollaries 2.16 and 2.15, we know that there is a compact subset C of $A \cap G_1$, such that $\mu(G_1) \subset \mu(H)C$. Therefore, Corollary 4.2 (with G_1 in the place of H_1) implies $d(H) \geqslant d(G_1) = \dim(G_1 \cap AN)$. Then, because $H \cap G_2 = e$ (and $H \subset AN$), we conclude that H is the graph of a homomorphism from $G_1 \cap AN$ to $G_2 \cap AN$, as desired. ■

Case 3. *Assume $H \cap G_1 = e$.* From Corollary 4.12, we know that $\dim H \geqslant d(G_2)$. Then, since $H \cap G_1 = e$, we conclude that H is the graph of a homomorphism from $G_2 \cap AN$ to $G_1 \cap AN$. Interchanging G_1 and G_2 yields the desired conclusion.

($\Leftarrow$) We verify the hypotheses of Theorem 9.1, with G_2 in the role of L.

Let $\overline{H}$ be the image of H under the natural homomorphism $G \to G/G_2$. Because $H \subset AN$, we know that $\overline{H}$ is closed (see 3.15(1)). It is well known (and follows easily from (2.3)) that any closed subgroup acts properly on the ambient group, so this implies that $\overline{H}$ acts properly on G/G_2. From the definition of H, we have $H \cap G_2 = e$, so we conclude that $H \cong \overline{H}$ acts properly on G/G_2; equivalently, G_2 acts properly on G/H (cf. 2.3).

Because $AN = (AN \cap G_1) \times (AN \cap G_2)$, we have $d(G) = d(G_1) + d(G_2)$. Also, we have $d(H) = \dim H$ (see 1.16) and, from the definition of H, we have $\dim H = \dim(AN \cap G_1) = d(G_1)$. Therefore

$$d(H) + d(G_2) = d(G_1) + d(G_2) = d(G).$$

There is a cocompact lattice in G_2 (cf. 1.5(2)).

So Theorem 9.1 implies that G/H has a tessellation. ■

9E. T. KOBAYASHI'S EXAMPLES OF HIGHER REAL RANK

T. Kobayashi observed that, besides the examples with $G = \mathrm{SO}(2, 2n)$ or $\mathrm{SU}(2, 2n)$ (see 1.11), Theorem 9.1 can also be used to construct tessellations of some homogeneous spaces G/H in which G and H are simple Lie groups with $\mathbb{R}\text{-rank}\, G > 2$. He found one pair of infinite families, and several isolated examples.

THEOREM 9.17 ([Kb6, Corollary 5.6]). *Each of the following homogeneous spaces has a tessellation:*

(1) $SO(4,4n)/Sp(1,n)$;
(2) $SO(4,4n)/SO(3,4n)$;
(3) $SO(8,8)/SO(8,7)$;
(4) $SO(8,8)/Spin(8,1)$;
(5) $SO(4,4)/SO(4,1)$;
(6) $SO(4,4)/Spin(4,3)$;
(7) $SO(4,3)/SO(4,1)$;
(8) $SO(4,3)/G_{2(2)}$.

It would be very interesting to find other examples of simple Lie groups G with reductive subgroups H and L that satisfy the hypotheses of Theorem 9.1.

Remark 9.18. Let $G = SO(4,4n)$ and $H' = Sp(1,n) \cap AN$. From 9.17(1), we know that G/H has a tessellation. H. Oh and D. Witte [OW2, Theorem 4.6(2)] pointed out that the deformations G/H_B (where H_B is as in Theorem 9.7, with $\mathbb{F} = \mathbb{C}$) also have tessellations, but it is not known whether there are other deformations of G/H' that also have tessellations.

It does not seem to be known whether the other examples in Theorem 9.17 lead to nontrivial deformations, after intersecting H with AN.

10. Large Subgroups of SO(2, *n*) and SU(2, *n*)

This section presents a short proof of the results we need from [OW1] and [IW]. Those papers provide an approximate calculation of $\mu(H)$, for every closed, connected subgroup H of $SO(2,n)$ or $SU(2,n)$, respectively, but here we consider only subgroups of large dimension. Also, we do not need a complete description of the entire set $\mu(H)$; we are only interested in whether or not there is a curve h^t, such that $\rho(h^t) \asymp \|h^t\|^k$, for some $k \in \{1,2\}$. The main results of this section are Theorem 10.14 (for $k=1$) and Theorem 10.21 (for $k=2$). They give a sharp upper bound on $d(H)$, for subgroups H that fail to contain such a curve, and, if n is even, also provide a fairly explicit description of all the subgroups of AN that attain the bound.

Because of the limited scope of this section, the proof here is shorter than the previous work, and we are able to give a fairly unified treatment of the two groups $SO(2,n)$ and $SU(2,n)$. The arguments are elementary, but they involve case-by-case analysis and a lot of details, so they are not pleasant to read.

STANDING ASSUMPTIONS 10.1. Throughout this section:

(1) We use the notation of Section 7. (In particular, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $q = \dim_{\mathbb{R}} \mathbb{F}$.)
(2) $G = SU(2,n;\mathbb{F})$.
(3) $n \geqslant 3$.

(4) H is a closed, connected subgroup of AN that is compatible with A (see 10.2), so $\dim H = d(H)$ (see 1.16).
(5) $U = H \cap N$. (Note that U is connected (see 3.15(2)).)
(6) $\mathfrak{u}_{\phi=0} = \{u \in \mathfrak{u} \mid \phi_u = 0\}$.
(7) We use the notation of Section 8. (In particular, $\|\rho(h)\|$ is defined in (8.4) and $\Delta(h)$ is defined in (8.5).)
(8) Except in Subsection 10A, H is compatible with A (see 10.2).

10A. SUBGROUPS COMPATIBLE WITH A

Recall that the Real Jordan Decomposition of an element of G is defined in (3.2); any element g of AN has a Real Jordan Decomposition $g = au$ (with c trivial). If a is an element of A, rather than only conjugate to an element of A, we could say that g is 'compatible with A.' We now define a similar, useful notion for subgroups of AN, and recall point out of its basic properties. Lemma 10.3 shows there is usually no loss of generality in assuming that H is compatible with A, and Lemma 10.5 shows that the compatible subgroups can be described fairly explicitly.

DEFINITION 10.2 ([OW1, Definition 2.2]). Let us say that H is *compatible* with A if $H \subset TUC_N(T)$, where $T = A \cap (HN)$, $U = H \cap N$, and $C_N(T)$ denotes the centralizer of T in N.

LEMMA 10.3 [OW1, Lemma 2.3]. *H is conjugate, via an element of N, to a subgroup that is compatible with A.*

The preceding proposition shows that H is conjugate to a subgroup H' that is compatible with A. The subgroup H' is usually not unique, however. The following lemma provides one way to change H', often to an even better subgroup.

LEMMA 10.4. *Assume that H is compatible with A, and let $T = A \cap (HN)$. If $u \in C_N(T)$, then $u^{-1}Hu$ is compatible with A.*

Proof. Let $H' = u^{-1}Hu$, $T' = A \cap (H'N)$, and $U' = H' \cap N$. Because u centralizes T, we have

$$u^{-1}Tu = T.$$

Also, because $u \in N$, and N is normal, we have $u^{-1}HuN = HN$, so

$$u^{-1}Tu = T = A \cap (HN) = A \cap (u^{-1}HuN) = A \cap (H'N) = T'.$$

Since $u \in N$, we have $u^{-1}Nu = N$, so

$$u^{-1}Uu = u^{-1}(H \cap N)u = (u^{-1}Hu) \cap (u^{-1}Nu) = H' \cap N = U',$$

and

$$u^{-1}C_N(T)u = C_{u^{-1}Nu}(u^{-1}Tu) = C_N(T').$$

Thus,

$$H' = u^{-1}Hu \subset u^{-1}TUC_N(T)u = \big(u^{-1}Tu\big)\big(u^{-1}Uu\big)\big(u^{-1}C_N(T)u\big) = T'U'C_N(T'),$$

as desired. □

LEMMA 10.5 ([OW1, Lemma 2.4]). *If H is compatible with A, then either*

(1) $H = (H \cap A) \ltimes (H \cap N)$; *or*
(2) *there is a positive root ω, a nontrivial group homomorphism ψ: $\ker \omega \to N_\omega N_{2\omega}$, and a closed, connected subgroup U of N, such that*
   (a) $H = \{a\psi(a) \mid a \in \ker \omega\}\, U$;
   (b) *U is normalized by both $\ker \omega$ and $\psi(\ker \omega)$; and*
   (c) $U \cap \psi(\ker \omega) = e$.

COROLLARY 10.6. *If H is compatible with A, then $A \cap (HN)$ normalizes $H \cap N$.*

LEMMA 10.7 ([OW1, Lemma 2.8]). *If $\dim(H/(H \cap N)) \geqslant \dim A$, then*

(1) *H contains a conjugate of A; and*
(2) *H is a Cartan-decomposition subgroup.*

10B. SUBGROUPS WITH NO NEARLY LINEAR CURVE

Our goal is to prove Theorem 10.14; we begin with some preliminary results.

First, an observation that simplifies the calculations in some cases, by allowing us to assume that $x_u = 0$.

LEMMA 10.8. *Let $u \in \mathfrak{u}$. If $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) \leqslant 1$ and $y_u \neq 0$, then there is some $g \in N_\alpha$, such that*

(1) $x_{g^{-1}ug} = 0$,
(2) $\phi_{g^{-1}ug} = \phi_u$, *and*
(3) $y_{g^{-1}ug} = y_u$.

Proof. Because $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) \leqslant 1$ and $y_u \neq 0$, there is some $\lambda \in \mathbb{F}$, such that $x_u = \lambda y_u$. Let

- v be the element of $\mathfrak{n}_\alpha$ with $\phi_\alpha = -\lambda$,
- $g = \exp(v) \in N_\alpha$, and
- $w = g^{-1}ug$.

From (7.23), we see that

- $\phi_w = \phi_u$,
- $x_w = x_u + \phi_v y_u = 0$, and
- $y_w = y_u$,

as desired. □

PROPOSITION 10.9. *If there does not exist a continuous curve $h^t \to \infty$ in U, such that $\rho(h^t) \asymp h^t$, then*

(1) *for every nonzero element z of $\mathfrak{d}_\mathfrak{h}$, we have $|\eta_z|^2 + \mathsf{x}_z\mathsf{y}_z \neq 0$; and*
(2) *for every element u of $\mathfrak{u}_{\phi=0}$, such that* $\dim_\mathbb{F}(\mathbb{F}x_u + \mathbb{F}y_u) = 1$, *we have*

$$\mathsf{x}_u|y_u|^2 + \mathsf{y}_u|x_u|^2 + 2\,\mathrm{Im}(x_u y_u^\dagger \eta_u^\dagger) \neq 0.$$

Proof of the Contrapositive. (1) Suppose there is a nonzero element z of $\mathfrak{d}_\mathfrak{h}$ with $\Delta(z) = 0$. Let $h^t = \exp(tz) = \mathrm{Id} + tz$ (see 7.18). We have

$$h_{j,k} = \begin{cases} \mathrm{O}(t) & \text{for all } j, k, \\ \mathrm{O}(1) & \text{if } (j,k) \notin \{1,2\} \times \{n+1, n+2\}. \end{cases}$$

Then, because $\Delta(h^t) = 0$, it is easy to see that $\rho(h) \asymp t$. Also, we have $h^t = \mathrm{Id} + tz \asymp t$, so $\rho(h^t) \asymp t \asymp h^t$, as desired.

(2) Suppose there is an element u of $\mathfrak{u}_{\phi=0}$, such that $\dim_\mathbb{F}(\mathbb{F}x_u + \mathbb{F}y_u) = 1$, and

$$\mathsf{x}_u|y_u|^2 + \mathsf{y}_u|x_u|^2 + 2\,\mathrm{Im}(x_u y_u^\dagger \eta_u^\dagger) = 0. \tag{10.10}$$

Let $h = h^t = \exp(tu)$.

Case 1. *Assume* $x_u = 0$. Because $\dim_\mathbb{F}(\mathbb{F}x_u + \mathbb{F}y_u) = 1$, we must have $y_u \neq 0$. Then, from (10.10), we know that $\mathsf{x}_u = 0$. So, from (7.18), we see that

- $h_{2,n+1} \asymp |y_u|^2 t^2 \asymp t^2$,
- $h_{j,k} = \mathrm{O}(t)$ whenever $(j,k) \neq (2, n+1)$, and
- $h_{j,k} = \mathrm{O}(1)$ whenever $j \neq 2$ and $k \neq n+1$.

This implies that $\rho(h) \asymp t^2 \asymp h$.

Case 2. *Assume* $y_u = 0$. This is similar to Case 1. (In fact, this can be obtained as a corollary of Case 1 by replacing H with its conjugate under the Weyl reflection corresponding to the root α.)

Case 3. *Assume* $y_u \neq 0$. Because $\dim_\mathbb{F}(\mathbb{F}x_u + \mathbb{F}y_u) = 1$, Lemma 10.8 implies there is some $g \bullet \mathrm{N}_\alpha$, such that, letting $w = g^{-1}ug$, we have $\phi_w = \phi_u = 0$, $x_w = 0$, and $y_w = y_u \neq 0$. We show below that (10.10) is satisfied with w in the place of u, so, from Case 1, we conclude that $\rho(\exp(tw)) \asymp \exp(tw)$. Thus, the desired conclusion follows from Lemma 8.25 (with $k = 1$).

To complete the proof, we now show that (10.10) is satisfied with w in the place of u. (This can be verified by direct calculation, but we give a more conceptual proof.) Because $g^{-1} \in N_\alpha$, multiplication by g^{-1} on the left performs a row operation on the first two rows of h; likewise, multiplication by g on the right performs a column operation on the last two columns of h. These operations do not change the determinant $\Delta(h)$: thus $\Delta(\exp(tw)) = \Delta(\exp(tu))$.

From (7.18) and the definition of Δ, we see that

$$\Delta(\exp(tu)) = -\tfrac{1}{4}(|x_u|^2|y_u|^2 - |x_u y_u^\dagger|^2)t^4 + \big(\mathsf{x}_u|y_u|^2 + \mathsf{y}_u|x_u|^2 + 2\,\mathrm{Im}(x_u y_u^\dagger \eta_u^\dagger)\big)t^3 + \mathrm{O}(t^2).$$

Because $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 1$, we have $|x_u|^2|y_u|^2 - |x_u y_u^\dagger|^2 = 0$, so this simplifies to

$$\Delta(\exp(tu)) = (\mathsf{x}_u|y_u|^2 + \mathsf{y}_u|x_u|^2 + 2\ \mathrm{Im}(x_u y_u^\dagger \eta_u^\dagger))t^3 + \mathrm{O}(t^2).$$

Thus, (10.10) is equivalent to the condition that $\Delta(\exp(tu)) = \mathrm{O}(t^2)$. Then, since

$$\Delta(\exp(tw)) = \Delta(\exp(tu)) = \mathrm{O}(t^2),$$

we conclude that (10.10) is also valid for w. $\square$

LEMMA 10.11. *If there does not exist a continuous curve $h^t \to \infty$ in U, such that $\rho(h^t) \asymp h^t$, then*

$$\dim \mathfrak{d}_{\mathfrak{h}} + \dim \mathfrak{u}/\mathfrak{u}_{\phi=0} \leqslant 2q - 1.$$

Furthermore, if equality holds, and $\mathbb{F} = \mathbb{C}$, then $\mathfrak{u} = \mathfrak{u}_{\phi=0}$ and $\dim \mathfrak{d}_{\mathfrak{h}} = 3$.

Proof. Case 1. *Assume* $\mathbb{F} = \mathbb{C}$. Because $|\eta_z|^2 + \mathsf{x}_z\mathsf{y}_z$ is a quadratic form of signature (1, 3) on $\mathfrak{d}$, we know, from 10.9(1), that $\dim \mathfrak{d}_{\mathfrak{h}} \leqslant 3 = 2q - 1$.

Thus, we may assume $\mathfrak{u}/\mathfrak{u}_{\phi=0} \neq 0$, so there is some $u \in \mathfrak{u}$, such that $\phi_u \neq 0$.

Subcase 1.1. *Assume there exists $z \in \mathfrak{d}_{\mathfrak{h}}$, such that $\mathsf{y}_z = 0$ and $\eta_z \notin \mathbb{R}\phi_u$.* From (7.22), we see that $[u, z] \in \mathfrak{d}_{\mathfrak{h}}$, with $\mathsf{x}_{[u,z]} = -\mathrm{Im}(\phi_u \overline{\eta_z}) \neq 0$ and $\mathsf{y}_{[u,z]} = \eta_{[u,z]} = 0$. This contradicts 10.9(1).

Subcase 1.2. *Assume there exists $z \in \mathfrak{d}_{\mathfrak{h}}$, such that* $\mathsf{y}_z \neq 0$. From (7.22), we see that $[u, z]$ is an element of $\mathfrak{d}_{\mathfrak{h}}$, such that $\mathsf{y}_{[u,z]} = 0$, and $\eta_{[u,z]} = \phi_u \mathsf{y}_z$ is a purely imaginary multiple of ϕ_u. So Subcase 1.1 applies (with $[u, z]$ in the place of z).

Subcase 1.3. *Assume $\mathsf{y}_z = 0$ and $\eta_z \in \mathbb{R}\phi_u$, for all $z \in \mathfrak{d}_{\mathfrak{h}}$.* From 10.9(1), we see that $\mathfrak{d}_{\mathfrak{h}} \cap \mathfrak{n}_{2\alpha+2\beta} = \{0\}$, so the assumption of this subcase implies $\dim \mathfrak{d}_{\mathfrak{h}} \leqslant 1$. Thus,

$$\dim \mathfrak{d}_{\mathfrak{h}} + \dim \mathfrak{u}/\mathfrak{u}_{\phi=0} \leqslant 1 + 2 = 3 = 2q - 1,$$

so the desired inequality holds.

If equality holds, then $\dim \mathfrak{d}_{\mathfrak{h}} = 1$ and $\dim \mathfrak{u}/\mathfrak{u}_{\phi=0} = 2$. Thus, we may choose $z \in \mathfrak{d}_{\mathfrak{h}}$, such that $z \neq 0$, and $u' \in \mathfrak{u}$, such that $\mathbb{R}\phi_u + \mathbb{R}\phi_{u'} = \mathbb{C}$. From the assumption of this subcase, we know that $\eta_z \in \mathbb{R}\phi_u$; thus, $\eta_z \notin \mathbb{R}\phi_{u'}$. Therefore, Subcase 1.1 applies, with u' in the place of u.

Case 2. *Assume* $\mathbb{F} = \mathbb{R}$. Because $\dim \mathfrak{d}_{\mathfrak{h}} \leqslant \dim \mathfrak{n}_{\alpha+2\beta} = 1$ and $\dim \mathfrak{u}/\mathfrak{u}_{\phi=0} \leqslant \dim \mathfrak{n}_\alpha = 1$, the desired inequality holds unless $\mathfrak{d}_{\mathfrak{h}} \neq 0$ and $\mathfrak{u}/\mathfrak{u}_{\phi=0} \neq 0$. Thus, we may assume there is some nonzero $z \in \mathfrak{d}_{\mathfrak{h}}$ and some $u \in \mathfrak{u}$, such that $\phi_u \neq 0$.

Subcase 2.1. *Assume* $y_u = 0$. We may assume $|x_u|^2 + \phi_u \eta_u \neq 0$ (by replacing u with $u + z$, if necessary). Let $h^t = \exp(tu)$. From (7.19), we see that $h^t_{1,n+2} \asymp t^2$, but

$$h^t_{j,k} = \begin{cases} O(t) & \text{if } (j,k) \neq (1, n+2), \\ O(1) & \text{if } j \neq 1 \text{ and } k \neq n+2. \end{cases}$$

Therefore $\rho(h^t) = O(t^2) = O(h^t)$, so Lemma 8.24(1) implies that $\rho(h^t) \asymp h^t$. This is a contradiction.

Subcase 2.2. *Assume* $y_u \neq 0$. Let v be the element of $\mathfrak{n}_\beta$ with $y_v = -(1/\phi_u)x_u$, and let $w = \exp(-v)u\exp(v)$. Then $x_w = 0$ (see 7.23 and 7.22). Thus, by replacing H with the conjugate $\exp(-v)H\exp(v)$ (see 8.25), we may assume $x_u = 0$. For any large real number t, let $h = h^t$ be the element of $\exp(tu + \mathfrak{n}_{\alpha+2\beta})$ that satisfies $\eta_h = -\phi_h \|y_h\|^2/12$. Then, from (7.17), we see that

$$h = \begin{pmatrix} 1 & \phi_h & \frac{1}{2}\phi_h y_h & -\frac{1}{4}\phi_h |y_h|^2 & \frac{1}{8}\phi_h^2 |y_h|^2 \\ & 1 & y_h & -\frac{1}{2}|y_h|^2 & \frac{1}{4}\phi |y_h|^2 \\ & & 1 & -y_h^\dagger & \frac{1}{2}\phi_h y_h^\dagger \\ & & & 1 & -\phi_h \\ & & & & 1 \end{pmatrix}.$$

Clearly, we have $h \asymp \phi_h^2 |y_h|^2$.

A calculation shows that $\Delta(h) = 0$, and certain other 2×2 minors also have cancellation. With this in mind, it is not difficult to verify that $\rho(h) \asymp \phi_h^2 |y_h|^2 \asymp h$ (see [OW1, Case 3 of proof of 5.12(3 $\Rightarrow$ 2)] for details). This is a contradiction.

PROPOSITION 10.12. *If there does not exist a continuous curve $h^t \to \infty$ in $U_{\phi=0}$, such that $\rho(h^t) \asymp h^t$, then*

$$\dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} \leqslant \begin{cases} q(n-2) & \text{if } n \text{ is even}, \\ q(n-3) & \text{if } n \text{ is odd and } n \neq 3, \\ q-1 & \text{if } n = 3. \end{cases}$$

Furthermore,

(a) *if equality holds, and n is even, then* $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$, *for every* $u \in \mathfrak{u}_{\phi=0} \setminus \mathfrak{d}_{\mathfrak{h}}$;
(b) *if equality holds, and $n = 3$, then* $\dim \mathfrak{d}_{\mathfrak{h}} \leqslant q$.

Proof. By passing to a subgroup, we may assume $\mathfrak{u} = \mathfrak{u}_{\phi=0}$. Let V be the projection of $\mathfrak{u}$ to $\mathfrak{n}_\beta + \mathfrak{n}_{\alpha+\beta}$; then $\dim V = \dim \mathfrak{u}/\dim \mathfrak{d}_{\mathfrak{h}}$.

Case 1. *Assume* $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$ *for every* $u \in \mathfrak{u} \setminus \mathfrak{d}_{\mathfrak{h}}$.

Subcase 1.1. *Assume n is even.* From Theorem 10.9(1), we know that V does not intersect $\mathfrak{n}_\beta$ (or $\mathfrak{n}_{\alpha+\beta}$, either, for that matter), so

$$\dim V + \dim \mathfrak{n}_\beta \leqslant \dim(\mathfrak{n}_\beta + \mathfrak{n}_{\alpha+\beta}) = \dim \mathfrak{n}_\beta + \dim \mathfrak{n}_{\alpha+\beta}.$$

Therefore

$$\dim \mathfrak{u} / \dim \mathfrak{d}_{\mathfrak{h}} = \dim V \leqslant \dim \mathfrak{n}_{\alpha+\beta} = q(n-2),$$

as desired. (If equality holds, then we have Conclusion (a).)

Subcase 1.2. *Assume n is odd.*

Step 1.2.1. *We have* $\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} \leqslant q(n-3)$. Suppose not: then

$$\dim V \geqslant q(n-3) + 1.$$

(This will lead to a contradiction.) Let $X = \{x_v \mid v \in V\}$, so X is a $\mathbb{R}$-subspace of $\mathbb{F}^{n-2}$. For each $x \in X$, there is some $v = v(x) \in V$, such that $x_v = x$; define $f(x) = y_{v(x)}$. By the assumption of this case, we know

$$V \cap \mathfrak{n}_\beta = \{0\},$$

so $v(x)$ is uniquely determined by x; thus, $f\colon X \to \mathbb{F}^{n-2}$ is a well-defined $\mathbb{R}$-linear map. Also, again from the assumption of this case, we know that

$$f(x) \notin \mathbb{F}x \quad \text{for every nonzero } x \in X. \tag{10.13}$$

Because $V \cap \mathfrak{n}_\beta = 0$, we have

$$\dim X = \dim V \geqslant q(n-3) + 1 = \dim \mathbb{F}^{n-2} - (q-1).$$

If $\mathbb{F} = \mathbb{R}$ (that is, if $q = 1$), this implies $X = \mathbb{R}^{n-2}$, so f is defined on all of $\mathbb{R}^{n-2}$. Because n is odd, this implies that f has a real eigenvalue, which contradicts (10.13).

We may now assume $\mathbb{F} = \mathbb{C}$. Let

- $E = (X \times \mathbb{C}^{n-2})/\!\equiv$, where $(x, v) \equiv (-x, -v)$,
- $\mathbb{P}X$ be the projective space of the real vector space X, and
- $\zeta(x, v) = [x] \in \mathbb{P}X$, for $(x, v) \in E$,

so (E, ζ) is a vector bundle over $\mathbb{P}X$.

Define $g\colon X \to \mathbb{C}^{n-2}$ by $g(x) = ix$. Any $\mathbb{R}$-linear transformation $Q\colon X \to \mathbb{C}^{n-2}$ is a continuous function, such that $Q(-x) = -Q(x)$ for all $x \in X$; that is, a section of (E, ζ). Thus, Id, f, and g each define a section of (E, ζ). Furthermore, these three sections are pointwise linearly independent over $\mathbb{R}$, because (10.13) implies that x, $f(x)$, and ix are linearly independent over $\mathbb{R}$, for every nonzero $x \in X$. On the other hand, the theory of characteristic classes [MS, Proposition 4, p. 39] implies that (E, ζ) does not have three pointwise $\mathbb{R}$-linearly independent sections (see [IW, Lemma 8.2] for details). This is a contradiction.

Step 1.2.2. *Completion of the proof of Subcase* 1.2. From Step 1.2.1, we see that the desired inequality holds.

We may now assume $n = 3$ and $\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} = q - 1$. Since $\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} \leqslant q(n-3) = 0$, we must have $q = 1$, so $\mathbb{F} = \mathbb{R}$. Therefore $\mathfrak{n}_{2\alpha} = \mathfrak{n}_{2\beta} = 0$, so

$$\dim \mathfrak{d}_{\mathfrak{h}} \leqslant \dim \mathfrak{n}_{\alpha+2\beta} = q,$$

as desired.

Case 2. *Assume there is some* $v \in \mathfrak{u} \setminus \mathfrak{d}_{\mathfrak{h}}$, *such that* $\dim_{\mathbb{F}}(\mathbb{F}x_v + \mathbb{F}y_v) = 1$.

Subcase 2.1. *Assume* $x_v = 0$. Since $v \notin \mathfrak{d}_{\mathfrak{h}}$, we must have $y_v \neq 0$. Then $\mathsf{x}_{v+z} \neq 0$ for every $z \in \mathfrak{d}_{\mathfrak{h}}$ (otherwise 10.9(2) yields a contradiction); this implies

$$\mathsf{x}_v \neq 0,$$

and

$$\mathsf{x}_z = 0 \quad \text{for every } z \in \mathfrak{d}_{\mathfrak{h}}.$$

Because $\mathsf{x}_v \neq 0$, we know that $\mathbb{F} \neq \mathbb{R}$; so

$$\mathbb{F} = \mathbb{C}.$$

Since $\mathsf{x}_z = 0$ for every $z \in \mathfrak{d}_{\mathfrak{h}}$, but $\mathfrak{d}_{\mathfrak{h}} \cap \mathfrak{n}_{2\beta} = 0$ (see 10.9(1)), we must have $\eta_z \neq 0$ for every $z \in \mathfrak{d}_{\mathfrak{h}}$. Therefore

$$\dim \mathfrak{d}_{\mathfrak{h}} \leqslant \dim \mathfrak{n}_{\alpha+2\beta} = q = 2.$$

Let $p\colon V \to \mathfrak{n}_{\alpha+\beta}$ be the natural projection. Note that

$$\dim \ker p = 1.$$

(If $v' \in \mathfrak{u}$, with $x_{v'} = 0$, then there is some $t \in \mathbb{R}$, such that $\mathsf{x}_{v'+tv} = \mathsf{x}_{v'} + t\mathsf{x}_v = 0$. We also have $x_{v'+tv} = 0$, so, from 10.9(2), we see that $v' + tv \in \mathfrak{d}_{\mathfrak{h}}$. Thus $v' \in \mathbb{R}v + \mathfrak{d}_{\mathfrak{h}}$. So $\ker p = (\mathbb{R}v + \mathfrak{d}_{\mathfrak{h}})/\mathfrak{d}_{\mathfrak{h}}$ is 1-dimensional.)

Because $\mathsf{x}_z = 0$ for every $z \in \mathfrak{d}_{\mathfrak{h}}$, and $\mathfrak{u}$ is a Lie algebra, we see, from (7.22), that $p(V)$ must be a totally isotropic subspace for the symplectic form $i\,\mathrm{Im}(x\tilde{x}^{\dagger})$, so

$$\dim p(V) \leqslant \tfrac{1}{2}\dim \mathfrak{n}_{\alpha+\beta} = n - 2.$$

Therefore

$$\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} = \dim V = \dim p(V) + \dim \ker p \leqslant (n-2) + 1 = n - 1.$$

This completes the proof if $n \neq 3$:

- If n is even, then, because $n \geqslant 4$, we have $n - 1 < 2(n-2) = q(n-2)$.
- If $n > 3$ is odd, then $n \geqslant 5$, so $n - 1 \leqslant 2(n-3) = q(n-3)$.

Now let $n = 3$, and suppose $\dim V = 2$. (This will lead to a contradiction.) Because equality is attained in the proof above, we must have $\dim p(V) = n - 2 = 1$, so there exists $w \in \mathfrak{u}$ with $x_w \neq 0$. For $t \in \mathbb{R}$, let $w_t = w + tv$. Then

$$\begin{aligned} \mathsf{x}_{w_t}|y_{w_t}|^2 + \mathsf{y}_{w_t}|x_{w_t}|^2 + 2\,\mathrm{Im}\,(x_{w_t}y_{w_t}^{\dagger}\eta_{w_t}) &= t^3\mathsf{x}_v|y_v|^2 + O(t^2) \\ &\to \begin{cases} +\mathsf{x}_v\infty & \text{as } t \to \infty \\ -\mathsf{x}_v\infty & \text{as } t \to -\infty. \end{cases} \end{aligned}$$

Thus, this expression changes sign, so it must vanish for some t. On the other hand, since $n = 3$, we have $\dim_{\mathbb{C}}(\mathbb{C}x + \mathbb{C}y) \leqslant 1$ for every $x, y \in \mathbb{C}^{n-2} = \mathbb{C}$. Thus 10.9(2) yields a contradiction.

Subcase 2.2. *Assume* $y_v = 0$. This is similar to Subcase 2.1. (In fact, this can be obtained as a corollary of Subcase 2.1 by replacing H with its conjugate under the Weyl reflection corresponding to the root α.)

Subcase 2.3. *Assume* $y_v \neq 0$. Because $\dim_{\mathbb{F}}(\mathbb{F}x_v + \mathbb{F}y_v) = 1$, Lemma 10.8 implies there is some $g \in N_\alpha$, such that, letting $w = g^{-1}vg$, we have $\phi_w = \phi_u = 0$, $x_w = 0$, and $y_w = y_v \neq 0$. There is no harm in replacing H with $g^{-1}Hg$ (see 8.25). Then Subcase 2.1 applies (with w in the place of v). $\blacksquare$

THEOREM 10.14. *Recall that Assumptions* 10.1 *are in effect.*

If there does not exist a continuous curve $h^t \to \infty$ in H, such that $\rho(h^t) \asymp h^t$, then

$$\dim H \leqslant \begin{cases} qn & \text{if } n \text{ is even,} \\ q(n-1) & \text{if } n \text{ is odd.} \end{cases} \tag{10.15}$$

Furthermore, if equality holds, and n is even, then

(1) $\mathfrak{h} = (\ker \alpha) \ltimes \mathfrak{u}$;
(2) $\phi_u = 0$ *for every* $u \in \mathfrak{u}$;
(3) $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$, *for every* $u \in \mathfrak{u} \setminus \mathfrak{d}_{\mathfrak{h}}$;
(4) $|\eta_z|^2 + \mathsf{x}_z\mathsf{y}_z \neq 0$ *for every nonzero* $z \in \mathfrak{d}_{\mathfrak{h}}$;
(5) $\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} = q(n-2)$; *and*
(6) $\dim \mathfrak{d}_{\mathfrak{h}} = 2q - 1$.

Proof. Let

$$m = \begin{cases} q(n-2) & \text{if } n \text{ is even,} \\ q(n-3) & \text{if } n \geqslant 5 \text{ is odd,} \\ q-1 & \text{if } n = 3. \end{cases}$$

From Lemmas 10.7(1) and 10.11, and Proposition 10.12, we have

$$\begin{aligned} \dim H &\leqslant \dim \mathfrak{h}/\mathfrak{u} + (\dim \mathfrak{u}/\mathfrak{u}_{\phi=0} + \dim \mathfrak{d}_{\mathfrak{h}}) + \dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} \\ &\leqslant 1 + (2q-1) + m \\ &= 2q + m. \end{aligned} \tag{10.16}$$

This implies the desired inequality, unless $n = 3$, $\mathbb{F} = \mathbb{C}$, and we have equality in both Lemma 10.11 and Proposition 10.12. This is impossible, because equality in Lemma 10.11 requires $\dim \mathfrak{d}_{\mathfrak{h}} = 3$, but Proposition 10.12(b) implies $\dim \mathfrak{d}_{\mathfrak{h}} \leqslant 2$.

ASSUMPTION. *In the remainder of the proof, we assume that equality holds in* (10.15), *and that n is even.* Proposition 10.9(1) implies (4).

Case 1. *Assume* $\mathbb{F} = \mathbb{C}$. Because equality holds, Lemma 10.11 implies (2) and (6). Then Proposition 10.12(a) implies (3) (because $\mathfrak{u} = \mathfrak{u}_{\phi=0}$). Since $\mathfrak{u} = \mathfrak{u}_{\phi=0}$ (see 2) and equality holds in (10.15), we have

$$\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} = \dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} = m = q(n-2) \tag{10.17}$$

and

$$\dim \mathfrak{d}_{\mathfrak{h}} = \dim \mathfrak{u}/\mathfrak{u}_{\phi=0} + \dim \mathfrak{d}_{\mathfrak{h}} = 2q - 1, \tag{10.18}$$

so (5) and (6) hold.

Let $T = A \cap (HN)$. Corollary 10.6 implies that T normalizes $\mathfrak{u}$, so, from (3) and Lemma 3.21, we see that $T \subset \ker \alpha$. On the other hand, $\dim T = \dim \mathfrak{h}/\mathfrak{u}$, so, from equality in (10.16), we conclude that $\dim T = 1$. Therefore $T = \ker \alpha$.

Suppose ψ: $\ker \alpha \to N_\alpha$ is any continuous group homomorphism, such that $\psi(\ker \alpha)$ normalizes U. From (3) and (7.22), we see that $N_{N_\alpha}(U) = e$, so ψ must be trivial. This implies that 10.5(2) cannot apply here, so 10.5(1) yields (1).

Case 2. *Assume* $\mathbb{F} = \mathbb{R}$. Proposition 10.12(a) implies that $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$ for every $u \in \mathfrak{u}_{\phi=0} \smallsetminus \mathfrak{d}_{\mathfrak{h}}$.

Suppose (2) is false. Then there is some $u \in \mathfrak{u}$, such that $\phi_u \neq 0$. Also, because $\dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} = m > 0$, we may fix some $v \in \mathfrak{u}_{\phi=0} \smallsetminus \mathfrak{d}_{\mathfrak{h}}$. Then, letting $w = [u, v]$, we see, from (7.22), that $y_w = 0$ and $x_w \neq 0$, so $\dim_{\mathbb{F}}(\mathbb{F}x_w + \mathbb{F}y_w) = 1$. This contradicts the conclusion of the preceding paragraph.

Conclusion (3) follows from (2) and 10.12(a).

Conclusion (1) can be established by arguing as in the last two paragraphs of Case 1.

Equations (10.17) and (10.18) establish (5) and (6).

10C. SUBGROUPS WITH NO NEARLY QUADRATIC CURVE

Our goal is to prove Theorem 10.21; we start with two preliminary results.

LEMMA 10.19. *If there does not exist a continuous curve $h^t \to \infty$ in U, such that $\rho(h^t) \asymp \|h^t\|^2$, then*

(1) *for every element u of $\mathfrak{u}_{\phi=0}$, we have* $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) \leqslant 1$*;*
(2) *for every element z of $\mathfrak{d}_{\mathfrak{h}}$, we have $|\eta_z|^2 + \mathsf{x}_z \mathsf{y}_z = 0$; and*
(3) *for every element u of $\mathfrak{u}$, such that $\phi_u \neq 0$, $y_u = 0$, and $\mathsf{y}_u = 0$, we have* $|x_u|^2 + 2\operatorname{Re}(\phi_u \overline{\eta_u}) \neq 0$.

Proof of the Contrapositive. (1) Suppose there is an element u of $\mathfrak{u}_{\phi=0}$, such that $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$. Let $h^t = \exp(tu)$. Then, from (7.18), we see that $h^t = O(t^2)$. Furthermore,

$$\Delta(h^t) = \det\begin{pmatrix} \eta_u t - \frac{1}{2}x_u y_u^\dagger t^2 & \mathsf{x}_u t - \frac{1}{2}|x_u|^2 t^2 \\ \mathsf{y}_u t - \frac{1}{2}|y_u|^2 t^2 & -\overline{\eta_u} t - \frac{1}{2}y_u x_u^\dagger t^2 \end{pmatrix} = \tfrac{1}{4}\big||x_u||y_u|^2 - |x_u y_u^\dagger|^2\big| t^4 + O(t^3).$$

Because $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$, we have $|x_u||y_u| > |x_u y_u^\dagger|$, so $|x_u|^2|y_u|^2 - |x_u y_u^\dagger|^2 \neq 0$; therefore $\Delta(h^t) \asymp t^4$, so

$$\|h^t\|^2 = O(t^4) = O\big(\Delta(h^t)\big) = O\big(\rho(h^t)\big),$$

so Lemma 8.24(2) implies that $\rho(h^t) \asymp \|h^t\|^2$, as desired.

(2) Suppose there is an element z of $\mathfrak{d}_{\mathfrak{h}}$, such that $|\eta_z|^2 + \mathsf{x}_z\mathsf{y}_z \neq 0$; in other words, we have $\Delta(z) \neq 0$. Let $h^t = \exp(tz) = \mathrm{Id} + tz$ (see 7.18). Then $h^t = \mathrm{O}(t)$ and

$$t^2 \asymp \Delta(z)t^2 = \Delta(h^t) = \mathrm{O}\big(\rho(h^t)\big),$$

so

$$\|h^t\|^2 = \mathrm{O}(t^2) = \mathrm{O}\big(\rho(h^t)\big),$$

so Lemma 8.24(2) implies that $\rho(h^t) \asymp \|h^t\|^2$, as desired.

(3) Suppose there is an element u of $\mathfrak{u}$, such that $\phi_u \neq 0$, $y_u = 0$, $\mathsf{y}_u = 0$, and $|x_u|^2 + 2\,\mathrm{Re}(\phi_u\overline{\eta}_u) = 0$. Let $h^t = \exp(tu)$. From (7.19), we see that $h^t = \mathrm{Id} + tu$ (note that, because $|x_u|^2 + 2\,\mathrm{Re}(\phi_u\overline{\eta}_u) = 0$, we have $\mathrm{Re}\, h^2_{1,n+2} = 0$). Then $h^t = \mathrm{O}(t)$ and

$$\|\rho(h^t)\| \geqslant \left|\det\begin{pmatrix} h^t_{1,2} & h^t_{1,n+2} \\ h^t_{n+1,2} & h^t_{n+1,n+2}\end{pmatrix}\right| = \left|\det\begin{pmatrix} t\phi_u & * \\ 0 & -t\phi_u^\dagger\end{pmatrix}\right| \asymp t^2.$$

So $\|h^t\|^2 = \mathrm{O}(t^2) = \mathrm{O}(\rho(h^t))$. Thus, Lemma 8.24(2) implies that $\rho(h^t) \asymp \|h^t\|^2$, as desired. □

The following lemma obtains a dimension bound from Condition 10.19(1).

LEMMA 10.20. *If V is a $\mathbb{R}$-subspace of $\mathbb{F}^{n-2} \oplus \mathbb{F}^{n-2}$, such that $\dim_{\mathbb{F}}(\mathbb{F}x + \mathbb{F}y) \leqslant 1$ for every $(x, y) \in V$, then either*

(1) $\dim V \leqslant q(n-2)$; *or*
(2) $n = 3$ *and* $\dim V \leqslant 2q$.

Proof. Because $\dim_{\mathbb{R}} \mathbb{F}^{n-2} = q(n-2)$, we may assume that there exist nonzero $x_0, y_0 \in \mathbb{F}^{n-2}$, such that $(x_0, 0) \in V$ and $(0, y_0) \in V$ (otherwise, the projection to one of the factors of $\mathbb{F}^{n-2} \oplus \mathbb{F}^{n-2}$ is injective when restricted to V, so (1) holds). Then $(x_0, y_0) \in V$, so, by assumption, we have $\dim_{\mathbb{F}}(\mathbb{F}x_0 + \mathbb{F}y_0) \leqslant 1$. Because x_0 and y_0 are nonzero, this implies $\mathbb{F}x_0 = \mathbb{F}y_0$.

Step 1. *For all $(x, y) \in V$, we have $y \in \mathbb{F}x_0$.* We may assume $y \neq 0$ (otherwise the desired conclusion is obvious). Then, since $\dim_{\mathbb{F}}(\mathbb{F}x + \mathbb{F}y) \leqslant 1$, we conclude that $x \in \mathbb{F}y$. Similarly, because

$$(x + x_0, y) = (x, y) + (x_0, 0) \in V + V = V,$$

we must have $x + x_0 \in \mathbb{F}y$. Therefore

$$x_0 = (x + x_0) - x \in \mathbb{F}y - \mathbb{F}y = \mathbb{F}y.$$

Since $x_0 \neq 0$, this implies $\mathbb{F}x_0 = \mathbb{F}y$, so $y \in \mathbb{F}x_0$, as desired.

Step 2. *We have $V \subset \mathbb{F}y_0 \oplus \mathbb{F}x_0$.* Given $(x, y) \in V$, Step 1 asserts that $y \in \mathbb{F}x_0$. By symmetry (interchanging the two factors of $\mathbb{F}^{n-2} \oplus \mathbb{F}^{n-2}$), we must also have $x \in \mathbb{F}y_0$. So $(x, y) \in \mathbb{F}y_0 \oplus \mathbb{F}x_0$, as desired.

Step 3. *Completion of the proof.* From Step 2, we have

$$\dim V \leqslant \dim_{\mathbb{R}}(\mathbb{F}y_0 \oplus \mathbb{F}x_0) = 2q.$$

If $n \geqslant 4$, then (1) holds; otherwise, (2) holds. □

THEOREM 10.21. *Recall that Assumptions* 10.1 *are in effect.*

If there does not exist a continuous curve $h^t \to \infty$ in H, such that $\rho(h^t) \asymp \|h^t\|^2$, then $\dim H \leqslant qn$.

Furthermore, if equality holds, then H is of the form $H = T \ltimes U$, where

(1) $T = \ker \beta$,
(2) $\mathfrak{u} = ((\mathfrak{n}_\alpha + \mathfrak{n}_{\alpha+\beta} + \mathfrak{n}_{\alpha+2\beta}) \cap \mathfrak{u}) + \mathfrak{n}_{2\alpha+2\beta}$, *and*
(3) $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$ *for every* $u \in \mathfrak{u} \smallsetminus \mathfrak{n}_{2\alpha+2\beta}$.

Proof. Note that

$$\dim \mathfrak{h}/\mathfrak{u} \leqslant 1$$

(see 10.7(1)) and

$$\dim \mathfrak{u}/\mathfrak{u}_{\phi=0} \leqslant \dim \mathfrak{n}_\alpha = q.$$

Step 1. *We have* $\dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} \leqslant q(n-2)$. Suppose not. Let V be the projection of $\mathfrak{u}_{\phi=0}$ to $\mathfrak{n}_\beta + \mathfrak{n}_{\alpha+\beta}$. We have

$$\dim V = \dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} > q(n-2),$$

and, for every $u \in \mathfrak{u}_{\phi=0}$ with $x_u \neq 0$, we have $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) \leqslant 1$ (see 10.19(1)), so Lemma 10.20 implies that $n = 3$. Therefore, $\dim \mathfrak{n}_\beta = \dim \mathfrak{n}_{\alpha+\beta} = q$. Then, because $\dim V > q(n-2) = q$, we know that $V \cap \mathfrak{n}_\beta \neq 0$ and $V \cap \mathfrak{n}_{\alpha+\beta} \neq 0$; thus, there exist $u, v \in \mathfrak{u}_{\phi=0}$, such that

- $x_u = 0$, $y_u \neq 0$; and
- $x_v \neq 0$, $y_v = 0$.

Therefore $[u, v]$ is a nonzero element of $\mathfrak{n}_{\alpha+2\beta}$ (see 7.22), so $\Delta([u, v]) \neq 0$. This contradicts Lemma 10.19(2).

Step 2. *We have* $\dim \mathfrak{d}_{\mathfrak{h}} \leqslant q - 1$. Suppose not: then, because $\dim \mathfrak{n}_{2\alpha+2\beta} = q - 1$, there is some $u \in \mathfrak{d}_{\mathfrak{h}} \smallsetminus \mathfrak{n}_{2\alpha+2\beta}$, and, because $\dim \mathfrak{n}_{2\beta} = q - 1$, there is some nonzero $v \in \mathfrak{d}_{\mathfrak{h}}$, such that $y_v = 0$. We must have $\eta_v = 0$ (otherwise 10.19(2) yields a contradiction); thus $v \in \mathfrak{n}_{2\alpha+2\beta}$. We must have $y_u \neq 0$ (otherwise 10.19(2) yields a contradiction). Thus, we see that

$$|\eta|^2 + \mathsf{x}_{u+tv}\mathsf{y}_{u+tv} = |\eta_u|^2 + (\mathsf{x}_u + t\mathsf{x}_v)(t\mathsf{y}_u)$$

is nonconstant as a function of $t \in \mathbb{R}$, so 10.19(2) yields a contradiction.

Step 3. *The desired inequality.* We have

$$\begin{aligned}\dim \mathfrak{h} &\leqslant \dim \mathfrak{h}/\mathfrak{u} + \dim \mathfrak{u}/\mathfrak{u}_{\phi=0} + \dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} + \dim \mathfrak{d}_{\mathfrak{h}}\\ &\leqslant 1 + q + q(n-2) + (q-1)\\ &= qn,\end{aligned}$$

as desired. □

ASSUMPTION. *In the remainder of the proof, we assume that* $\dim H = qn$. *We must have equality throughout the preceding paragraphs.*

Step 4. *We have* $V \subset \mathfrak{n}_{\alpha+\beta}$. Suppose not: then there is some $v \in \mathfrak{u}_{\phi=0}$, such that $y_v \neq 0$. Let $u \bullet \mathfrak{u} \setminus \mathfrak{u}_{\phi=0}$ and $w = [u, v]$. Then, from (7.22), we see that $y_w = 0$ and $x_w \neq 0$, and that $[v, w] \bullet \mathfrak{n}_{\alpha+2\beta} + \mathfrak{n}_{2\alpha+2\beta}$. From 10.19(1), we have $x_v \in \mathbb{F}y_v$ and $x_{v+2} \in \mathbb{F}y_{v+w} = \mathbb{F}y_v$, so

$$x_w = x_{v+w} - x_v \in \mathbb{F}y_v - \mathbb{F}y_v = \mathbb{F}y_v.$$

Therefore $x_w y_w^{\dagger} \neq 0$, so $\eta_{[v,w]} \neq 0$ (see 7.22), so 10.19(2) yields a contradiction.

Step 5. *We have* $\mathfrak{d}_{\mathfrak{h}} = \mathfrak{n}_{2\alpha+2\beta}$. From Step 4, together with the fact that

$$\dim V = \dim \mathfrak{u}_{\phi=0}/\mathfrak{d}_{\mathfrak{h}} = q(n-2) = \dim \mathfrak{n}_{\alpha+\beta},$$

we conclude that $V = \mathfrak{n}_{\alpha+\beta}$. Therefore,

$$\mathfrak{u}_{\phi=0} + \mathfrak{d} = V + \mathfrak{d} = \mathfrak{n}_{\alpha+\beta} + \mathfrak{d},$$

so

$$\begin{aligned}\mathfrak{u} &\supset [\mathfrak{u}_{\phi=0}, \mathfrak{u}_{\phi=0}]\\ &= [\mathfrak{u}_{\phi=0} + \mathfrak{d}, \mathfrak{u}_{\phi=0} + \mathfrak{d}]\\ &= [\mathfrak{n}_{\alpha+\beta} + \mathfrak{d}, \mathfrak{n}_{\alpha+\beta} + \mathfrak{d}]\\ &= [\mathfrak{n}_{\alpha+\beta}, \mathfrak{n}_{\alpha+\beta}]\\ &= \mathfrak{n}_{2\alpha+2\beta}.\end{aligned}$$

Because $\dim \mathfrak{d}_{\mathfrak{h}} = q - 1 = \dim \mathfrak{n}_{2\alpha+2\beta}$, we must have $\mathfrak{d}_{\mathfrak{h}} = \mathfrak{n}_{2\alpha+2\beta}$.

Step 6. *We have* $\mathfrak{d}_{\mathfrak{h}} = \mathfrak{n}_{2\alpha+2\beta}$. Let $T = (HN) \cap A$ be the projection of H to A. Then there exists $\sigma \bullet \{\beta, \alpha+\beta, \alpha+2\beta\}$, such that $T = \ker(\alpha - \sigma)$, and, in the notation of Lemma 3.21, we have

$$\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{n}^{=\alpha}) + (\mathfrak{u} \cap \mathfrak{n}^{\neq\alpha}).$$

Because T normalizes $\mathfrak{u}$, (see 10.6), we know, from Lemma 3.21, that $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{n}^{=\alpha}) + (\mathfrak{u} \cap \mathfrak{n}^{\neq\alpha})$. Since $\dim \mathfrak{u}/\mathfrak{u}_{\phi=0} = q$, we know that $\mathfrak{u} \cap \mathfrak{n}^{=\alpha}$ projects nontrivially (in fact, surjectively) to $\mathfrak{n}_\alpha$. On the other hand, we know that $\mathfrak{u} \cap \mathfrak{n}_\alpha = 0$ (otherwise 10.19(3) yields a contradiction). Therefore $\mathfrak{n}^{=\alpha} \neq \mathfrak{n}_\alpha$, so there must be a positive root $\sigma \neq \alpha$, such that $\sigma|_T = \alpha|_T$. Then $T \subset \ker(\alpha - \sigma)$; since $\dim T = \dim H/U = 1$, we must have $T = \ker(\alpha - \sigma)$.

Because $\mathfrak{u} \cap \mathfrak{n}^{\neq\alpha} \subset \mathfrak{u}_{\phi=0}$, we have

$$\dim(\mathfrak{u} \cap \mathfrak{n}^{=\alpha}) \geqslant \dim \frac{\mathfrak{u} \cap \mathfrak{n}^{=\alpha}}{\mathfrak{u}_{\phi=0} \cap \mathfrak{n}^{=\alpha}} = \dim \frac{\mathfrak{u}/(\mathfrak{u} \cap \mathfrak{n}^{\neq\alpha})}{\mathfrak{u}_{\phi=0}/(\mathfrak{u} \cap \mathfrak{n}^{\neq\alpha})} = \dim \frac{\mathfrak{u}}{\mathfrak{u}_{\phi=0}} = q.$$

Then, since $\mathfrak{u} \cap \mathfrak{n}_\alpha = 0$, we must have

$$\dim \mathfrak{n}^{=\alpha} \geqslant \dim(\mathfrak{u} \cap \mathfrak{n}^{=\alpha}) + \dim \mathfrak{n}_\alpha \geqslant q + q = 2q.$$

By inspection, we see that this implies $\sigma \notin \{2\alpha, 2\beta\}$, so we conclude that $\sigma \in \{\beta, \alpha+\beta, \alpha+2\beta\}$, as desired.

Step 7. We have $\sigma \in \{\alpha+\beta, \alpha+2\beta\}$, $T = \ker\beta$, and $\mathfrak{n}^{=\alpha} = \mathfrak{n}_\alpha + \mathfrak{n}_{\alpha+\beta} + \mathfrak{n}_{\alpha+2\beta}$. Since $\ker\beta = \ker 2\beta$, it suffices to show $\sigma \neq \beta$. Thus, let us suppose $\sigma = \beta$. (This will lead to a contradiction.) We have $\mathfrak{n}^{=\alpha} = \mathfrak{n}_\alpha + \mathfrak{n}_\beta$ (and recall that $\mathfrak{u} \cap \mathfrak{n}_\alpha = \{0\}$), so there is some $u \in \mathfrak{u}$, such that $\phi_u \neq 0$ and $y_u \neq 0$. Because $V = \mathfrak{n}_{\alpha+\beta}$, we have

$$\dim\{v \in V \mid x_v \in \mathbb{F} y_u\} = q > \dim \mathbb{F}_{\mathrm{imag}},$$

so there is some $v \in \mathfrak{u}_{\phi=0}$, such that $0 \neq x_v \in \mathbb{F} y_u$ and $\mathrm{y}_v = 0$. Then $[u, v] \in \mathfrak{n}_{\alpha+2\beta} + \mathfrak{n}_{2\alpha+2\beta}$, with $\eta_{[u,v]} \neq 0$ (see 7.22), so 10.19(2) yields a contradiction.

Step 8. *We have* $H = (H \cap A) \ltimes (H \cap N)$. Suppose not: because $T = \ker\beta$, we conclude that there is some nonzero $w \in \mathfrak{n}_\beta + \mathfrak{n}_{2\beta}$, such that w normalizes $\mathfrak{u}$ (see 10.5).

If $y_w \neq 0$, then, because $V = \mathfrak{n}_{\alpha+\beta}$, there is some $v \in \mathfrak{u}_{\phi=0}$, such that $y_w \in \mathbb{F} x_v$ and $y_v = 0$. Then $[w, v] \in \mathfrak{n}_{\alpha+2\beta} + \mathfrak{n}_{2\alpha+2\beta}$, with $\eta_{[w,v]} \neq 0$ (see 7.22), so 10.19(2) yields a contradiction.

If $y_w = 0$, then, since $w \neq 0$, we must have $\mathrm{y}_w \neq 0$. There is some $v \in \mathfrak{u}$ with $\phi_v \neq 0$. Then $[w, v] \in \mathfrak{n}_{\alpha+2\beta} + \mathfrak{n}_{2\alpha+2\beta}$, with $\eta_{[w,v]} \neq 0$ (see 7.22), so 10.19(2) yields a contradiction.

Step 9. *Completion of the proof*. (1) From Step 8, We know that $H = T \ltimes U$, and, from Step 7, that $T = \ker\beta$.

(2) Since $\mathfrak{n}^{=\alpha} = \mathfrak{n}_\alpha + \mathfrak{n}_{\alpha+\beta} + \mathfrak{n}_{\alpha+2\beta}$, it suffices to show $\mathfrak{u} \cap \mathfrak{n}^{\neq\alpha} = \mathfrak{n}_{2\alpha+2\beta}$: given $v \in \mathfrak{u} \cap \mathfrak{n}^{\neq\alpha}$, we wish to show $v \in \mathfrak{n}_{2\alpha+2\beta}$. Because $V = \mathfrak{n}_{\alpha+\beta}$, we know that $y_v = 0$. Thus, all that remains is to show that $\mathrm{y}_v = 0$. If not, then choosing $u \in \mathfrak{u}$ with $\phi_u \neq 0$, we see that $\eta_{[u,v]} \neq 0$ (see 7.22). So 10.19(2) yields a contradiction.

(3) From Lemma 10.19(3), we know that $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$ for every $u \in \mathfrak{u} \smallsetminus \mathfrak{n}_{2\alpha+2\beta}$. ∎

11. Homogeneous Spaces of SO(2, *n*) and SU(2, *n*)

This section proves two main results. Both assume that G is either $\mathrm{SO}(2,n)$ or $\mathrm{SU}(2,n)$.

(1) Theorem 11.1 shows that if n is odd, and one or two specific homogeneous spaces of G do not have tessellations, then no interesting homogeneous space of G has a tessellation.
(2) Theorem 11.2 shows that if n is even, then certain deformations of the examples found by Kulkarni and Kobayashi (see 1.11) are essentially the only interesting homogeneous spaces of G that have tessellations.

The classification results of Section 10 (specifically, Theorems 10.21 and 10.14) play a crucial role in the proofs.

We use the notation $\mathrm{SU}(2, n; \mathbb{F})$ of Section 7, to provide a fairly unified treatment of $\mathrm{SO}(2, n)$ and $\mathrm{SU}(2, n)$.

THEOREM 11.1. *Assume* $G = \mathrm{SU}(2, 2m+1; \mathbb{F})$ *with* $m \geqslant 1$, *and let* H *be any closed, connected subgroup of* G, *such that neither* H *nor* G/H *is compact.*

If Conjecture 1.19 *is true, then* G/H *does not have a tessellation.*

Proof. Assume Conjecture 1.19 is true, and suppose Γ is a crystallographic group for G/H. (This will lead to a contradiction.) Let

$$H_1 = \mathrm{SU}(1, 2m+1; \mathbb{F}) \quad \text{and} \quad H_2 = \mathrm{Sp}(1, m; \mathbb{F})$$

(see 7.12). From (7.15), we have $d(H_1) = q(2m+1)$ and $d(H_2) = q(2m)$, where $q = \dim_{\mathbb{R}} \mathbb{F}$. We may assume that $H \subset AN$ (see 3.5), and that H is compatible with A (see 10.3).

Because H is not a Cartan decomposition subgroup (see 2.9), the contrapositive of Proposition 8.19 implies, for some $k \in \{1, 2\}$, that there does **not** exist a continuous curve $h^t \to \infty$ in H, such that $\rho(h^t) \asymp \|h^t\|^k$. Therefore, either Theorem 10.14 (if $k = 1$) or Theorem 10.21 (if $k = 2$) implies that $d(H) \leqslant q(2m+1) = d(H_1)$.

We consider two cases.

Case 1. *Assume that* Γ *acts properly discontinuously on* G/H_1. Theorem 4.1(2) (combined with the fact that $d(H) \leqslant d(H_1)$) implies that G/H_1 has a tessellation. This contradicts either Theorem 1.20 (if $\mathbb{F} = \mathbb{R}$) or Conjecture 1.19c (if $\mathbb{F} = \mathbb{C}$).

Case 2. *Assume that* Γ *does not act properly discontinuously on* G/H_1. From Lemma 8.21, we know that $\mu(H_1)$ and $\mu(H_2)$ are the two walls of A^+, so Corollary 4.11 (combined with the assumption of this case) implies that Γ acts properly discontinuously on G/H_2. Therefore, since Conjecture 1.19ab asserts that G/H_2 does not have a tessellation, the contrapositive of Theorem 4.1(2) (with H_2 in the role of H_1) implies that $d(H) > d(H_2) = q(2m)$. Hence, the contrapositive of Theorem 10.14 implies there is a continuous curve $h^t \to \infty$ in H, such that $\rho(h^t) \asymp h^t$. Thus, there is a compact subset C of G, such that $H_1 \subset CHC$ (see 8.22). Since Γ acts properly discontinuously on G/H, this implies that Γ acts properly discontinuously on G/H_1 (see 2.4). This contradicts the assumption of this case. $\square$

THEOREM 11.2. *Assume* $G = \mathrm{SU}(2, 2m; \mathbb{F})$ *with* $m \geqslant 2$, *and let* H *be a closed, connected, nontrivial, proper subgroup of* AN.

The homogeneous space G/H *has a tessellation if and only if either*

(1) *there is an* $\mathbb{R}$*-linear map* $B\colon \mathbb{F}^{n-2} \to \mathbb{F}^{n-2}$, *such that*
  (a) $\mathrm{Im}((vB)(wB)^{\dagger}) = -\mathrm{Im}(vw^{\dagger})$ *for every* $v, w \in \mathbb{F}^{n-2}$ (*see* 9.8), *and*
  (b) $xB \notin \mathbb{F}x$, *for every nonzero* $x \bullet \mathbb{F}^{n-2}$ (*see* 9.9), *and*
  (c) H *is conjugate to* H_B (*see* 9.3 *and* 9.6); *or*
(2) $\mathbb{F} = \mathbb{R}$ *and* H *is conjugate to* $\mathrm{SU}(1, 2m; \mathbb{R}) \cap AN$ (*see* 7.12); *or*
(3) $\mathbb{F} = \mathbb{C}$ *and there exists* $c \in (0, 1]$, *such that* H *is conjugate to* $H_{[c]}$ (*see* 9.11).

Proof. ($\Leftarrow$) See (1) Theorem 9.7(2), (2) Theorem 9.2 (and 7.15), or (3) Theorem 9.14(2).

($\Rightarrow$) Let $n = 2m$, so $G = \mathrm{SU}(2, n; \mathbb{F})$. By combining Remark 1.16, Corollary 4.12, Lemma 8.21, and Remark 7.15, we see that

$$\dim H = d(H) \geqslant \min\{d(\mathrm{SU}(1, n; \mathbb{F})), d(\mathrm{Sp}(1, m; \mathbb{F}))\} = qn.$$

Also, we may assume H is compatible with A (see 10.3). Because H is not a Cartan decomposition subgroup (see 2.9), Proposition 8.19 implies that one of the following two cases applies.

Case 1. *Assume there does not exist a continuous curve* $h^t \to \infty$ *in* H, *such that* $\rho(h^t) \asymp h^t$. Since $\dim H \geqslant qn$, Theorem 10.14 implies that $\dim H = qn$, and that H is of the form $H = T \ltimes U$ (with $U \subset N$), where

(i) $T = \ker \alpha$;
(ii) $\phi_u = 0$ for every $u \in \mathfrak{u}$;
(iii) $\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = 2$, for every $u \bullet \mathfrak{u} \smallsetminus \mathfrak{d}_{\mathfrak{h}}$;
(iv) $|\eta_z|^2 + \mathsf{x}_z \mathsf{y}_z \neq 0$ for every nonzero $z \bullet \mathfrak{d}_{\mathfrak{h}}$;
(v) $\dim \mathfrak{u}/\mathfrak{d}_{\mathfrak{h}} = q(n-2)$; and
(vi) $\dim \mathfrak{d}_{\mathfrak{h}} = 2q - 1$.

Step 1.1. *We may assume that* $\mathfrak{d}_{\mathfrak{h}} = \{z \in \mathfrak{d} \mid \mathsf{x}_z = -\mathsf{y}_z\}$. Because $\dim \mathfrak{d}_{\mathfrak{h}} = 2q - 1$ (see vi), it suffices to show that $\mathsf{x}_z = -\mathsf{y}_z$ for all $z \in \mathfrak{d}_{\mathfrak{h}}$. This is trivially true if $\mathbb{F} = \mathbb{R}$, as $\mathsf{x}_z, \mathsf{y}_z \in \mathbb{F}_{\mathrm{imag}} = \{0\}$ in this case. Thus, we assume $\mathbb{F} = \mathbb{C}$.

For any $z \in \mathfrak{d}_{\mathfrak{h}}$ with $\eta_z = \mathsf{y}_z = 0$, we know, from (iv), that $z = 0$; therefore, Lemma 8.26(1) implies there exist $\mathbb{R}$-linear maps $R\colon \mathbb{C} \to i\mathbb{R}$ and $S\colon i\mathbb{R} \to i\mathbb{R}$, such that $\mathsf{x}_z = R(\eta_z) + S(\mathsf{y}_z)$ for all $z \in \mathfrak{d}_{\mathfrak{h}}$. More concretely, we may say that there exist $\lambda \bullet \mathbb{C}$ and $c \bullet \mathbb{R}$, such that $\mathsf{x}_z = \mathrm{Im}(\lambda\eta_z) + c\mathsf{y}_z$ for all $z \in \mathfrak{d}_{\mathfrak{h}}$.

Let v be the element of $\mathfrak{n}_{\alpha}$ with $\phi_v = \overline{\lambda}/2$, and let $H^* = \exp(-v) H \exp(v)$ be the conjugate of H by $\exp(v)$. Then H^* satisfies the conditions imposed on H (note that H^*, like H, is compatible with A (see 10.4)), so there exist $\lambda^* \in \mathbb{C}$ and $c^* \in \mathbb{R}$, such that $\mathsf{x}_{z^*} = \mathrm{Im}(\lambda^* \eta_{z^*}) + c^* \mathsf{y}_{z^*}$ for all $z^* \bullet \mathfrak{d}_{\mathfrak{h}}^*$. Given $z^* \in \mathfrak{d}_{\mathfrak{h}}^*$ with $\mathsf{y}_{z^*} = 0$, let $z = \exp(v)z\exp(-v)$. Because $\mathsf{y}_{z^*} = 0$, we have $[[z^*, -v], -v] = 0$, so, from Remark 7.23 and (7.22), we see that

$$\mathsf{y}_z = \mathsf{y}_{z^*} = 0,$$
$$\eta_z = \eta_{z^*} - \phi_{-v}\mathsf{y}_{z^*} = \eta_{z^*},$$

and

$$\mathsf{x}_z = \mathsf{x}_{z^*} + 2\,\mathrm{Im}(\phi_{-v}\overline{\eta_{z^*}}) = \mathsf{x}_{z^*} + 2\,\mathrm{Im}\big((-\overline{\lambda}/2)\overline{\eta_z}\big) = \mathsf{x}_{z^*} + \mathrm{Im}(\lambda\eta_z) = \mathsf{x}_{z^*} + \mathsf{x}_z.$$

Therefore

$$0 = \mathsf{x}_{z^*} = \mathrm{Im}(\lambda^*\eta_{z^*}) + c^*\mathsf{y}_{z^*} = \mathrm{Im}(\lambda^*\eta_{z^*}).$$

Since η_{z^*} is arbitrary, this implies $\lambda^* = 0$. Thus, by replacing H with H^*, we may assume that $\lambda = 0$. This means that $\mathsf{y}_z = c\mathsf{x}_z$ for all $z \in \mathfrak{d}_{\mathfrak{h}}$.

From (vi) (and because $\mathbb{F} = \mathbb{C}$, so $q = 2$), we know that $\dim \mathfrak{d}_{\mathfrak{h}} = 3 > 1$, so there is some nonzero $w \in \mathfrak{d}_{\mathfrak{h}}$, such that $\mathsf{y}_w = 0$. (So $\mathsf{x}_w = c\mathsf{y}_w = 0$.) Then $|\eta_w|^2 + \mathsf{x}_w\mathsf{y}_w = |\eta_w|^2 > 0$, so we see, from (iv), that $|\eta_z|^2 + \mathsf{x}_z\mathsf{y}_z > 0$ for every nonzero $z \in \mathfrak{d}_{\mathfrak{h}}$. Now, since

$$\dim \mathfrak{d}_{\mathfrak{h}} = 3 > 2 = \dim \mathfrak{n}_{\alpha+2\beta},$$

there is some nonzero $z \in \mathfrak{d}_{\mathfrak{h}}$, such that $\eta_z = 0$. We have

$$0 < |\eta_z|^2 + \mathsf{x}_z\mathsf{y}_z = 0 + c\mathsf{y}_z^2.$$

Because y_z is pure imaginary, we know that $\mathsf{y}_z^2 < 0$, so this implies that $c < 0$. Thus, replacing H by a conjugate under a diagonal matrix, we may assume $c = -1$, as desired.

Step 1.2. *Setting* $\mathfrak{u}' = (\mathfrak{n}_\alpha + \mathfrak{n}_{\alpha+\beta}) \cap \mathfrak{u}$, *we have* $\mathfrak{u} = \mathfrak{u}' + \mathfrak{d}_{\mathfrak{h}}$. Since $T = \ker\alpha$ (see i), we have

- $\beta|_T = (\alpha+\beta)|_T$,
- $2\beta|_T = (\alpha+2\beta)|_T = (2\alpha+2\beta)|_T$, and
- $\beta|_T \neq 2\beta|_T$.

Thus, in the notation of Lemma 3.21, we have $\mathfrak{n}^{=\beta} \cap \mathfrak{u} = \mathfrak{u}'$ and $\mathfrak{n}^{\neq\beta} \cap \mathfrak{u} = \mathfrak{d}_{\mathfrak{h}}$, so $\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{d}_{\mathfrak{h}}$, as desired. (Note that this is a direct sum of vector spaces, not of Lie algebras: we have $[\mathfrak{u}', \mathfrak{u}'] \subset \mathfrak{d}_{\mathfrak{h}}$.)

Step 1.3. *Completion of the proof of Case* 1. For any $u \in \mathfrak{u}'$ with $x_u = 0$, we have

$$\dim_{\mathbb{F}}(\mathbb{F}x_u + \mathbb{F}y_u) = \dim_{\mathbb{F}} \mathbb{F}y_u \leqslant 1 < 2,$$

so $u \in \mathfrak{u}' \cap \mathfrak{d}_{\mathfrak{h}} = \{0\}$ (see iii); therefore, Lemma 8.26(1) implies there is a $\mathbb{R}$-linear map $B\colon \mathbb{F}^{n-2} \to \mathbb{F}^{n-2}$, such that $y_u = x_uB$ for all $u \in \mathfrak{u}'$. Then, because

$$\dim \mathfrak{u}_{\alpha+\beta} = \dim_{\mathbb{R}} \mathbb{F}^{n-2} = q(n-2) = \dim \mathfrak{u}'$$

(see v), we must have

$$\mathfrak{u}' = \{ u \bullet \mathfrak{n}_\beta + \mathfrak{n}_{\alpha+\beta} \mid y_u = x_u B \}.$$

Combining this with (i) and the conclusions of Steps 1.1 and 1.2, we see that $\mathfrak{h} = \mathfrak{h}_B$. Therefore $H = H_B$, so Conclusion (1c) holds.

From (iii), we see that Conclusion (1b) holds.

Letting $z = [u, v]$, for any $u, v \bullet \mathfrak{u}'$, we see, from (7.22), that

$$\mathsf{x}_z = -2\,\mathrm{Im}(x_u x_v^\dagger)$$

and

$$\mathsf{y}_z = -2\,\mathrm{Im}(y_u y_v^\dagger) = -2\,\mathrm{Im}\big((x_u B)(x_v B)^\dagger\big).$$

From Step 1.1, we know that $\mathsf{y}_z = -\mathsf{x}_z$, so this implies that Conclusion (1a) holds.

Case 2. *Assume there does not exist a continuous curve $h^t \to \infty$ in H, such that* $\rho(h^t) \asymp \|h^t\|^2$. Since $\dim H \geqslant qn$, Theorem 10.21 implies that $\dim H = qn$, and that H is of the form $H = T \ltimes U$, where

(i) $T = \ker \beta$,
(ii) $\mathfrak{u} = ((\mathfrak{n}_\alpha + \mathfrak{n}_{\alpha+\beta} + \mathfrak{n}_{\alpha+2\beta}) \cap \mathfrak{u}) + \mathfrak{n}_{2\alpha+2\beta}$, and
(iii) $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$ for every $u \in \mathfrak{u} \smallsetminus \mathfrak{n}_{2\alpha+2\beta}$.

Let

$$\mathfrak{u}' = (\mathfrak{n}_\alpha + \mathfrak{n}_{\alpha+\beta} + \mathfrak{n}_{\alpha+2\beta}) \cap \mathfrak{u}$$

(so $\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{n}_{2\alpha+2\beta}$). Let Q be the sesquilinear form (or bilinear form, if $\mathbb{F} = \mathbb{R}$) on $\mathbb{F} \oplus \mathbb{F}^{n-2} \oplus \mathbb{F}$ defined by

$$Q((\phi_1, x_1, \eta_1), (\phi_2, x_2, \eta_2)) = \phi_1 \overline{\eta_2} + x_1 x_2^\dagger + \eta_1 \overline{\phi_2}.$$

Let

$$V_\mathfrak{h} = \{ (\phi_u, x_u, \eta_u) \in \mathbb{F} \oplus \mathbb{F}^{n-2} \oplus \mathbb{F} \mid u \in \mathfrak{u}' \}.$$

From (iii), we see that the restriction of $\mathrm{Re}\,Q$ to $V_\mathfrak{h}$ is a (positive-definite) inner product.

Let $V_\mathfrak{h}^\perp$ be the $(\mathrm{Re}\,Q)$-orthogonal complement to $V_\mathfrak{h}$. As a form over $\mathbb{F}$, Q has signature $(1, n-1)$. Thus, as a form over $\mathbb{R}$, $\mathrm{Re}\,Q$ has signature $(q, q(n-1))$. Since

$$\dim V_\mathfrak{h} = \dim \mathfrak{h} - \dim \mathfrak{t} - \dim \mathfrak{n}_{2\alpha+2\beta} = qn - 1 - (q-1) = q(n-1),$$

we conclude that $V_\mathfrak{h}^\perp$ is a q-dimensional $\mathbb{R}$-subspace on which $\mathrm{Re}\,Q$ is negative-definite.

Choose some nonzero $u \bullet V_\mathfrak{h}^\perp$. Multiplying by a real scalar to normalize, we may assume $Q(u, u) = -2$. Because $\mathrm{SU}(1, n-1)$ is transitive on the vectors of norm -1, there is some $g \bullet \mathrm{SU}(\mathrm{Re}\,Q)$, such that $g(u) = (1, 0, -1)$. Thus, letting

$$\hat{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(2, n; \mathbb{F}), \tag{11.3}$$

and $\mathfrak{h}^\sharp = \hat{g}^{-1}\mathfrak{h}g$, we have $(1,0,-1) \bullet V_{\mathfrak{h}^\sharp}$, so, by replacing $\mathfrak{h}$ with the conjugate $\mathfrak{h}^\sharp$, we may assume $u = (1,0,-1)$.

Then

$$\begin{aligned} V_{\mathfrak{h}} &\subset u^\perp \\ &= (1,0,-1)^\perp \\ &= \{\, (\phi, x, \eta) \mid \operatorname{Re} Q((\phi, x, \eta), (1,0,-1)) = 0 \,\} \\ &= \{\, (\phi, x, \eta) \mid \operatorname{Re}\bigl(\phi(-1) + x(0^\dagger) + \eta(1)\bigr) = 0 \,\} \\ &= \{\, (\phi, x, \eta) \mid \operatorname{Re}\eta = \operatorname{Re}\phi \,\}. \end{aligned} \tag{11.4}$$

Subcase 2.1. *Assume* $\mathbb{F} = \mathbb{R}$. By comparing (11.4) and (7.13) (with $\mathbb{F} = \mathbb{R}$), we conclude that

$$\mathfrak{h} \subset \mathrm{SU}(1,n;\mathbb{R}) \cap (\mathfrak{a} + \mathfrak{n}).$$

By comparing dimensions, we see that equality must hold; this establishes Conclusion (2).

Subcase 2.2. *Assume* $\mathbb{F} = \mathbb{C}$. Choose some nonzero $v \in V_{\mathfrak{h}}^\perp$, such that v is $(\operatorname{Re} Q)$-orthogonal to u. Multiplying by a real scalar to normalize, we may assume $Q(v,v) = -2$. By replacing v with $-v$ if necessary, we may assume $(\operatorname{Im} Q(u,v))/i \geqslant 0$.

Because v is $(\operatorname{Re} Q)$-orthogonal to $u = (1,0,-1)$, we have $\operatorname{Re}\eta_v = \operatorname{Re}\phi_v$ (see 11.4). Let $s = (\operatorname{Im}\phi_v)/i$ and $t = (\operatorname{Im}\eta_v)/i$. Then

$$\begin{aligned} 0 &\leqslant (\operatorname{Im} Q(u,v))/i \\ &= (\operatorname{Im}((1)\overline{\eta_v} + 0(x_v^\dagger) + (-1)\overline{\phi_v}))/i \\ &= (-\operatorname{Im}\eta_v + \operatorname{Im}\phi_v)/i \\ &= -t + s, \end{aligned}$$

so

$$(\operatorname{Im} Q(u,v))/i = |s - t|.$$

Also,

$$\begin{aligned} -2 &= Q(v,v) \\ &= |x_v|^2 + 2\operatorname{Re}(\phi_v\overline{\eta_v}) \\ &= |x_v|^2 + 2(\operatorname{Re}\phi_v)^2 - 2(\operatorname{Im}\phi_v)(\operatorname{Im}\eta_v) \\ &\geqslant \; - 2(\operatorname{Im}\phi_v)(\operatorname{Im}\eta_v) \\ &= 2st, \end{aligned}$$

so $st \leqslant -1$. Thus, s and t are of opposite signs so, because $|s|\,|t| \geqslant 1$, we have

$$(\operatorname{Im} Q(u,v))/i = |s - t| = |s| + |t| \geqslant 2.$$

Therefore, we may choose $c \in (0, 1]$, such that

$$\operatorname{Im} Q(u, v) = i\left(c + \frac{1}{c}\right).$$

Let

$$w = \left(\frac{i}{c}, 0, -ic\right).$$

Then

$$Q(w, w) = |x_w|^2 + 2\operatorname{Re}(\phi_w \overline{\eta_w}) = 0^2 + 2(i/c)(ic) = -2 = Q(v, v),$$

and

$$Q(u, w) = \phi_u \overline{\eta_w} + x_u x_w^\dagger + \eta_u \overline{\phi_w} = (1)(ic) + 0 + (-1)(-i/c) = i\left(c + \frac{1}{c}\right) = \operatorname{Im} Q(u, v).$$

Hence, there is some $h \in \mathrm{SU}(Q)$, such that $h(u) = u$ and $h(v) = w$. Thus, replacing $\mathfrak{h}$ with the conjugate $\hat{h}^{-1}\mathfrak{h}\hat{h}$ (cf. 11.3), we may assume $v = w$.

Therefore

$$\begin{aligned} V_{\mathfrak{h}} &\subset v^\perp \\ &= (cv)^\perp \\ &= (i, 0, -ic^2)^\perp \\ &= \{(\phi, x, \eta) \mid \operatorname{Re} Q((\phi, x, \eta), (i, 0, -ic^2)) = 0\} \\ &= \{(\phi, x, \eta) \mid \operatorname{Re}(\phi(ic^2) + x(0^\dagger) + \eta(-i)) = 0\} \\ &= \{(\phi, x, \eta) \mid \operatorname{Im} \eta = c^2 \operatorname{Im} \phi\}. \end{aligned}$$

By combining this with (11.4) and comparing with (9.11) (with $\mathbb{F} = \mathbb{C}$), we conclude that $\mathfrak{h} \subset \mathfrak{h}_{[c^2]}$. By comparing dimensions, we see that equality must hold; this establishes Conclusion (3) (because $0 < c^2 \leqslant 1$). ■

Theorem 11.2″ can be restated in the following more elementary (but less precise) form. □

COROLLARY 11.5. *Let H be a closed, connected subgroup of $G = \mathrm{SU}(2, 2m; \mathbb{F})$ with $m \geqslant 2$, such that neither H nor G/H is compact, and let $q = \dim_{\mathbb{R}} \mathbb{F}$.*

The homogeneous space G/H has a tessellation if and only if

(1) *$d(H) = 2qm$; and*
(2) *either $H \sim \mathrm{SU}(1, 2m; \mathbb{F})$ or $H \sim \mathrm{Sp}(1, m; \mathbb{F})$.*

Proof. ($\Leftarrow$) This is Theorem 9.2.

($\Rightarrow$) Theorem 11.2($\Rightarrow$) provides us with three possibilities.

(1) In each case, we have $d(H) = 2qm$ (see 9.5, 7.15, and 9.12).
(2) In each case, there is some $k \in \{1, 2\}$, such that $\rho(h) \asymp \|h\|^k$ for $h \in H$ (see 9.7(1), 8.21(1), and 9.14(1)). Then Corollary 8.23 implies either that $H \sim \mathrm{SU}(1, 2m; \mathbb{F})$ (if $k = 1$) or that $H \sim \mathrm{Sp}(1, m; \mathbb{F})$ (if $k = 2$). □

The following proposition shows that no further restriction can be placed on c in the statement of Theorem 11.2(3).

PROPOSITION 11.6. *If* $\mathbb{F} = \mathbb{C}$, *then* $H_{[c]}$ *is not conjugate to* $H_{[c']}$, *unless* $c = c'$ (*for* $c, c' \in (0, 1]$).

Proof. Suppose $g^{-1}H_{[c]}g = H_{[c']}$, for some $g \bullet G = \mathrm{SU}(2, 2m)$. Because all maximal split tori in $H_{[c']}$ are conjugate, we may assume that g normalizes $\ker\beta$. Since all roots of $\ker\beta$ on both $\mathfrak{h}_{[c]}$ and $\mathfrak{h}_{[c']}$ are positive, g cannot invert $\ker\beta$, so we conclude that g centralizes $\ker\beta$; that is, $g \bullet C_G(\ker\beta)$.

In the notation of Case 2 of the proof of Theorem 11.2, define

$$S = \{\hat{h} \mid h \in \mathrm{SU}(Q)\}$$

(cf. 11.3). Then $C_G(\ker\beta) = (\ker\beta)S$, so we may assume $g \in S$ (because $\ker\beta$, being a subgroup of $H_{[c]}$, obviously normalizes $H_{[c]}$). Write $g = \hat{h}$. Then, because $g^{-1}H_{[c]}g = H_{[c']}$, we must have $h(V_{\mathfrak{h}_{[c]}}) = V_{\mathfrak{h}_{[c']}}$; hence $h(V_{\mathfrak{h}_{[c]}}^{\perp}) = V_{\mathfrak{h}_{[c']}}^{\perp}$.

For any basis $\{u, v\}$ of $V_{\mathfrak{h}_{[c]}}^{\perp}$ with $Q(u, u) = Q(v, v) = -2$ and $\operatorname{Re} Q(u, v) = 0$, we have

$$\operatorname{Im} Q(u, v) = \pm i(c + (1/c)).$$

Similarly for any $(\operatorname{Re} Q)$-orthonormal basis $\{u', v'\}$ of $V_{\mathfrak{h}_{[c']}}^{\perp}$. Because $h \in \mathrm{SU}(Q)$, this implies $c + (1/c) = c' + (1/c')$. Because $c, c' \in (0, 1]$, we conclude that $c = c'$, as desired. □

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References

[Abe] Abels, H.: Properly discontinuous groups of affine transformations: a survey, *Geom. Dedicata* **87** (2001), 309–333.

[AMS] Abels, H., Margulis, G. A., and Soifer, G. A.: Properly discontinuous groups of affine transformations with orthogonal linear part, *CR Acad. Sci. Paris* **324 I** (1997), 253–258.

[Ben] Benoist, Y.: Actions propres sur les espaces homogènes réductifs, *Ann. Math.* **144** (1996), 315–347.

[BL] Benoist, Y. and Labourie, F.: Sur les espaces homogènes modèles de variétés compactes, *Publ. Math. IHES* **76** (1992), 99–109.

[Br1] Borel, A.: Compact Clifford-Klein forms of symmetric spaces, *Topology* **2** (1963), 111–122.

[Br2] Borel, A.: *Linear Algebraic Groups*, 2nd edn, Springer, New York, 1991.

[BT] Borel, A., Tits, J.: Groupes réductifs, *Publ. Math. IHES* **27** (1965), 55–150.

[Cha] Charlap, L. S.: *Bieberbach Groups and Flat Manifolds*, Springer, New York, 1986.

[Coh] Cohen, D. E.: *Groups of Cohomological Dimension One*, Lecture Notes in Math. 245, Springer, New York, 1972.

[Cow] Cowling, M.: Sur les coefficients des représentations unitaires des groupes de Lie simple, In: P. Eymard, J. Faraut, G. Schiffmann, and R. Takahashi (eds), *Analyse harmonique sur les groupes de Lie II* (*Séminaire Nancy-Strasbourg 1976–78*), Lecture Notes in Math. 739, Springer, New York, 1979, pp. 132–178.

[Dix] Dixmier, J.: L'application exponentielle dans les groupes de Lie résolubles, *Bull. Soc. Math. France* **85** (1957), 113–121.

[Dol] Dold, A.: *Lectures on Algebraic Topology*, 2nd edn, Springer, New York, 1980.

[FG] Fried, D. and Goldman, W. M.: Three-dimensional affine crystallographic groups, *Adv. Math.* **47** (1983), 1–49.

[Gol] Goldman, W.: Nonstandard Lorentz space forms, *J. Differential Geom.* **21** (1985), 301–308.

[GdW] Goodman, R. and Wallach, N.: *Representations and Invariants of the Classical Groups*, Cambridge Univ. Press, Cambridge, 1998.

[GtW] Goto, M. and Wang, H.-C.: Non-discrete uniform subgroups of semisimple Lie groups, *Math. Ann.* **198** (1972), 259–286.

[Gro] Gromov, M.: Asymptotic invariants of infinite groups, In: G. A. Niblo and M. A. Roller (eds), *Geometric Group Theory*, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Notes 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[Hel] Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Hc1] Hochschild, G. P.: *Basic Theory of Algebraic Groups and Lie Algebras*, Springer, New York, 1981.

[Hc2] Hochschild, G. P.: *The Structure of Lie Groups*, Holden-Day, San Francisco, 1965.

[How] Howe, R.: A notion of rank for unitary representations of the classical groups, In: A. Figà Talamanca, (ed.), *Harmonic Analysis and Group Representations*, (CIME 1980), Liguori, Naples, 1982, pp. 223–331.

[Hm1] Humphreys, J. E.: *Linear Algebraic Groups*, Springer, New York, 1975.

[Hm2] Humphreys, J. E.: *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1980.

[Hus] Husemoller, D.: *Fibre Bundles*, 2nd edn, Springer, New York, 1966.

[IW] Iozzi, A. and Witte, D.: Cartan decomposition subgroups of SU(2,*n*), *J. Lie Theory* **11** (2001), 505–543.

[Iwa] Iwasawa, K.: On some types of topological groups, *Ann. Math.* **50** (1949), 507–558.

[Jac] Jacobson, N.: *Lie Algebras*, Dover, New York, 1979.

[KS] Katok, A. and Spatzier, R.: First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity, *IHES Publ. Math.* **79** (1994), 131–156.

[KN] Kobayashi, S. and Nomizu, K.: *Foundations of Differential Geometry*, vol. 1, Interscience, New York, 1963.

[Kb1] Kobayashi, T.: Proper action on a homogeneous space of reductive type, *Math. Ann.* **285** (1989), 249–263.

[Kb2] Kobayashi, T.: Discontinuous groups acting on homogeneous spaces of reductive type, In: T. Kawazoe, T. Oshima and S. Sano (eds), *Proceedings of ICM–90 Satellite Conference on Representation Theory of Lie Groups and Lie Algebras at Fuji–Kawaguchiko* (*31 August –3 September, 1990*), World Scientific, Singapore, 1992, pp. 59–75.

[Kb3] Kobayashi, T.: A necessary condition for the existence of compact Clifford–Klein forms of homogeneous spaces of reductive type, *Duke Math. J.* **67** (1992), 653–664.

[Kb4] Kobayashi, T.: On discontinuous groups acting on homogeneous spaces with non-compact isotropy groups, *J. Geom. Phys.* **12** (1993), 133–144.

[Kb5] Kobayashi, T.: Criterion of proper actions on homogeneous spaces of reductive groups, *J. Lie Theory* **6** (1996), 147–163.

[Kb6] Kobayashi, T.: Discontinuous groups and Clifford–Klein forms of pseudo-Riemannian homogeneous manifolds, In: B. Ørsted and H. Schlichtkrull, (eds), *Algebraic and Analytic Methods in Representation Theory*, Academic Press, New York, 1997, pp. 99–165.

[Kb7] Kobayashi, T.: Deformation of compact Clifford–Klein forms of indefinite Riemannian homogeneous manifolds, *Math. Ann.* **310** (1998), 395–409.

[Kb8] Kobayashi, T.: Discontinuous groups for non-Riemannian homogeneous spaces, In: B. Engquist and W. Schmidt (eds), *Mathematics Unlimited—2001 and Beyond*, Springer, New York, 2001, pp. 723–747.

[KO] Kobayashi, T. and Ono, K.: Note on Hirzebruch's proportionality principle, *J. Fac. Sci. Univ. Tokyo, Math.* **37** (1990), 71–87.

[Kos] Kostant, B.: On convexity, the Weyl group, and the Iwasawa decomposition, *Ann. Sci. Ecole Norm. Sup.* **6** (1973), 413–455.

[Kul] Kulkarni, R.: Proper actions and pseudo-Riemannian space forms, *Adv. Math.* **40** (1981), 10–51.

[Lab] Labourie, F.: Quelques résultats récents sur les espaces localement homogènes compacts, In: P. de Bartolomeis, F. Tricerri and E. Vesentini (eds), *Manifolds and Geometry*, Sympos. Math. 36, Cambridge Univ. Press, 1996.

[Mr1] Margulis, G. A.: Free properly discontinuous groups of affine transformations, *Dokl. Akad. Nauk SSSR* **273** (1983), 937–940.

[Mr2] Margulis, G. A.: Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients, *Bull. Soc. Math. France* **125** (1997), 447–456.

[MS] Milnor, J. W. and Stasheff, J. D.: *Characteristic Classes*, Princeton Univ. Press, Princeton, 1974.

[Mos] Mostow, G. D.: Factor spaces of solvable groups, *Ann. Math.* **60** (1954), 1–27.

[Oh] Oh, H.: Tempered subgroups and representations with minimal decay of matrix coefficients, *Bull. Soc. Math. France* **126** (1998), 355–380.

[OW1] Oh, H. and Witte, D.: Cartan-decomposition subgroups of SO(2,n), *Trans. Amer. Math. Soc.* (to appear).

[OW2] Oh, H. and Witte, D.: New examples of compact Clifford–Klein forms of homogeneous spaces of SO(2,n), *Internat. Math. Res. Not.* 2000 (8 March 2000), No. 5, 235–251.

[OW3] Oh, H. and Witte, D.: Compact Clifford-Klein forms of homogeneous spaces of SO(2,n), *Geom. Dedicata* (to appear).

[Pal] Palais, R. S.: On the existence of slices for actions of non-compact Lie groups, *Ann. Math.* **73**(2) (1961), 295–323.

[PR] Platonov, V. and Rapinchuk, A.: *Algebraic Groups and Number Theory*, Academic Press, Boston, 1994.

[Rag] Raghunathan, M. S.: *Discrete Subgroups of Lie Groups*, Springer, New York, 1972.

[Sai] Saito, M.: Sur certains groupes de Lie résolubles II, *Sci. Papers Coll. Gen. Ed. Univ. Tokyo* **7** (1957), 157–168.

[Sal] Salein, F.: Variétés anti-deSitter de dimension 3 possédant un champ de Killing non trivial, *CR Acad. Sci. Paris* **324 I** (1997), 525–530.

[Tom] Tomanov, G.: The virtual solvability of the fundamental group of a generalized Lorentz space form, *J. Differential Geom.* **32** (1990), 539–47.

[Var] Varadarajan, V. S.: *Lie Groups, Lie Algebras, and their Representations*, Springer, New York, 1984.

[Whi] Whitehead, G. W.: *Elements of Homotopy Theory*, Springer, New York, 1978.

[Wit] Witte, D.: Tessellations of solvmanifolds, *Trans. Amer. Math. Soc.* **350**(9) (1998), 3767–3796.

[ZS] Zariski, O. and Samuel, P.: *Commutative Algebra*, vol. 1, Springer, New York, 1958.

[Zeg] Zeghib, A.: On closed anti-deSitter spacetimes, *Math. Ann.* **310**(4) (1998), 695–716.

[Zm1] Zimmer, R. J.: Orbit spaces of unitary representations, ergodic theory, and simple Lie groups, *Ann. Math.* **106** (1977), 573–588.

[Zm2] Zimmer, R. J.: *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.